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Some New Results on Bipolar-Valued Fuzzy d-Algebras

S. Praveena[1,*] and M. Kamaraj[1]

1 Department of Mathematics, Government Arts and Science College, Sivakasi, Tamilnadu, India.

Abstract: In this paper we introduce definitions of generalized bipolar-valued fuzzy d-algebra and investigate some associate results.

Keywords: Fuzzy d-algebra, bipolar-valued fuzzy d-algebra, bipolar-valued fuzzy d-subalgebra and upper bipolar-valued fuzzy sets, t-level-cut of bipolar-valued fuzzy sets and some results.

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1. Introduction

In this paper we first introduced two classes of abstract algebras BCK-algebras and BCI-algebra [3]. It is known that the class of BCK-algebra is a proper subclass of the class of BCK-algebras. A wide class of abstract algebras BCH-algebras were introduced [5]. They showed that the class of BCI-algebra is a proper subclass of the class of BCH-algebras. J.Neggers and H.S.Kim [3] introduced a new notion called a d-algebra, which is another generalization of BCK-algebras and investigate relation between d-algebra and BCK-algebra. Then the author extended it to the notion of fuzzy d-algebras, fuzzy d-ideal [4] and investigate result among them. The bipolar-valued fuzzy d-algebra is considered and related results are investigate. The notion of equivalent relation on the family of all bipolar-valued fuzzy d-algebra of a d-algebra is introduced and then some properties are discussed.

2. Preliminaries

Definition 2.1 ([3]). *A d-algebra is a non-empty set X with binary operation $*$ satisfying the following axioms*

1. $x * x = 0$,
2. $0 * x = 0$,
3. $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$ *for all* $x, y \in X$.

A BCK-algebra is a d-algebra $(X; *0)$ satisfying the following additional axioms

4. $((x * y) * (x * z)) * (z * y) = 0$,
5. $(x * (x * y)) * y = 0 \; \forall \; x, y \in X$.

[*] *E-mail: spreveena91@gmail.com (Research Scholar)*

*	0	1	2	3
0	0	0	0	0
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Example 2.2 ([4]). *Let $X = \{0, 1, 2, 3\}$ be a set with the following table*

- *The diagonal elements are zero therefore $x * x = 0 \ \forall \ x \in X$.*
- *The row corresponding to 0 is 0. Therefore $0 * x = 0$.*
- *If $x_i * y_i + y_i * x_i \neq 0 \Longrightarrow x * y$ and $y * x = 0 \Longrightarrow x = y$. There is no element $x * y = y * x = 0$ for $x \neq y$.*
- *If $x = 0, y = 0, z = 0$ and if we take same value for x, y, z than the conditions $((x * y) * (x * z)) * (z * y) = 0$ and $(x * (x * y)) * y = 0$ is trivially true.*

Now we can proceeded for different value for x, y, z we get

Case 1: $x = 1, y = 1, z = 2$

1. $x * y = 1 * 1 = 0,\ x * z = 1 * 2 = 3,\ z * y = 2 * 1 = 3.$

$$(x * y) * (x * z) = 0 * 3 = 0 \Longrightarrow ((x * y) * (x * z)) * (z * y) = 0 * 3 = 0$$

2. $(x * (x * y)) * y = (1 * (1 * 1)) * 1 = (1 * 0) * 1 = 0 * 1 = 0.$

Case 2: $x = 1,\ y = 1,\ z = 3$

1. $x * y = 1 * 1 = 0,\ x * z = 1 * 3 = 2,\ z * y = 3 * 1 = 3.$

$$((x * y) * (x * z)) * (z * y) = (0 * 2) * 3 = 0 * 3 = 0$$

2. $(x * (x * y)) * y = (1 * (1 * 1)) * 1 = (1 * 1) * 1 = 0 * 1 = 0.$

Case 3: $x = 1,\ y = 2,\ z = 1$

1. $(x * y) * (x * z) = 3 * 0 = 3 \Longrightarrow ((x * y) * (x * z)) * (z * y) = 3 * 3 = 0.$
2. $(x * (x * y)) * y = (1 * (1 * 2)) * 2 = (1 * 3) * 2 = 2 * 2 = 0.$

