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Exact Zero-Divisor Graph of a Commutative Ring

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Abstract: The aim of this article is to continue the study of exact zero-divisor graph of a commutative ring with nonzero identity. We discuss the properties and nature of exact zero-divisor graph and compare some of its properties with zero-divisor graph.

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1. Introduction

The study of graphs associated with algebraic structures was initiated in 1878 when Arthur Cayley introduced Cayley graph of finite groups in [6]. After this, many graphs associated with algebraic structures were introduced. I. Beck defined the zero-divisor graphs in [5]. The definition of Beck was later modified by Anderson and Livingston in [3]. In the definition of I. Beck, the vertices are the elements of R, while Anderson and Livingston restricted the vertex set to only nonzero zero divisors of R. This graph is denoted by $\Gamma(R)$. Exact zero divisors were introduced by I. B. Henriques and I. M. Sega in [11]. Motivated by the study of zero-divisor graphs $\Gamma(R)$ in [3], we begun the study of exact zero-divisor graph in [13]. In [13], we have discussed several examples and properties of $E\Gamma(R)$ and compare some of its properties with $\Gamma(R)$.

Through out the article, the rings considered are commutative rings with nonzero identity. Following [11], we say that an element x is an exact zero-divisor of R, if there exists $y \in R^*$ such that $Ann(x) = \{r \in R | rx = 0\}$ is a principal ideal yR whose annihilator is xR, i.e. $Ann(x) = yR$ and $Ann(y) = xR$. We say that $EZ(R)$ is the set of exact zero-divisors of R. We associate a simple graph $E\Gamma(R)$ to R with the vertex set $EZ(R)^* = EZ(R) - \{0\}$, the set of nonzero exact zero divisors of R. Two vertices x and y are adjacent if and only if (x, y) is a pair of exact zero-divisors of R, i.e. $Ann(x) = yR$ and $Ann(y) = xR$. The zero-divisor graph defined in [3] has the vertex set $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero divisors of R and two vertices x and y are adjacent if $xy = 0$. In this paper, we continue our investigation of exact zero-divisor graphs begun in [13]. In section 2, we define basic terminologies and discuss some examples of $E\Gamma(R)$. In section 3, we discuss the properties of exact zero-divisor graphs for rings of the form $\mathbb{Z}_n$, with specific values of n. In section 4, we continue investing properties of $E\Gamma(R)$ and comparing with the properties of $\Gamma(R)$. In section 5, we define compressed exact zero-divisor graph defined using equivalence classes in R.

We call a graph G is connected if there is a path between any two distinct vertices. The length of the shortest path between any two vertices x and y is denoted by $d(x, y)$, and $d(x, y) = \infty$ if no such path exists. The diameter of a graph G is defined

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as $diam(G) = sup\{d(x,y) \mid x \ \& \ y \ are \ distinct \ vertices \ of \ G\}$. A cycle in a graph is a path of length at least 3 through distinct vertices with same begin and end vertices. The girth of a graph G is denoted by $g(G)$ and is defined to be the length of the shortest cycle in G. $g(G) = \infty$ if G contains no cycle. A graph is said to be complete if each vertex in the graph is adjacent to every other vertex. A complete graph with n vertices is denoted by K_n. A complete bipartite graph is a graph such that every vertex in one partitioning subset is adjacent to every vertex in the other partitioning subset. If the partitioning subsets have cardinalities m and n respectively, then the graph is denoted by $K_{m,n}$. By a null graph, we mean the edgeless graph, while by an empty graph, we mean a graph with no vertices. For a subset $A \subset R$, $A^* = A - \{0\}$. $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{F}_m$ indicates ring of integers, ring of integers modulo n and field with m elements, respectively. We follow [4] for other standard notations. To avoid trivialities, we assume that R is not an integral domain unless otherwise stated.

2. Examples and Preliminaries

In this section we recall several definitions from [13], and discuss a variety of examples of $E\Gamma(R)$. Also we mention the properties of $E\Gamma(R)$ studied in [13]. As discussed in introduction, R is a commutative ring with nonzero identity.

