A Study on Dominator Coloring of Friendship and Barbell Graphs

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Abstract: A dominator coloring of a graph $G$ is a proper coloring of $G$ in which every vertex should dominate every vertex of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of $G$. In this paper we study the dominator chromatic number of the barbell graph $B_n$ and friendship graph $F_n$.

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1. Introduction

For general notations and concepts of graphs we refer to [1, 3, 9]. For definitions of coloring and domination refer to [2, 10–13]. Unless mentioned otherwise, all the graphs mentioned in this paper are simple, finite and connected undirected graphs.

A graph coloring is an assignment of colors to the vertices of $G$. The vertex coloring is said to be proper if no two adjacent vertices of $G$ receive the same color. Various coloring derivatives are found in the literature, one such problem is dominator coloring. Dominator colorings were introduced in [7] and they were motivated by [4]. The aim in this paper is to study about the dominator coloring of the barbell graph $B_n$ and friendship graph $F_n$ under the operations of shadow, middle, center, total, line, subdivision and related subdivision.

Definition 1.1 ([8]). The $n$-barbell graph is obtained by connecting two copies of $K_n$ by a bridge. And it is denoted by $B_n$.

Figure 1. Barbell Graph $B_6$

Definition 1.2 ([6]). The friendship graph $F_n$ can be constructed by joining $n$ copies of the cycle graph $C_3$ with a common verte.

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Figure 2. Friendship Graph $F_4$

Definition 1.3 ([15]). For a connected graph $G$ with $V(G) = \{v_1, v_2, ..., v_n\}$ the shadow graph $D_2(G)$ is obtained by taking two copies of $G$, say $G'$ and $G''$ with $V(G') = \{v'_1, v'_2, ..., v'_n\}$ and $V(G'') = \{v''_1, v''_2, ..., v''_n\}$ such that,

(1). $v'_i \sim v'_j$ in $G'$ and $v''_i \sim v''_j$ in $G''$ if and only if $v_i \sim v_j$ in $G$

(2). $v'_i \sim v''_j$ in $D_2(G)$ if and only if $v_i \sim v_j$ in $G$

Definition 1.4 ([16]). For a graph $G(V, E)$ the middle graph $M(G)$ with the vertex set $V(G) \cup E(G)$ is defined as follows, two vertices $x, y$ of $M(G)$ if either

(1). $x, y \in V(G)$ and $x \sim y$ in $G$, or

(2). $x \in V(G), y \in E(G)$ and $x, y$ are incident in $G$

Definition 1.5 ([5]). The line graph $L(G)$ of the graph $G$ is the graph such that,

(1). $V(L(G)) = E(G)$ and

(2). $x \sim y$ in $L(G)$ if $x \sim y$ in $G$ for $x, y \in E(G)$

Definition 1.6 ([5]). The total graph $T(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of $G$ are adjacent or incident.

Definition 1.7 ([5]). The subdivision graph $S(G)$ of a graph $G$ is the graph obtained from $G$ by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of $G$.

Definition 1.8 ([5]). The related subdivision graph $R(G)$ is the graph obtained from $G$ by adding a new vertex corresponding to each edge of $G$ and by joining each new vertex to the end vertices of the edge corresponding to it.

2. Dominator Coloring

A dominator coloring [14] of a graph $G$ is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph $G$.

Example 2.1 ([14]). The graph $G$ in Figure 1 has $\chi_d(G) = 7$. 
Theorem 2.2 ([14]). The path $P_n$ of order $n \geq 2$,  
$$\chi_d(P_n) = \begin{cases} 
1 + \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 2, 3, 4, 5, 7 \\
2 + \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}
\end{cases}$$

Theorem 2.3 ([14]). The cycle $C_n$ of order $n \geq 3$,  
$$\chi_d(C_n) = \begin{cases} 
\left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 4 \\
1 + \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 5 \\
2 + \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}
\end{cases}$$

Theorem 2.4 ([14]). The complete graph $K_n$ of order $n \geq 3$,  
$$\chi_d(K_n) = n.$$  

3. A Study on Dominator Coloring Number of Friendship graphs

Theorem 3.1. For any Friendship graph $F_n$, $n \geq 2$, $\chi_d(F_n) = 3$.