Case 4: $x = 2,\ y = 1,\ z = 3$

1. $(x * y) * (x * z) = 3 * 1 = 2 \Longrightarrow ((x * y) * (x * z)) * (z * y) = 2 * 3 = 1 \neq 0.$
2. $(x * (x * y)) * y = (2 * (2 * 1)) * 3 = (2 * 3) * 3 = 2 \neq 0.$

Definition 2.3 ([2]). *Let X be a BCK-algebra and I be a subset of X, then I is called an ideal of X if*

1. $0 \in I$,
2. *y and $x * y \in I \Longrightarrow x \in I$ for all $x, y \in X$.*

Example 2.4 ([4]). *Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table*

*	0	1	2	3
0	0	0	0	0
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Let $I = \{0, 1\}$ be a subset of X. Then $0 \in I$ is obviously true.

1. *y and $x * y \in I \Longrightarrow x \in I\ \forall\ x, y \in X$.*

The row corresponding to 0 is 0 and I is a subset of X. Then the condition is holds.

Definition 2.5 ([3]). *A nonempty set X with a constant 0 and a binary operation $*$ is called a d-algebra if it satisfies the following axioms*

1. *$x * x = 0$,*
2. *$0 * x = 0$,*
3. *$x * y = 0$ and $y * x = 0 \Longrightarrow x = y\ \forall\ x, y \in X$.*

Example 2.6 ([3]).

a. *Every BCK-algebra is a d-algebra*

b. *Let $X = \{0, 1, 2\}$ be a set with the following table*

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

- *The diagonal elements are zero. Therefore $x * x = 0\ \forall\ x \in X$.*
- *The row corresponding to 0 is 0. Therefore $0 * x = 0$.*
- *If $x_i * y_i + y_i * x_i \neq 0 \Longrightarrow x * y = 0, y * x = 0 \Longrightarrow x = y$. There is no element $x * y = y * x$ for $x = y$.*

Then $(X; *, 0)$ is a d-algebra but not BCK-algebra since $(2 * (2 * 2)) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

Definition 2.7 ([2, 4]). *Let S be a nonempty subset of d-algebra X, then S is called subalgebra of X if $x * y \in S$, for all $x, y \in S$.*

Example 2.8 ([2]). *Let $X = \{0, a, b, c\}$ be a d-algebra with the following table*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	a	0	0
c	c	c	c	0

Let $S = \{0, a, b\}$ be a nonempty subset of a d-algebra X.

1. *In the case of same value for x, y. The diagonal elements are zero. Therefore $x * y = 0 \in S$.*

2. *The row corresponding to 0 is 0. When $x = 0$ is a fixed value then different value for $y \Longrightarrow x * y = 0 \in S$.*

Case 1: $x = a,\ y = b \Longrightarrow x * y = a * b = 0 \in S$.

Case 2: $x = b,\ y = a \Longrightarrow x * y = b * a = a \in S$.

Definition 2.9 ([2, 4])**.** *Let X be a d-algebra and I be a subset X, then I is called d-ideal of X if*

1. $0 \in I$,

2. $x * y \in I$ *and* $y \in I \Longrightarrow x \in I$,

3. $x \in I$ *and* $y \in X \Longrightarrow x * y \in I$ *i.e.,* $I \times X \subseteq I$.

Example 2.10 ([2])**.** *Let $X = \{0, a, b, c, d\}$ be a d-algebra with the following cayley table Let $I = \{0, a, c\}$ be a subset X.*

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	c
c	c	c	a	0	c
d	c	c	a	a	0

Then we can easily prove that I is a d-ideal.

Definition 2.11 ([4])**.** *A mapping $f : X \to Y$ of d-algebras is called a homomorphism if $f(x * y) = f(x) * f(y),\ \forall\, x, y \in X$.*

Note: If $f : X \to Y$ is homomorphism of d-algebras, then $f(0) = 0$.