Definition 2.1. *An element x of R is exact zero divisor if there exists $y \in R^*$ such that $Ann(x) = \{r \in R | rx = 0\}$ is a principal ideal yR whose annihilator is xR, i.e. $Ann(x) = yR$ and $Ann(y) = xR$.*

In this case, we say that (x, y) is a pair of exact zero divisors. It can be seen that an exact zero divisor is a zero divisor.

Definition 2.2. *Let $EZ^*(R)$ be the set of nonzero exact zero divisors of R. We associate a simple graph $E\Gamma(R)$ to R with vertex set $EZ(R)^*$, and two vertices x and y are adjacent if (x, y) is a pair of exact zero divisors, i.e. $Ann(x) = yR$ and $Ann(y) = xR$.*

Clearly, $E\Gamma(R)$ is an empty graph if R is an integral domain. We discuss a variety of examples of $E\Gamma(R)$ by showing the graphs of several rings only. Being an easy exercise, we omit the calculation part in the examples.

Example 2.3. *The exact zero-divisor graphs of several commutative rings shown in the Figure 1, are the graphs such that there is a vertex which is adjacent to every other vertex.*

Figure 1.

Example 2.4. *We can observe from Figure 2 that exact zero-divisor graph of a ring need not be connected. Note that the zero-divisor graph of a commutative ring is always connected.*

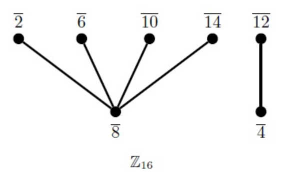

Figure 2.

Example 2.5. *The exact zero-divisor graph of $R = \mathbb{Z}_5[X]/(X^2)$ is a complete graph, which is shown in figure 3.*

Example 2.6. *The exact zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is shown in figure 3. This example indicates that a zero-divisor may not be an exact zero-divisor of a commutative ring R. For $R = \mathbb{Z}_2 \times \mathbb{Z}_4$, $(0,2) \in Z(R)^*$ but $(0,2) \notin EZ(R)^*$.*

Example 2.7. *The exact zero-divisor graph of $\mathbb{Z} \times \mathbb{Z}$ is shown in figure 3. This is an example of an infinite commutative ring with its exact zero-divisor graph to be finite. We note that for a commutative ring R, its zero-divisor graph is finite if and the ring R is finite or an integral domain ([3], Theorem 2.2).*

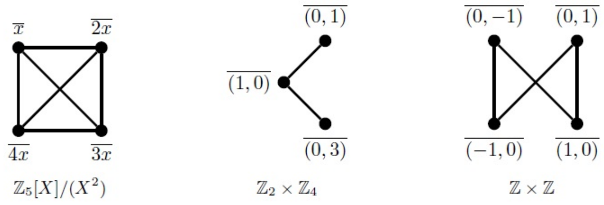

Figure 3.

We have discussed some properties of $E\Gamma(R)$ for a commutative ring R in [13]. We end this section by noting down some facts from [13].

(1). A zero-divisor graph of R is always connected ([3], theorem 2.3). But the result is not true for exact zero-divisor graph of a commutative ring R ([13], remark 3.1). It can be observed also from example 2.4.

(2). The zero-divisor graph $\Gamma(R)$ of R is finite if and only if R is finite or an integral domain ([3] theorem 2.2). This is not true in the case of exact zero-divisor graph of R ([13], remark 3.2). It can be observed also from example 2.7.

(3). If $E\Gamma(R)$ is connected, then the length of the shortest path between any two vertices is at most two ([13], theorem 3.3). Since $E\Gamma(R)$ is not connected, we can modify this fact as if there is a path between any two distinct vertices of $E\Gamma(R)$, then the length of the path cannot exceed two.

(4). If $E\Gamma(R)$ contains a cycle, then $g(E\Gamma(R)) \leq 4$ ([13], theorem 3.4).

(5). If R is a ring of the form $\mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. Then $E\Gamma(R)$ is connected and complete bipartite graph ([13], theorem 3.5). The converse of this statement is not true. (example 2.5)

(6). If $R = \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. Then $E\Gamma(R)$ and $\Gamma(R)$ coincide in this case ([13], remark 3.5).