Proof. Let $v_1$ be the common vertex of the $n$ copies of $C_3$ in $F_n$. Assign the color $c_1$ to $v_1$. Then the color class $c_1$ dominates every vertices of $F_n$. Since all the other vertices of $F_n$ in each copy of $C_3$ are non adjacent and $\chi(C_3) = 3$, we can assign rest of two colors to the remaining vertices other than $v_1$ in each copy of $C_3$. Hence $\chi_d(F_n) = 3$.

Corollary 3.2. For any Friendship graph $F_n$, $n \geq 2$, $\chi_d(F_n) = \chi(F_n)$.

Theorem 3.3. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(D_2(F_n)) = 3$.

Proof. Let $G = D_2(F_n)$ be the shadow graph of friendship graph $F_n$, and let $v'_1, v'_2, \ldots, v'_{2n+1}$ be the vertices of $F_n'$, the first copy of $F_n$ and $v''_1, v''_2, \ldots, v''_{2n+1}$ be the vertices of $F''_n$, the second copy of $F_n$. Hence $|V(G)| = 4n + 2$.

Define the dominator coloring $f : V(G) \to \{1, 2, 3, \ldots, \chi_d(G)\}$ as follows: Let $v_1$ be the common vertex of the $n$ copies of $C_3$ in $F_n$, $f(v'_1) = f(v''_1) = 1$, which dominates every vertex of $G$ other than $v'_1$ and $v''_1$. Therefore $f(v'_{2i+1}) = f(v''_{2i+1}) = 2$ for $1 \leq i \leq n - 1$ and $f(v'_{2i}) = f(v''_{2i}) = 3$ for $1 \leq i \leq n - 1$. Hence $\chi_d(D_2(F_n)) = 3$.
Corollary 3.4. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(D_2(F_n)) = \chi_d(F_n) = \chi(F_n)$.

Theorem 3.5. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(M(F_n)) = 2n + 2$.

Proof. Let $G = M(F_n)$ be the middle graph of friendship graph $F_n$, then $V(F_n) = \{v_1, v_2, \ldots, v_{2n+1}\}$ and $E(F_n) = \{e_1, e_2, \ldots, e_{3n}\}$. Hence $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, \ldots, v_{2n+1}, e_1, e_2, \ldots, e_{3n}\}$, hence $|V(G)| = 5n + 1$. Let $v_1$ be the common vertex and $e_i$ for $1 \leq i \leq n$ be the edges which is not adjacent to $v_1$ in $F_n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, \ldots, \chi_d(G)\}$ as follows: $f(v_i) = 1$, $f(e_{i+1}) = 2 + i$ for $0 \leq i \leq n - 1$, $f(e_{3i+2}) = n + 2 + i$ for $0 \leq i \leq n - 1$ and $f(e_{3i}) = 2n + 2$ for $1 \leq i \leq n$. Hence $\chi_d(M(F_n)) = 2n + 2$. \hfill $\square$

Corollary 3.6. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(M(F_n)) = \chi(K_n) + 1$.

Theorem 3.7. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(C(F_n)) = n + 3$.

Proof. Let $G = C(F_n)$ be the center graph of friendship graph $F_n$. Here, $V(F_n) = \{v_1, v_2, \ldots, v_{2n+1}\}$ and $E(F_n) = \{e_1, e_2, \ldots, e_{3n}\}$. Hence $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, \ldots, v_{2n+1}, e_1, e_2, \ldots, e_{3n}\}$, hence $|V(G)| = 5n + 1$. Let $v_1$ be the common vertex, $v_{i1}$ and $v_{i+1}$ be the vertices of $i^{th}$ vertex which is not have $e_{i1}$ for $1 \leq i \leq n$ copy of $C_1$ in $F_n$ and $e_{3i}$, $1 \leq i \leq n$ be the edges which is not adjacent to $v_1$ in $F_n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, \ldots, \chi_d(G)\}$ as follows: $f(v_1) = 1$, $f(e_{i+3i}) = f(e_{i+3i}) = 2$, $f(v_{i2}) = f(v_{i2+1}) = i + 1$ and $f(e_{3i}) = n + 3$. Hence $\chi_d(C(F_n)) = n + 3$. \hfill $\square$

Corollary 3.8. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(C(F_n)) = \chi_d(K_n) + \chi_d(F_n)$.

Theorem 3.9. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(L(F_n)) = 2n$.