Definition 2.12 ([4])**.** *Let μ be the fuzzy set of a set X. For a fixed $s \in [0, 1]$ the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ.*

Definition 2.13 ([4])**.** *A fuzzy set μ in d-algebra X is called fuzzy sub algebra of X if it satisfies $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Example 2.14 ([4])**.** *Let $X = \{0, 1, 2\}$ be a set given by the following table Then $(X; *, 0)$ is a d-algebra. We define a fuzzy*

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

set $\mu : X \to [0, 1]$ by $\mu(0) = 0.7,\ \mu(x) = 0.02$, where for all $x \neq 0$.

Cases	x	y	$\mu(x*y)$	$\mu(x)$	$\mu(y)$	$\min\{\mu(x), \mu(y)\}$
1	0	0	0.7	0.7	0.7	0.7
2	0	1	0.7	0.7	0.02	0.02
3	0	2	0.7	0.7	0.02	0.02
4	1	0	0.02	0.02	0.7	0.02
5	1	1	0.7	0.02	0.02	0.02
6	1	2	0.02	0.02	0.02	0.02
7	2	0	0.02	0.02	0.7	0.02
8	2	1	0.02	0.02	0.02	0.02
9	2	2	0.7	0.02	0.02	0.02

Then μ is a fuzzy sub algebra of X.

Definition 2.15 ([1]). *A bipolar-valued fuzzy set D in X is an object having the form $D = \{(x, \mu_D^P(x), \mu_D^N(x)) / x \in X\}$ where $\mu_D^P : X \longrightarrow [0,1]$ and $\mu_D^N : X \to [-1,0]$ are mappings. The position of membership degree $\mu_D^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $D = \{(x, \mu_D^P(x), \mu_D^N(x)) / x \in X\}$ and the negative membership degree $\mu_D^N(x)$ denote the satisfaction degree of x to some implicit counter-property of $D = \{(x, \mu_D^P(x), \mu_D^N(x)) / x \in X\}$.*

Definition 2.16 ([7]). *Let f be a mapping from a set X to a set Y. If $B = \{(y, \mu_B^+(y), \mu_B^-(y)) / y \in Y\}$ is an bipolar-valued fuzzy set in Y, then the preimage of B under f denoted by $f^-(B)$, is the bipolar-valued fuzzy set in X defined by $f^-(B) = \{(x, f^-(\mu_B^+)(x), f^-(\mu_B^-)(x)) / x \in X\}$, and if $D = \{(x, \mu_D^+(x), \mu_D^-(x)) / x \in X\}$ is an bipolar-valued fuzzy set in X, then the image of D under f, denoted by $f(D)$, is the bipolar-valued fuzzy set by*

$$f(D) = \{(y, f_{\sup}(\mu_D^+)(y), f_{\inf}(\mu_D^-)(y)) / y \in Y\},$$

where

$$f_{\sup}(\mu_D^+)(y) = \begin{cases} \sup\limits_{x \in f^-(y)} \mu_D^+(x), & \text{if } f^-(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

$$f_{\inf}(\mu_D^-)(y) = \begin{cases} \inf\limits_{x \in f^-(y)} \mu_D^-(x), & \text{if } f^-(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

3. Bipolar-Valued Fuzzy d-Algebra

Definition 3.1. *Let X be a d-algebra. An bipolar-valued fuzzy set $D = (x, \mu_D^+, \mu_D^-)$ in X is called an bipolar-valued fuzzy d-algebra if it satisfies*

$$\mu_D^+(x*y) \geq \min\{\mu_D^+(x), \mu_D^+(y)\},$$
$$\mu_D^-(x*y) \leq \max\{\mu_D^-(x), \mu_D^-(y)\}, \quad \text{for all } x, y \in X.$$

Example 3.2.