3. Exact Zero-Divisor Graph of $\mathbb{Z}_n$

In this section, we will focus on the exact zero-divisor graphs of a commutative ring of the form $\mathbb{Z}_n$. We will discuss the nature of $E\Gamma(R)$ for particular values of n. Clearly for $R = \mathbb{Z}_p$, where p is a prime, $E\Gamma(R)$ is an empty graph. We note the fact that for a ring of the form $R = \mathbb{Z}_{p^n}$, the zero divisors of R are precisely the elements divisible by p.

Theorem 3.1. *Let $R = \mathbb{Z}_{p^2}$, where p is a prime number. Then $E\Gamma(R)$ is complete graph K_{p-1} with $p-1$ vertices.*

Proof. Let $R = \mathbb{Z}_{p^2}$, where p is a prime. Then $Z(R)^* = \left\{\overline{p}, \overline{2p}, \overline{3p}, \ldots, \overline{(p-1)p}\right\}$. Now, $Ann(\overline{p}) = \overline{p}R$. But since $\left\{\overline{1}, \overline{2}, \overline{3}, \ldots, \overline{p-1}\right\} \subset U^*(R)$, we have $\overline{p}R = \overline{2p}R = \overline{3p}R = \ldots = \overline{(p-1)p}R$. Therefore $Ann(\overline{p}) = \overline{p}R = \overline{2p}R = \overline{3p}R = \ldots = \overline{(p-1)p}R$. Also $Ann(\overline{p}) = Ann(\overline{2p}) = Ann(\overline{3p}) \ldots = Ann(\overline{(p-1)p})$. Hence $Z(R)^* = EZ(R)^*$ and each of the

$\overline{p}, \overline{2p}, \overline{3p}, \ldots, \overline{(p-1)p}$ are adjacent with each other in $E\Gamma(R)$. Thus $E\Gamma(R)$ is a complete graph with $p-1$ vertices, i.e. K_{p-1}. □

[9], theorem 3.1 indicates that the zero-divisor graph $\Gamma(R)$ of a commutative ring $\mathbb{Z}_{p^2}$ is also K_{p-1}. So in this case $\Gamma(R)$ and $E\Gamma(R)$ coincide. We have seen that, for a prime number p, $E\Gamma(R)$ of $\mathbb{Z}_{p^2}$ is a complete graph. Example of $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is a disjoint union of two complete bipartite graphs (example 2.4). Also the exact zero-divisor graph $E\Gamma(R)$ of $\mathbb{Z}_{32}$ is as in figure 4, which is also a disjoint union of two complete bipartite graphs. We generalize this fact in the next theorem.

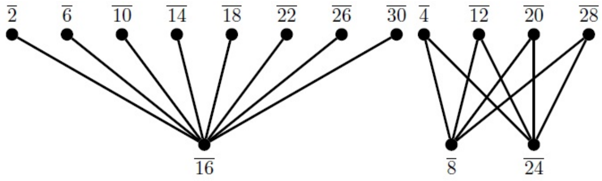

Figure 4.

Theorem 3.2. *If $R = \mathbb{Z}_{p^n} (n \geq 3)$, then $E\Gamma(R)$ is disjoint union of $[n/2]$ number of complete bipartite graphs, where $[n/2]$ is integer part of $\frac{n}{2}$.*

Proof. Let $R = \mathbb{Z}_{p^n} (p \geq 3), n \in \mathbb{N}$. Therefore the zero divisors in R are precisely the elements divisible by p, i.e. $u_1 p,\ u_2 p^2,\ \ldots,\ u_{n-1} p^{n-1}$; where each u_i $(1 \leq i \leq n-1)$ are units in R. Now, $Ann(\overline{u_1 p}) = (\overline{u_{n-1} p^{n-1}})R$ and $Ann(\overline{u_{n-1} p^{n-1}}) = (\overline{u_1 p})R$. Similarly, $Ann(\overline{u_2 p^2}) = (\overline{u_{n-2} p^{n-2}})R$ and $Ann(\overline{u_{n-2} p^{n-2}}) = (\overline{u_2 p^2})R$. This process (say $*$) will continue up to $n/2$ or $(n-1)/2$ depending upon the value of n, whether it is even or odd.