Proof. Let $G = L(F_n)$ be the line graph of friendship graph $F_n$, from the definition of line graph, $V(G) = E(F_n) = \{e_1, e_2, \ldots, e_{3n}\}$. Let $v_1$ be the common vertex and $e_{i1}$ for $1 \leq i \leq n$ be the edges which are not adjacent to $v_1$ in $F_n$. Hence the vertex other than $e_{3i}, 1 \leq i \leq n$ of $G$ forms a complete graph of order $2n$, which induces $2n$ colors. Hence every vertices of $G$ is dominated and it remains to color the vertices $e_{3i}, 1 \leq i \leq n$ which consumes any of the $2n$ colors other than the colors of the two adjacent vertices. \hfill $\square$

Corollary 3.10. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(L(F_n)) = \chi_d(K_{2n})$.

Theorem 3.11. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(T(F_n)) = 3n + 1$.

Proof. Let $G = T(F_n)$ be the total graph of friendship graph $F_n$. By the definition of total graph, $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, \ldots, v_{2n+1}, e_1, e_2, \ldots, e_{3n}\}$, hence $|V(G)| = 5n + 1$. Let $v_1$ be the common vertex, $v_{i1}$ and $v_{i+1}$ be the vertices of $i^{th}$ for $1 \leq i \leq n$ copy of $C_1$ in $F_n$ and $e_{3i}, 1 \leq i \leq n$ be the edges which are not adjacent to $v_1$ in $F_n$. Consider the vertices, $e_{3i+1}$ and $e_{3i+2}$ for $0 \leq i \leq n - 1$ which forms a complete graph of order $2n$, and it induces $2n$ colors, since $v_1$ is adjacent to every vertex of $G$ other than $e_{3i}, 1 \leq i \leq n$. Hence $f(v_1) = 2n + 1$. Hence the color class $[2n + 1]$ dominates every vertex of $G$ other than $e_{3i}$. Then $f(v_{i2}) = f(e_{i+3i}), f(v_{i2+1}) = f(e_{i+3i}), f(3i) = 2n + 1 + i$. Therefore $\chi_d(T(F_n)) = 3n + 1$. \hfill $\square$

Corollary 3.12. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(T(F_n)) = \chi_d(K_{2n+1}) + n$.

Theorem 3.13. For any friendship graph $F_n$, $n \geq 2$, $\chi_d(S(F_n)) = n + 3$.

Proof. Let $G$ be the subdivision graph of $F_n$. Here $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, \ldots, v_{2n+1}, e_1, e_2, \ldots, e_{3n}\}$. Let $v_1$ be the common vertex, $v_{i1}, v_{i+1}$ be the vertices of $i^{th}$ for $1 \leq i \leq n$ copy of $C_1$ in $F_n$, $e_{3i+1}, e_{3i+2}, 1 \leq i \leq n$ be the edges incident to $v_1$ and $e_{3i}, 1 \leq i \leq n$ be the edges which is not incident to $v_1$ in $F_n$. Define the dominator coloring
Lemma 3.15. For any friendship graph $G$ with $n$ vertices, $\chi_d(G) = \chi_d(K_2) + \chi_d(F_n)$. 

For the barbell graph $B_n$, the second copy of $K_n$, the bridge $e$, dominates the color class of $n - 1$. If $f(v_{n-1}) = f(v'_{n-1}) = n - 2$, implies the vertices $v_1, v_2, v_{n-2}$ and $v_1', v_2', v_{n-2}'$ doesn’t dominate any color class. Hence $f(v_{n-2}) = n$ and $f(v'_{n-2}) = n + 1$, implies the vertices $v_1, v_2, v_{n-2}$ and $v_1', v_2', v_{n-2}'$ dominates the color class of $n$ and $v_1, v_2, v_{n-2}'$ dominates the color class of $n + 1$.

$\Rightarrow \chi_d(B_n) = n + 1$. 

Corollary 4.2. For the barbell graph $B_n$, $n \geq 3$, $\chi_d(B_n) = \chi(B_n) + 1$. 

Proof. For a proper coloring of $B_n$, each color can be used maximum twice, hence $\chi(B_n) = n$. Hence $\chi_d(B_n) = n + 1 = \chi(B_n) + 1$. 