1. *Consider a d-algebra* $X = \{0, a, b, c\}$ *with the following table*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Let $D = \left(x, \mu_D^+(x), \mu_D^-(x)\right)$ *be an bipolar-valued fuzzy set in* X *defined by* $\mu_D^+(0) = 0.8 = \mu_D^+(a), \mu_D^+(b) = \mu_D^+(c) = 0.3$ *and* $\mu_D^-(0) = \mu_D^-(a) = 0.03, \mu_D^-(b) = \mu_D^-(c) = 0.08$. *Then* $D = \left(x, \mu_D^+(x), \mu_D^-(x)\right)$ *is an bipolar-valued fuzzy d-algebra.*

2. *Consider a d-algebra* $X = \{0, a, b, c\}$. *Let* $D = \left(x, \mu_D^+, \mu_D^-\right)$ *be an bipolar-valued fuzzy set in* X *defined by* $\mu_D^+(0) = \mu_D^+(a) = t_1, \mu_D^+(b) = \mu_D^+(c) = t_2$ *and* $\mu_D^-(0) = \mu_D^-(a) = s_1 = \mu_D^-(c), \mu_D^-(b) = s_2$, *where* $t_1 > t_2$, $s_1 < s_2$ *and* $s_i + t_i \in [0, 1]$ *for* $i = 1, 2$. *Then* $D = \left(x, \mu_D^+, \mu_D^-\right)$ *is an bipolar-valued fuzzy d-algebra.*

Definition 3.3. *Let* $D = \left(x, \mu_D^+, \mu_D^-\right)$ *be the bipolar-valued fuzzy d-algebra of a set* X. *For a fixed* $s \in [0, 1]$, *the set* $\mu_D^+ = \left\{x \in X; \mu_D^+(x) \geq s\right\}$ *and* $\mu_D^- = \left\{x \in X; \mu_D^-(x) \leq s\right\}$ *is called an upper level of bipolar-valued fuzzy set* $D = \left(x, \mu_D^+, \mu_D^-\right)$.

Definition 3.4. *A bipolar-valued fuzzy set* $D = \left(x, \mu_D^+, \mu_D^-\right)$ *in a d-algebra* X *is called a bipolar-valued sub algebra of* X *if it satisfies*

$$\mu_D^+(x * y) \geq \min\left\{\mu_D^+(x), \mu_D^+(y)\right\}, \mu_D^-(x * y) \leq \max\left\{\mu_D^-(x), \mu_D^-(y)\right\}.$$

Example 3.5. *Let* $X = \{0, 1, 2\}$ *be a set given by the following table*

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Let $D = \left(x, \mu_D^+, \mu_D^-\right)$ *be an bipolar-valued fuzzy set in* X *defined by* $\mu_D^+(0) = 0.8$, $\mu_D^+(x) = 0.4$ *and* $\mu_D^-(0) = 0.05$, $\mu_D^-(x) = 0.09$. *Then* $D = (x, \mu_D^+, \mu_D^-)$ *is a bipolar-valued fuzzy sub algebra of* X.

Proposition 3.6. *If an bipolar-valued fuzzy set* $D = \left(x, \mu_D^+, \mu_D^-\right)$ *in* X *is an bipolar-valued fuzzy d-algebra of* X, *then* $\mu_D^+(0) \geq \mu_D^+(x)$ *and* $\mu_D^-(0) \leq \mu_D^-(x)$ $\forall$ $x \in X$.

Proof. Let $x \in X$, since $x * x = 0$ by definition of d-algebra [4] we have $\mu_D^+(0) = \mu_D^+(x * x) \geq \min\left\{\mu_D^+(x), \mu_D^+(x)\right\} = \mu_D^+(x)$. Therefore $\mu_D^+(0) \geq \mu_D^+(x)$. Similarly, we can prove that $\mu_D^-(0) = \mu_D^-(x * x) \leq \max\left\{\mu_D^-(x), \mu_D^-(x)\right\} = \mu_D^-(x)$. Therefore $\mu_D^-(0) \leq \mu_D^-(x)$ $\forall$ $x \in X$. □

Theorem 3.7. *If* $\{D_i / i \in \Lambda\}$ *is an arbitrary family of bipolar-valued fuzzy d-algebra of* X, *then* $\bigcap D_i$ *is an bipolar-valued fuzzy d-algebra of* X, *where* $\bigcap D_i = \left\{x, \bigwedge \mu_{D_i}^+(x), \bigvee \mu_{D_i}^-(x)\right\}$.