Case I: n is even.

If n is even, the process $*$ will end with $Ann(\overline{u_{n/2} p^{n/2}}) = (\overline{u_{n/2} p^{n/2}})R$. Thus the vertex set of $E\Gamma(R)$ will be disjoint union of $n/2$ sets. Also each $\overline{u_i p^i}$ is adjacent to each $\overline{u_{n-i} p^{n-i}}$ in $E\Gamma(R)$. Therefore each vertex set gives a complete bipartite graph. Hence $E\Gamma(R)$ is disjoint union of $n/2 = [n/2]$ number of complete bipartite graphs, where $[n/2]$ indicates the integer part of $\frac{n}{2}$.

Case II: n is odd.

If n is odd, the process $*$ will end with $Ann(\overline{u_{(n-1)/2} p^{(n-1)/2}}) = (\overline{u_{(n+1)/2} p^{(n+1)/2}})R$ and $Ann(\overline{u_{(n+1)/2} p^{(n+1)/2}}) = (\overline{u_{(n-1)/2} p^{(n-1)/2}})R$. Thus the vertex set of $E\Gamma(R)$ will be disjoint union of $(n-1)/2$ sets. Also each $\overline{u_i p^i}$ is adjacent to each $\overline{u_{n-i} p^{n-i}}$ in $E\Gamma(R)$. Therefore each vertex set gives a complete bipartite graph. Hence $E\Gamma(R)$ is disjoint union of $(n-1)/2 = [n/2]$ number of complete bipartite graphs, where $[n/2]$ indicates the integer part of $\frac{n}{2}$. □

We end the section with following result.

Theorem 3.3. *Let $R = \mathbb{Z}_{pq}$, where p and q are distinct primes. Then $E\Gamma(R)$ is a complete bipartite graph $K_{p-1,q-1}$.*

Proof. Let $R = \mathbb{Z}_{pq}$, where p and q are distinct primes. Then $\mathbb{Z}_{pq}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$. But since p and q are primes, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are fields. Therefore by ([13], theorem 3.5), $E\Gamma(R)$ is complete bipartite graph. Also in this case $Z(R)^* = EZ(R)^* = A \cup B$, where $A = \left\{\overline{(1,0)}, \overline{(2,0)}, \ldots, \overline{(p-1,0)}\right\}$ and $B = \left\{\overline{(0,1)}, \overline{(0,2)}, \ldots, \overline{(0,q-1)}\right\}$, and $A \cap B = \phi$. Thus $E\Gamma(R) = K_{p-1,q-1}$. □

4. Some Properties of $E\Gamma(R)$

In section 3, we have discussed several properties of $E\Gamma(R)$ for rings of the form $\mathbb{Z}_n$. In this section, we will discuss some properties of $E\Gamma(R)$ for R to be a commutative ring. We begin the section with a result that generalizes the theorem 3.5 of [13] for integral domains.

Theorem 4.1. *Let $R = D_1 \times D_2$, where D_1 and D_2 are integral domains. Then $E\Gamma(R)$ is connected and complete bipartite graph.*

Proof. Let $R = D_1 \times D_2$, where D_1 and D_2 are integral domains. Then $Z(R)^* = X \cup Y$, where $X = \{(x, 0)|x \in D_1\}$ and $Y = \{(0, y)|y \in D_2\}$. Clearly $X \cap Y = \phi$. Let $(u, 0), (0, v) \in R$ such that $u \in U(D_1)^*,\ v \in U(D_2)^*$. Then $Ann((u, 0)) = \{0\} \times D_2 = (0, v)R$ and $Ann((0, v)) = D_1 \times \{0\} = (u, 0)R$. Therefore $(u, 0), (0, v) \in EZ(R)^*$ and $(u, 0)-(0, v)$ are adjacent in $E\Gamma(R)$ for $u \in U(D_1)^*,\ v \in U(D_2)^*$. Now let $(x, 0) \in R$ such that $x \in D_1 - U(D_1)^*$. Then $Ann((x, 0)) = \{0\} \times D_2 = (0, 1)R$. But $Ann(0, 1) = D_1 \times \{0\} \neq (x, 0)R$. Thus $(x, 0)$ is not an exact zero divisor of R. Similarly $(0, y)$ such that $y \in D_2 - U(D_2)^*$ is not an exact zero divisor of R. Also for $u, u' \in U(D_1)^*$ and $v, v' \in U(D_2)^*$, $(u, 0)-(u', 0)$ and $(0, v)-(0, v')$ are not adjacent in $E\Gamma(R)$. Hence vertex set of $E\Gamma(R)$ is $A \cup B$, where $A = U(D_1)^*$ and $B = U(D_2)^*$. And each $(u, 0)-(0, v)$ are adjacent in $E\Gamma(R)$, where $u \in U(D_1)^*,\ v \in U(D_2)^*$. Thus $E\Gamma(R)$ is a connected and complete bipartite graph. □