Figure 5. Dominator coloring of $B_n$ 

Theorem 4.3. For any barbell graph $B_n$, $n \geq 3$, $\chi_d(D_2(B_n)) = 2n - 1$. 

Proof. Let $G = D_2(B_n)$ be the shadow graph of the barbell graph $B_n$, and let $v_1, v_2, ..., v_{2n}$ be the vertices of $B_n$, the first copy of $B_n$ and $v_1', v_2', ..., v_{2n}'$ be the vertices of $B_n'$, the second copy of $B_n$. Hence $|V(G)| = 4n$. Define the dominator coloring $f : V(G) \rightarrow \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_1') = f(v_2') = 1$, $f(v_{2n}') = f(v_{2n-1}) = 1$ and $f(v_i') = f(v'_i) = i$, if $2 \leq i \leq 2n - 1$. Hence $\chi_d(D_2(B_n)) = 2n - 1$. 

Corollary 4.4. For any barbell graph $B_n$, $n \geq 3$, $\chi_d(D_2(B_n)) = \chi_d(B_n) + \chi(B_n) - \chi(B_n)$. 

$\boxdot$
Theorem 4.5. For any barbell graph $B_n$, $n \geq 3$, $\chi_d(S(B_n)) = 2n + 1$.

Proof. Let $G = S(B_n)$ be the subdivision graph of the barbell graph $B_n$. Here, $V(B_n) = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(B_n) = \{e_1, e_2, \ldots, e_{n(n-1)+1}\}$. Hence $V(G) = V(B_n) \cup E(B_n) = \{v_1, v_2, \ldots, v_{2n}, e_1, e_2, \ldots, e_{n(n-1)+1}\}$, where $e_1, e_2, \ldots, e_{n(n-1)+1}$ are the newly introduced vertices to obtain the subdivision of $B_n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, \ldots, \chi_d(G)\}$ as follows: $f(e_i) = 1$ for $1 \leq i \leq n(n-1)+1$ and $f(v_i) = i + 1$ for $1 \leq i \leq 2n$. Hence, $\chi_d(S(B_n)) = 2n + 1$. □

Corollary 4.6. For any barbell graph $B_n$, $n \geq 3$, $\chi_d(S(B_n)) = \chi_d(B_n) + \chi(B_n)$.

Theorem 4.7. For any barbell graph $B_n$, $n \geq 3$, $\chi_d(R(B_n)) = 2n$.

Proof. Let $G = R(B_n)$ be the related subdivision graph of the barbell graph $B_n$. Here, $V(B_n) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ such that $e = v_1 v_1'$ be the bridge and $E(B_n) = \{e_1, e_2, \ldots, e_{n(n-1)+1}\}$. Hence $V(G) = V(B_n) \cup E(B_n) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n', e_1, e_2, \ldots, e_{n(n-1)+1}\}$, where $e_1, e_2, \ldots, e_{n(n-1)+1}$ are the newly introduced vertices to obtain the related subdivision of $B_n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, \ldots, \chi_d(G)\}$ as follows: $f(e_i) = 1$ for $1 \leq i \leq n(n-1)+1$, $f(v_i) = i + 1$ for $1 \leq i \leq n$, $f(v_1') = n + 1$ and $f(v_i') = n + i$ for $2 \leq i \leq n$. Hence $\chi_d(R(B_n)) = 2n$. □

Observation 4.8.

(1). $\chi_d(L(B_3)) = 4$.
(2). $\chi_d(L(B_4)) = 6$.
(3). $\chi_d(L(B_5)) = 9$.

Observation 4.9.

(1). $\chi_d(M(B_3)) = 6$.
(2). $\chi_d(M(B_4)) = 7$.
(3). $\chi_d(M(B_5)) = 11$.

Observation 4.10.

(1). $\chi_d(C(B_3)) = 5$.
(2). $\chi_d(C(B_4)) = 7$.

Observation 4.11.

(1). $\chi_d(T(B_3)) = 7$.
(2). $\chi_d(T(B_4)) = 9$.

5. Conclusion

In this paper, we obtain the dominator coloring number of the barbell graph $B_n$ and friendship graph $F_n$ and successfully derived expressions for different operations such as shadow, center, line, total, middle, subdivision and related subdivision. There are many interesting graph operations such as Corona, Cartesian and Strong Cartesian of graphs etc. to which this study can be extended.
References