Proof. Let $x, y \in X$. Then

$$\bigwedge \mu_{D_i}^+(x * y) \geq \bigwedge \left(\min\left\{\mu_{D_i}^+(x), \mu_{D_i}^+(y)\right\}\right)$$

$$= \min\left\{\bigwedge \mu^{+}_{D_i}(x), \bigwedge \mu^{+}_{D_i}(y)\right\}$$

And

$$\bigvee \mu^{-}_{D_i}(x*y) \leq \bigvee \left(\max\left\{\mu^{-}_{D_i}(x), \mu^{-}_{D_i}(y)\right\}\right)$$
$$= \max\left\{\bigvee \mu^{-}_{D_i}(x), \bigvee \mu^{-}_{D_i}(y)\right\}$$

Hence $\bigcap D_i = \left\{x, \bigwedge \mu^{+}_{D_i}(x), \bigvee \mu^{-}_{D_i}(x)\right\}$ is an bipolar-valued fuzzy d-algebra of X. □

Theorem 3.8. *If an bipolar-valued fuzzy set $D = (x, \mu^{+}_{D}, \mu^{-}_{D})$ in X is an bipolar-valued fuzzy d-algebra of X, then the sets $X_{\mu^+} = \left\{x \in X/\mu^{+}_{D}(x) = \mu^{+}_{D}(0)\right\}$ and $X_{\mu^-} = \left\{x \in X/\mu^{-}_{D}(x) = \mu^{-}_{D}(0)\right\}$ are d-sub algebras of X.*

Proof. Let $x, y \in U(\mu^{+}_{D}, t)$. Then $\mu^{+}_{D}(x) \geq t$ and $\mu^{+}_{D}(y) \geq t$. It follows that $\mu^{+}_{D}(x*y) \geq \min\left\{\mu^{+}_{D}(x), \mu^{+}_{D}(y)\right\} \geq t$ so that $x*y \in U(\mu^{+}_{D}, t)$ is a d-sub algebra of X. Now let $x, y \in L(\mu^{-}_{D}, t)$, then $\mu^{-}_{D}(x), \mu^{-}_{D}(y) \leq t$. It follows that $\mu^{-}_{D}(x*y) \leq \max\left\{\mu^{-}_{D}(x), \mu^{-}_{D}(y)\right\} \leq t$ and so $x*y \in L(\mu^{-}_{D}, t)$. Therefore $L(\mu^{-}_{D}, t)$ is a d-sub algebra. □

Theorem 3.9. *Let $D = (x, \mu^{+}_{D}, \mu^{-}_{D})$ be an bipolar-valued fuzzy set in X such that the set $U(\mu^{+}_{D}, t)$ and $L(\mu^{-}_{D}, t)$ are d-sub algebras of X. Then $D = (x, \mu^{+}_{D}, \mu^{-}_{D})$ is an bipolar-valued fuzzy d-algebra of X.*

Proof. Assume that there exist $x_0, y_0 \in X$ such that $\mu^{+}_{D}(x_0 * y_0) < \min\left\{\mu^{+}_{D}(x_0), \mu^{+}_{D}(y_0)\right\}$. Let $t_0 = \frac{1}{2}\left(\mu^{+}_{D}(x_0 * y_0) + \min\left\{\mu^{+}_{D}(x_0), \mu^{+}_{D}(y_0)\right\}\right)$.
Then $\mu^{+}_{D}(x_0 * y_0) < t_0 < \min\left\{\mu^{+}_{D}(x_0), \mu^{+}_{D}(y_0)\right\}$ and so $x_0 * y_0 \notin U(\mu^{+}_{D}, t_0)$, but $x_0, y_0 \in U(\mu^{+}_{D}, t_0)$. This is a contradiction and therefore $\mu^{+}_{D}(x*y) \geq \min\left\{\mu^{+}_{D}(x), \mu^{+}_{D}(y)\right\}$ for all $x, y \in X$. Now suppose that $\mu^{-}_{D}(x_0 * y_0) > \max\left\{\mu^{-}_{D}(x_0), \mu^{-}_{D}(y_0)\right\}$ for some $x_0, y_0 \in X$. Taking $s_0 = \frac{1}{2}\left(\mu^{-}_{D}(x_0 * y_0) + \max\left\{\mu^{-}_{D}(x_0), \mu^{-}_{D}(y_0)\right\}\right)$, then $\max\left\{\mu^{-}_{D}(x_0), \mu^{-}_{D}(y_0)\right\} < s_0 < \mu^{-}_{D}(x_0 * y_0)$. It follows that $x_0, y_0 \in L(\mu^{-}_{D}, s_0)$ and $x_0 * y_0 \notin L(\mu^{-}_{D}, s_0)$, a contradiction. Hence $\mu^{-}_{D}(x*y) \leq \max\left\{\mu^{-}_{D}(x), \mu^{-}_{D}(y)\right\}$ for all $x, y \in X$. This completes the proof. □