We know that for fields $\mathbb{F}_1$ and $\mathbb{F}_2$, if $R = \mathbb{F}_1 \times \mathbb{F}_2$, then $E\Gamma(R)$ is connected. In next theorem, we will show that if $E\Gamma(R)$ is connected for R to be Von Neumann Regular Ring, then $R \simeq \mathbb{F}_1 \times \mathbb{F}_2$.

Theorem 4.2. *Let R to be Von Neumann Regular Ring. If $E\Gamma(R)$ is connected, then $R \simeq \mathbb{F}_1 \times \mathbb{F}_2$.*

Proof. Let R to be Von Neumann Regular Ring. Suppose that R admits more than two prime ideals. Let P_1, P_2, P_3 be prime ideals of R such that $P_2 \cap P_3 \nsubseteq P_1$ and $P_1 \cap P_3 \nsubseteq P_2$. Let $x \in (P_2 \cap P_3) - P_1$ and $y \in (P_1 \cap P_3) - P_2$. Therefore $x = ue,\ y = vf$, where $u, v \in U(R)$ and $e,\ f$ are idempotent elements. Since $E\Gamma(R)$ is connected, let $x-z-y$ be the shortest path between x and y. Also $z = u_1e_1$, where $u_1 \in U(R)$, and e_1 is an idempotent element. Now by the definition of $E\Gamma(R)$, $Ann(x) = zR,\ \&\ Ann(z) = xR$. Since $x = ue,\ z = u_1e_1$, we have $e_1 = 1 - e$. Similarly, since $y = vf$, and $z-y$ are adjacent in $E\Gamma(R)$, we have $e_1 = 1 - f$. But then $e = f$, which gives $Rx = Ry$, a contradiction. Therefore R admits exactly two prime ideals. Thus $R \simeq \mathbb{F}_1 \times \mathbb{F}_2$. □

Corollary 4.3. *Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \ldots \times \mathbb{F}_n$, where each $\mathbb{F}_i (1 \le i \le n)$ are fields. If $E\Gamma(R)$ is connected, then $n = 2$.*

Proof. Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \ldots \times \mathbb{F}_n$, where each $\mathbb{F}_i (1 \le i \le n)$ are fields. Then R is Von Neumann Regular Ring. Hence by theorem 4.2, $n = 2$. □

Remark 4.4. *Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where each $\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$ are fields. We know that $E\Gamma(R)$ is not connected. Here we will discuss about the number of connected components of $E\Gamma(R)$. Let $\alpha_1, \alpha_2, \alpha_3$ be arbitrary elements from $\mathbb{F}_1^*, \mathbb{F}_2^*, \mathbb{F}_3^*$, respectively. Then $Ann((\alpha_1, 0, 0)R) = (0, \alpha_2, \alpha_3)R$ and $Ann((0, \alpha_2, \alpha_3)R) = (\alpha_1, 0, 0)R$. $Ann((0, \alpha_2, 0)R) = (\alpha_1, 0, \alpha_3)R$ and $Ann((\alpha_1, 0, \alpha_3)R) = (0, \alpha_2, 0)R$; $Ann((0, 0, \alpha_3)R) = (\alpha_1, \alpha_2, 0)R$ and $Ann((\alpha_1, \alpha_2, 0)R) = (0, 0, \alpha_3)R$. Therefore we can observe that $E\Gamma(R)$ is disjoint union of three complete bipartite graphs. We generalize this fact in next theorem.*