Theorem 3.10. *Any d-sub algebra of X can be realized as both a μ^{+} level d-sub algebra and a μ^{-} level d-sub algebra of some bipolar-valued fuzzy d-algebra of X.*

Proof. Let S be a d-sub algebra of X and let μ^{+}_{D} and μ^{-}_{D} be fuzzy sets in X defined by

$$\mu^{+}_{D} = \begin{cases} t, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$
$$\mu^{-}_{D} = \begin{cases} s, & \text{if } x \in S \\ 1, & \text{otherwise} \end{cases}$$

For all $x \in X$ where t and s are fixed numbers in $(0,1)$ such that $t + s < 1$. Let $x, y \in X$. If $x, y \in S$, then $x*y \in S$. Hence $\mu^{+}_{D}(x*y) = \min\left\{\mu^{+}_{D}(x), \mu^{-}_{D}(y)\right\}$ and $\mu^{-}_{D}(x*y) = \max\left\{\mu^{-}_{D}(x), \mu^{-}_{D}(y)\right\}$. If at least one of x and y dose not belongs to S, then at least one of $\mu^{+}_{D}(x)$ and $\mu^{+}_{D}(y)$ is equal to 0 and at least one of $\mu^{-}_{D}(x)$ and $\mu^{-}_{D}(y)$ is equal to 1. It follows that $\mu^{+}_{D}(x*y) \geq 0 = \min\left\{\mu^{+}_{D}(x), \mu^{+}_{D}(y)\right\}, \mu^{-}_{D}(x*y) \leq 1 = \max\left\{\mu^{-}_{D}(x), \mu^{-}_{D}(y)\right\}$. Hence $D = (x, \mu^{+}_{D}, \mu^{-}_{D})$ is an bipolar-valued fuzzy d-algebra of X. Obviously $U(\mu^{+}_{D}, t) = S = L(\mu^{-}_{D}, t)$. This completes the proof. □

Definition 3.11. *Let $D = (x, \mu^{+}_{D}, \mu^{-}_{D})$ be an bipolar-valued fuzzy set in X and let $t \in [0,1]$. Then the set $U(\mu^{+}_{D}, t) = \left\{x \in X/\mu^{+}_{D}(x) \geq t\right\}$ and $L(\mu^{-}_{D}, t) = \left\{x \in X/\mu^{-}_{D}(x) \leq t\right\}$ is called a μ^{+} level t-cut and μ^{-} level t-cut of D.*

Theorem 3.12. *If an bipolar-valued fuzzy set $D = \left(x, \mu_D^+, \mu_D^-\right)$ in X is an bipolar-valued fuzzy d-algebra of X, then the μ^+ level t-cut and μ^- level t-cut of D are d-sub algebras of X for every $t \in [0,1]$ such that $t \in Im\left(\mu_D^+\right) \cap Im\left(\mu_D^-\right)$, which are called a μ^+ level d-sub algebra and a μ^- level d-sub algebra respectively.*

Proof. Let $x, y \in U\left(\mu_D^+, t\right)$. Then $\mu_D^+(x) \geq t, \mu_D^-(y) \geq t$, it follows that $\mu_D^+(x*y) \geq \min\left\{\mu_D^+(x), \mu_D^+(y)\right\} \geq t$ so that $x*y \in U\left(\mu_D^+, t\right)$. Hence $U\left(\mu_D^+, t\right)$ is a d-sub algebra of X. Now let $x, y \in L\left(\mu_D^-, t\right)$. Then $\mu_D^-(x) \leq t, \mu_D^-(y) \leq t$. It follows that $\mu_D^-(x*y) \leq \max\left\{\mu_D^-(x), \mu_D^-(y)\right\} \leq t$ and so $x*y \in L\left(\mu_D^-, t\right)$. Therefore $L(\mu_D^-, t)$ is a d-sub algebra. □

Theorem 3.13. *Let α be a d-homomorphism of a d-algebra X into a d-algebra Y and D an bipolar-valued fuzzy d-algebra of Y. Then $\alpha^{-1}(D)$ is an bipolar-valued fuzzy d-algebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
\mu_{\alpha^{-1}}^+(D)(x*y) &= \mu_D^+(\alpha(x*y)) \\
&= \mu_D^+(\alpha(x)*\alpha(y)) \\
&\geq \min\left\{\mu_D^+(\alpha(x)), \mu_D^+(\alpha(y))\right\} \\
&= \min\left\{\mu_{\alpha^{-1}(D)}^+(x), \mu_{\alpha^{-1}(D)}^+(y)\right\} \\
\mu_{\alpha^{-1}(D)}^-(x*y) &= \mu_D^-(\alpha(x*y)) \\
&= \mu_D^-(\alpha(x)*\alpha(y)) \\
&\leq \max\left\{\mu_D^-(\alpha(x)), \mu_D^-(\alpha(y))\right\} \\
&= \max\left\{\mu_{\alpha^{-1}(D)}^-(x), \mu_{\alpha^{-1}(D)}^-(y)\right\}.
\end{aligned}$$

Hence $\alpha^{-1}(D)$ is an bipolar-valued fuzzy d-algebra in X. □

Theorem 3.14. *If an bipolar-valued fuzzy set $D = (x, \mu_D^+, \mu_D^-)$ in X is an bipolar-valued fuzzy d-algebra of X, then so is D^c, where $D^c = \left\{\left(x, 1-\mu_D^+(x), -1-\mu_D^-(x)\right)/x \in X\right\}$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
\overline{\mu}_D^+(x*y) &= 1-\mu_D^+(x*y) \\
&\geq 1-\min\left\{\mu_D^+(x), \mu_D^+(y)\right\} \\
&= \min\left\{1-\mu_D^+(x), 1-\mu_D^+(y)\right\} \\
&\geq \min\left\{\overline{\mu}_D^+(x), \overline{\mu}_D^+(y)\right\} \\
\overline{\mu}_D^-(x*y) &= -1-\mu_D^-(x*y) \leq -1-\max\left\{\mu_D^-(x), \mu_D^-(y)\right\} \\
&= \max\left\{-1-\mu_D^-(x), -1-\mu_D^-(y)\right\} \\
&\leq \max\left\{\overline{\mu}_D^-(x), \overline{\mu}_D^-(y)\right\}.
\end{aligned}$$

Hence D^c is an bipolar-valued fuzzy d-algebra of X. □

Theorem 3.15. *Let α be a d-homomorphism of a d-algebra X onto a d-algebra Y. If $D = \left(x, \mu_D^+, \mu_D^-\right)$ is an bipolar-valued fuzzy d-algebra of X, then $\alpha(D) = \left(y, \alpha_{\sup}\left(\mu_D^+\right), \alpha_{\inf}\left(\mu_D^-\right)\right)$ is an bipolar-valued fuzzy d-algebra of Y.*