Theorem 4.5. *Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \ldots \times \mathbb{F}_n$, where each $\mathbb{F}_i$, $(1 \le i \le n)$ is a field. Then the exact zero-divisor graph $E\Gamma(R)$ is a disjoint union of $2^{n-1} - 1$ number of complete bipartite graphs.*

Proof. Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \ldots \times \mathbb{F}_n$, where each $\mathbb{F}_i$, $(1 \le i \le n)$ is a field. Let $\alpha_i \in \mathbb{F}_i$, then vertices of $E\Gamma(R)$ are n-tuples of $\alpha_i \in \mathbb{F}_i$ with at least one $\alpha_i \neq 0$. Suppose that n is odd. Then we can observe that for each $(1 \le i \le n)$, the vertex of the form $(0,0,\ldots,0,\alpha_i,0,\ldots,0)R$ with $\alpha_i(\neq 0)\in\mathbb{F}_i$ is adjacent with $(\alpha_1,\alpha_2,\ldots,\alpha_{i-1},0,\alpha_{i+1},\ldots,\alpha_n)R$, which gives $\binom{n}{1}$ number of complete bipartite components. Similarly, the vertices with exactly two nonzero $\alpha_i's$ gives $\binom{n}{2}$ number of complete bipartite components. Since n is odd, the total number of components of $E\Gamma(R)$ is $\sum_{i=1}^{n-1}\binom{n}{i} = 2^{n-1}-1$. Thus if n is odd, $E\Gamma(R)$ is disjoint union of $2^{n-1}-1$ number of complete bipartite graphs. Similarly, if n is even, then the number of components are $\sum_{i=1}^{\frac{n}{2}}\binom{n}{i} = 2^{n-1}-1$. Thus $E\Gamma(R)$ is disjoint union of $2^{n-1}-1$ number of complete bipartite graphs. □

Theorem 4.6. *Let R be a commutative ring with nonzero identity. If zero-divisor graph $\Gamma(R)$ of R is complete, then for exact zero-divisor graph $E\Gamma(R)$, $\Gamma(R)=E\Gamma(R)$.*

Proof. Let R be a commutative ring with nonzero identity such that zero-divisor graph $\Gamma(R)$ of R is complete. Therefore either $R\simeq\mathbb{Z}_2\times\mathbb{Z}_2$ or $xy=0$ for all $x,y\in Z(R)$ ([3], theorem 2.8). Clearly if $R\simeq\mathbb{Z}_2\times\mathbb{Z}_2$, then $\Gamma(R)=E\Gamma(R)$. Now let $xy=0$ for all $x,y\in Z(R)$. If possible suppose that $\Gamma(R)\neq E\Gamma(R)$. Therefore either $V(\Gamma(R))\neq V(E\Gamma(R))$ and/or $E(\Gamma(R))\neq E(E\Gamma(R))$. If $V(\Gamma(R))\neq V(E\Gamma(R))$, then there exists a zero divisor $x\in Z(R)^*$ such that $x\notin EZ(R)^*$. Therefore either $Ann(x)\neq(y)$ or $Ann(y)\neq(x)$ for any $y\in R^*$. In any of the case, we get for $r\in R^*, rxy\neq 0$, which contradicts the fact that $xy=0$. Thus $\Gamma(R)=E\Gamma(R)$. Thus $V(\Gamma(R))=V(E\Gamma(R))$. Similarly, we can show that $E(\Gamma(R))=E(E\Gamma(R))$. Thus $\Gamma(R)=E\Gamma(R)$. □

We recall that the chromatic number of a graph G is the minimum number of colours needed to produce a proper colouring of G. It is denoted by $\chi(G)$. The clique is a subset of vertices of an undirected graph G such that every two vertices are adjacent, i.e. its induced subgraph is complete. The number of vertices in a maximum clique of G is denoted by $\omega(G)$.