Proof. Let $D = \left(x, \mu_D^+, \mu_D^-\right)$ be a bipolar-valued fuzzy topological d-algebra in X and let $y_1, y_2 \in Y$. Noticing that, $\{x_1 * x_2 / x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2)\} \subseteq \{x \in X / x \in \alpha^{-1}(y_1 * y_2)\}$, we have

$$\begin{aligned}
\alpha_{\sup}\left(\mu_D^+\right)(y_1 * y_2) &= \sup\left\{\mu_D^+(x)/x \in \alpha^{-1}(y_1 * y_2)\right\} \\
&\geq \sup\left\{\mu_D^+(x_1 * x_2)/x_1 \in \alpha^{-1}(y_1),\ x_2 \in \alpha^{-1}(y_2)\right\} \\
&\geq \sup\left\{\min\left\{\mu_D^+(x_1), \mu_D^+(x_2)\right\}/x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2)\right\} \\
&= \min\left\{\sup\left\{\mu_D^+(x_1)/x_1 \in \alpha^{-1}(y_1)\right\}, \sup\left\{\mu_D^+(x_2)/x_2 \in \alpha^{-1}(y_2)\right\}\right\} \\
&= \min\left\{\alpha_{\sup}\left(\mu_D^+\right)(y_1), \alpha_{\sup}\left(\mu_D^+\right)(y_2)\right\}.
\end{aligned}$$

And

$$\begin{aligned}
\alpha_{\inf}\left(\mu_D^-\right)(y_1 * y_2) &= \inf\left\{\mu_D^-(x)/x \in \alpha^{-1}(y_1 * y_2)\right\} \\
&\leq \inf\left\{\mu_D^-(x_1 * x_2)/x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2)\right\} \\
&\leq \inf\left\{\max\left\{\mu_D^-(x_1), \mu_D^-(x_2)\right\}/x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2)\right\} \\
&= \max\left\{\inf\left\{\mu_D^-(x_1)/x_1 \in \alpha^{-1}(y_1)\right\}, \inf\left\{\mu_D^-(x_2)/x_2 \in \alpha^{-1}(y_2)\right\}\right\} \\
&= \max\left\{\alpha_{\inf}\left(\mu_D^-\right)(y_1), \alpha_{\inf}\left(\mu_D^-\right)(y_2)\right\}.
\end{aligned}$$

Hence $\alpha(D) = \left(y, \alpha_{\sup}\left(\mu_D^+\right), \alpha_{\inf}\left(\mu_D^-\right)\right)$ is an bipolar-valued fuzzy d-algebra in Y.
Let $\Omega(X)$ denote the family of all bipolar-valued fuzzy d-algebra of X and let $t \in [0,1]$, then $A \sim_{\mu^+} B \Longleftrightarrow U\left(\mu_A^+, t\right) = U(\mu_B^+, t)$ and $A \sim_{\mu^-} B \Longleftrightarrow L\left(\mu_A^-, t\right) = L\left(\mu_B^-, t\right)$, respectively for $A = \left(x, \mu_A^+, \mu_B^-\right)$ and $B = \left(x, \mu_B^+, \mu_B^-\right)$ in $\Omega(X)$. Then clearly $\sim_{\mu^+}$ and $\sim_{\mu^-}$ are equivalent relations on $\Omega(X)$. For any $A = \left(x, \mu_A^+, \mu_A^-\right) \in \Omega(X)$, let $[A]_{\mu^+}$ denote the equivalence class of $A = (x, \mu_A^+, \mu_A^-)$ modulo $\sim_{\mu^+}$ and denote by $\Omega(X)/\sim_{\mu^+}$ the collection of all equivalence classes of A modulo $\sim_{\mu^+}$. $\Omega(X)/\sim_{\mu^+} = \left\{[A]_{\mu^+}/A = \left(x, \mu_A^+, \mu_A^-\right) \in \Omega(X)\right\}$ and $\Omega(X)/\sim_{\mu^-} = \left\{[A]_{\mu^-}/A = \left(x, \mu_A^+, \mu_A^-\right) \in \Omega(X)\right\}$. Now let $S(X)$ denote the family of all d-sub algebras of X and let $t \in [0,1]$. Define maps α_t and β_t from $\Omega(X)$ to $S(X) \bigcup \{\phi\}$ by $\alpha_t(A) = U(\mu_A^+, t)$ and $\beta_t(A) = L(\mu_A^-, t)$ respectively for all $A = \left(x, \mu_A^+, \mu_A^-\right) \in \Omega(X)$. Then α_t and β_t are clearly well-defined. □

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