Definition 4.7. *A perfect graph G is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph, i.e. for every subgraph $H\subseteq G$, $\omega(H)=\chi(H)$.*

We note that a graph P_n is the graph with n vertices such that the vertices u_i and the edges e_j form an alternating sequence $u_1,e_1,u_2,e_2,\cdots,u_{n-1},e_{n-1},u_n$, where $e_i=u_{i-1}u_i$ for $i=1,2,\cdots,n$ and $u_i\neq u_j$ for all $i\neq j$. The graph P_4 is shown in the figure. The following theorem provides a tool for proving that a graph is perfect.

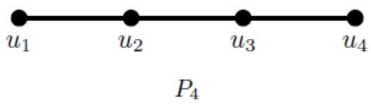

Figure 5.

Theorem 4.8 ([7])**.** *If a graph G does not contain P_4 as an induced subgraph, then G is perfect.*

Theorem 4.9. *For a commutative ring R, the exact zero-divisor graph $E\Gamma(R)$ of a commutative ring R is perfect.*

Proof. We know that the shortest path between any two vertices in $E\Gamma(R)$ for a commutative ring R cannot exceed two ([13], theorem 3.3). So if there is an alternating sequence $u_1,e_1,u_2,e_2,u_3,e_3,u_4$ of vertices u_1,u_2,u_3,u_4 and edges e_1,e_2,e_3 in $E\Gamma(R)$, then there is an edge between the vertices u_1 and u_4. So for any commutative ring R, $E\Gamma(R)$ does not contains P_4 as the induced subgraph. Therefore $E\Gamma(R)$ is perfect. □

Remark 4.10. *We can observe from ([10], theorem 1.2) that the zero-divisor graph of $\Gamma(\mathbb{Z}_{p^n})$, where p is prime, is perfect. Theorem 4.7 indicates that the fact also holds for exact zero-divisor graphs. Also the zero-divisor graph of $\Gamma(\mathbb{Z}_{p_1p_2})$, where p_1, p_2 are primes, is perfect which is also true in case of exact zero-divisor graphs.*

Remark 4.11. *([10], theorem 1.4) indicates that the zero-divisor graph $\mathbb{Z}_n$ is perfect if and only if $n=p^k$ for some prime p or $n=p_1p_2$ for some distinct primes p_1 and p_2. Theorem 4.9 indicates that for any commutative ring R, $E\Gamma(R)$ is perfect.*

5. Compressed Exact Zero-Divisor Graph

As in [2], for any element r and s of R, define $r \sim s$ if and only if $ann_R(r) = ann_R(s)$. Then $\sim$ is an equivalence relation on R. For any $r \in R$, let $[r]_R = \{s \in R | r \sim s\}$. Thus it is clear that $[0]_R = \{0\}$, $[1]_R = R - Z(R)$, & $[r]_R \subset Z(R) - \{0\}$, for every ring $R - ([0]_R \cup [1]_R)$. Furthermore, the operation on equivalence classes given by $[r]_R[s]_R = [rs]_R$ is well defined and thus makes the set $R_E = \{[r]_R | r \in R\}$ into a commutative monoid.

As in [14], $\Gamma(R_E)$ or $\Gamma_E(R)$ will denote the compressed zero-divisor graph of R, whose vertices are the elements of $Z(R_E) - \{[0]_R\}$ such that distinct vertices $[r]_R$ and $[s]_R$ are adjacent if and only if $[r]_R[s]_R = [0]_R$, if and only if $rs = 0$. In this section, we will define the compressed exact zero-divisor graph $E\Gamma_E(R)$ for a commutative ring R. We discuss the compressed exact zero-divisor graphs of several rings whose exact zero-divisor graphs are discussed in section 2. We also discuss some properties of $E\Gamma_E(R)$ and compare with the properties of $\Gamma_E(R)$.

The compressed zero-divisor graph $\Gamma_E(R)$ was first defined by S. B. Mulay in [12], where it has been noted that several graph-theoretic properties of $\Gamma(R)$ remain valid for $\Gamma_E(R)$. However, some properties of $\Gamma(R)$ does not hold for $\Gamma_E(R)$. For example, $\Gamma(R)$ is finite if and only if R is finite or an integral while $\Gamma_E(R)$ may be finite even if R is infinite and not an integral domain.

Definition 5.1. *The graph of equivalence classes of exact zero divisors of a ring R, denoted by $E\Gamma_E(R)$, is the graph associated to R whose vertices are the classes of elements in $EZ(R)^*$, and two distinct vertices x and y are adjacent if and only if $Ann(x) = yR$ and $Ann(y) = xR$.*

Example 5.2. *We have mentioned compressed exact zero-divisor graphs of some of the rings in figure 6, whose exact zero-divisor graphs are discussed in section 2.*

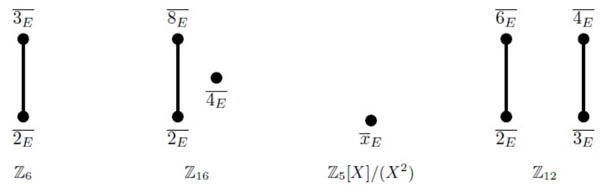

Figure 6.

In ([14], theorem 1.4), it has been shown that $\Gamma_E(R)$ is connected for every commutative ring with nonzero identity. Also $diam(\Gamma_E(R)) \leq 3$. From example 5.2, we can observe that $E\Gamma_E(R)$ need not be connected. In theorem 5.3, we will prove that if the compressed exact-zero divisor graph is connected, then it must be either K_1 and K_2.

Theorem 5.3. *If $E\Gamma_E(R)$ is connected, then $E\Gamma_E(R)$ is either K_1 or K_2.*

Proof. Let $E\Gamma_E(R)$ is connected. Suppose that $E\Gamma_E(R)$ is different from K_1 or K_2. Let $[x]_E$, $[y]_E$, *and* $[z]_E$ be three distinct vertices of $E\Gamma_E(R)$. Therefore there exists a path $[x]_E - [y]_E - [z]_E$ of shortest length between vertices $[x]_E, [y]_E, [z]_E$ in $E\Gamma_E(R)$. By the definition of $E\Gamma_E(R)$, we have $Ann(x) = yR$ and $Ann(y) = xR$. Similarly, $Ann(y) = zR$ and $Ann(z) = yR$. But then $Ann(x) = yR = Ann(z)$. Thus $[x]_E = [z]_E$. Therefore, there does not exist a path of length three between any two distinct vertices. Hence if $E\Gamma_E(R)$ is connected, then $E\Gamma_E(R)$ is either K_1 or K_2. □

Remark 5.4. *We have seen that $diam(\Gamma_E(R)) \leq 3$ for a commutative ring R. But if the compressed zero-divisor graph $E\Gamma_E(R)$ is connected, then $diam(E\Gamma_E(R)) \leq 1$.*

Remark 5.5. *From theorem 5.3, we can observe that any compressed zero-divisor graph with three distinct vertices cannot be connected. Hence we have the following theorem.*

Theorem 5.6. *For any commutative ring R, if the compressed exact zero-divisor graph $E\Gamma_E(R)$ is not connected, then $E\Gamma_E(R)$ is disjoint union of the complete graphs K_1 or K_2, i.e.* $E\Gamma(R) = \bigcup_{j=1}^{j=n} (K_i)_j$; *where* $i = 1$ *or* 2.

Proof. Suppose compressed exact zero-divisor graph $E\Gamma_E(R)$ of a commutative ring R is not connected. Let x, y, z from a connected component of $E\Gamma_E(R)$ such that $[x]_E - [y]_E - [z]_E$. But by definition of $E\Gamma_E(R)$, we can observe that $[x]_E = [y]_E$. Thus any connected component of $E\Gamma(R)$ can contain at most two vertices. Thus $E\Gamma_E(R)$ is disjoint union of the complete graphs K_1 or K_2. Hence $E\Gamma(R) = \bigcup_{j=1}^{j=n} (K_i)_j$; where $i = 1$ *or* 2. □

We end this section with an immediate corollary of theorem 5.1 and 5.2.

Corollary 5.7. *For any commutative ring R, $E\Gamma_E(R)$ does not contain a cycle.*

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