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A Study on Dominator Coloring of Friendship and Barbell Graphs

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Abstract: A dominator coloring of a graph G is a proper coloring of G in which every vertex should dominate every vertex of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G. In this paper we study the dominator chromatic number of the barbell graph B_n and friendship graph F_n.

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Keywords: Coloring, Domination, Dominator coloring.

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1. Introduction

For general notations and concepts of graphs we refer to [1, 3, 9]. For definitions of coloring and domination refer to [2, 10–13]. Unless mentioned otherwise, all the graphs mentioned in this paper are simple, finite and connected undirected graphs. A graph coloring is an assignment of colors to the vertices of G. The vertex coloring is said to be proper if no two adjacent vertices of G receive the same color. Various coloring derivatives are found in the literature, one such problem is dominator coloring. Dominator colorings were introduced in [7] and they were motivated by [4]. The aim in this paper is to study about the dominator coloring of the barbell graph B_n and friendship graph F_n under the operations of shadow, middle, center, total, line, subdivision and related subdivision.

Definition 1.1 ([8]). *The n-barbell graph is obtained by connecting two copies of K_n by a bridge. And it is denoted by B_n*

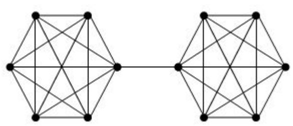

Figure 1. **Barbell Graph B_6**

Definition 1.2 ([6]). *The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common verte.*

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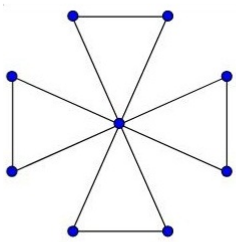

Figure 2. **Friendship Graph F_4**

Definition 1.3 ([15]). *For a connected graph G with $V(G) = \{v_1, v_2, ..., v_n\}$ the shadow graph $D_2(G)$ is obtained by taking two copies of G, say G' and $G"$ with $V(G') = \{v'_1, v'_2, ..., v'_n\}$ and $V(G") = \{v_1", v_2", ..., v_n"\}$ such that,*

(1). $v'_i \sim v'_j$ in G' and $v_i" \sim v_j"$ in $G"$ if and only if $v_i \sim v_j$ in G

(2). $v'_i \sim v_j"$ in $D_2(G)$ if and only if $v_i \sim v_j$ in G

Definition 1.4 ([16]). *For a graph $G(V, E)$ the middle graph $M(G)$ with the vertex set $V(G) \cup E(G)$ is defined as follows, two vertices x, y of $M(G)$ if either*

(1). $x, y \in V(G)$ and $x \sim y$ in G, or

(2). $x \in V(G), y \in E(G)$ and x, y are incident in G

Definition 1.5 ([5]). *The **line graph** $L(G)$ of the graph G is the graph such that,*

(1). $V(L(G)) = E(G)$ and

(2). $x \sim y$ in $L(G)$ if $x \sim y$ in G for $x, y \in E(G)$

Definition 1.6 ([5]). *The total graph $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.*

Definition 1.7 ([5]). *The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of G.*

Definition 1.8 ([5]). *The related subdivision grap $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.*

2. Dominator Coloring

A dominator coloring [14] of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G.

Example 2.1 ([14]). *The graph G in Figure 1 has $\chi_d(G) = 7$.*

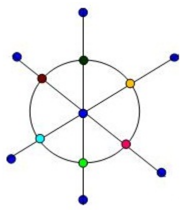

Figure 3.

Theorem 2.2 ([14]). *The path P_n of order $n \geq 2$,*

$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil, & if \ n = 2,3,4,5,7 \\ 2 + \lceil \frac{n}{3} \rceil, & otherwise \end{cases}$$

Theorem 2.3 ([14]). *The cycle C_n of order $n \geq 3$,*

$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & if \ n = 4 \\ 1 + \lceil \frac{n}{3} \rceil, & if \ n = 5 \\ 2 + \lceil \frac{n}{3} \rceil, & otherwise \end{cases}$$

Theorem 2.4 ([14]). *The complete graph K_n of order $n \geq 3$, $\chi_d(K_n) = n$.*

3. A Study on Dominator Coloring Number of Friendship graphs

Theorem 3.1. *For any Friendship graph $F_n, n \geq 2$, $\chi_d(F_n) = 3$.*

Proof. Let v_1 be the common vertex of the n copies of C_3 in F_n. Assign the color c_1 to v_1. Then the color class c_1 dominates every vertices of F_n. Since all the other vertices of F_n in each copy of C_3 are non adjacent and $\chi(C_3) = 3$, we can assign rest of two colors to the remaining vertices other than v_1 in each copy of C_3. Hence $\chi_d(F_n) = 3$. □

Corollary 3.2. *For any Friendship graph F_n $n \geq 2$, $\chi_d(F_n) = \chi(F_n)$.*

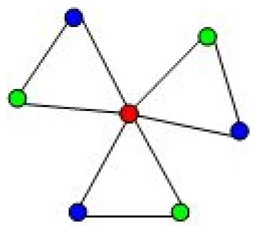

Figure 4. **Dominator coloring of F_3**

Theorem 3.3. *For any friendship graph F_n; $n \geq 2$, $\chi_d(D_2(F_n)) = 3$.*

Proof. Let $G = D_2((F_n))$ be the shadow graph of friendship graph F_n, and let $v'_1, v'_2, ..., v'_{2n+1}$ be the vertices of F'_n, the first copy of F_n and $v''_1, v''_2, ..., v''_{2n+1}$ be the vertices of F''_n, the second copy of F_n. Hence $| V(G) |= 4n + 2$.

Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: Let v_1 be the common vertex of the n copies of C_3 in F_n, $f(v'_1) = f(v''_1) = 1$, which dominates every vertex of G other than v'_1 and v''_1. Therefore $f(v'_{2i+1}) = f(v''_{2i+1}) = 2$ for $1 \leq i \leq n-1$ and $f(v'_{2i}) = f(v''_{2i}) = 3$ for $1 \leq i \leq n-1$. Hence $\chi_d(D_2(F_n)) = 3$. □

Corollary 3.4. *For any friendship graph F_n $n \geq 2$, $\chi_d(D_2(F_n)) = \chi_d(F_n) = \chi(F_n)$.*

Theorem 3.5. *For any friendship graph F_n $n \geq 2$, $\chi_d(M(F_n)) = 2n + 2$.*

Proof. Let $G = M((F_n))$ be the middle graph of friendship graph F_n, then $V(F_n) = \{v_1, v_2, \ldots, v_{2n+1}\}$ and $E(F_n) = \{e_1, e_2, ..., e_{3n}\}$. Hence $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, ..., v_{2n+1}, e_1, e_2, ..., e_{3n}\}$, hence $| V(G) |= 5n + 1$. Let v_1 be the common vertex and e_{3i} for $1 \leq i \leq n$ be the edges which is not adjacent to v_1 in F_n. Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_i) = 1$, $f(e_{3i+1}) = 2+i$ for $0 \leq i \leq n-1$, $f(e_{3i+2}) = n+2+i$ for $0 \leq i \leq n-1$ and $f(e_{3i}) = 2n + 2$ for $1 \leq i \leq n$. Hence $\chi_d(M(F_n)) = 2n + 2$. □

Corollary 3.6. *For any friendship graph F_n $n \geq 2$, $\chi_d(M(F_n)) = \chi(K_n) + 1$.*

Theorem 3.7. *For any friendship graph F_n $n \geq 2$, $\chi_d(C(F_n)) = n + 3$.*

Proof. Let $G = C((F_n))$ be the center graph of friendship graph F_n. Here, $V(F_n) = \{v_1, v_2, ..., v_{2n+1}\}$ and $E(F_n) = \{e_1, e_2, ..., e_{3n}\}$. Hence $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, ..., v_{2n+1}, e_1, e_2, ..., e_{3n}\}$, hence $| V(G) |= 5n + 1$. Let v_1 be the common vertex, v_{2i} and v_{2i+1} be the vertices of i^{th}(for $1 \leq i \leq n$) copy of C_3 in F_n and $e_{3i}, 1 \leq i \leq n$ be the edges which is not adjacent to v_1 in F_n. Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_1) = 1$, $f(e_{1+3i}) = f(e_{2+3i}) = 2$, $f(v_{2i}) = f(v_{2i+1}) = i + 1$ and $f(e_{3i}) = n + 3$. Hence $\chi_d(C(F_n)) = n + 3$. □

Corollary 3.8. *For any friendship graph F_n $n \geq 2$, $\chi_d(C(F_n)) = \chi_d(K_n) + \chi_d(F_n)$.*

Theorem 3.9. *For any friendship graph F_n $n \geq 2$, $\chi_d(L(F_n)) = 2n$.*

Proof. Let $G = L((F_n))$ be the line graph of friendship graph F_n, from the definition of line graph, $V(G) = E(F_n) = \{e_1, e_2, ..., e_{3n}\}$. Let v_1 be the common vertex and e_{3i} for $1 \leq i \leq n$ be the edges which are not adjacent to v_1 in F_n. Hence the vertex other than $e_{3i}, 1 \leq i \leq n$ of G forms a complete graph of order $2n$, which induces $2n$ colors. Hence every vertices of G is dominated and it remains to color the vetices $e_{3i}, 1 \leq i \leq n$ which consumes any of the $2n$ colors other than the colors of the two adjacent vertices. □

Corollary 3.10. *For any friendship graph F_n $n \geq 2$, $\chi_d(L(F_n)) = \chi_d(K_{2n})$.*

Theorem 3.11. *For any friendship graph F_n of $n \geq 2$, $\chi_d(T(F_n)) = 3n + 1$.*

Proof. Let $G = T((F_n))$ be the total graph of friendship graph F_n. By the definition of total graph, $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, ..., v_{2n+1}, e_1, e_2, ..., e_{3n}\}$, hence $| V(G) |= 5n + 1$. Let v_1 be the common vertex, v_{2i} and v_{2i+1} be the vertices of i^{th}(for $1 \leq i \leq n$) copy of C_3 in F_n and $e_{3i}, 1 \leq i \leq n$ be the edges which are not adjacent to v_1 in F_n. Consider the vertices, e_{3i+1} and e_{3i+2} for $0 \leq i \leq n - 1$ which forms a complete graph of order $2n$, and it induces $2n$ colors, since v_1 is adjacent to every vertex of G other than $e_{3i}, 1 \leq i \leq n$. Hence $f(v_1) = 2n + 1$. Hence the color class $[2n + 1]$ dominates every vertex of G other than e_{3i}. Then $f(v_{2i}) = f(e_{1+3i})$, $f(v_{2i+1}) = f(e_{2+3i})$, $f(3i) = 2n + 1 + i$. Therefore $\chi_d(T(F_n)) = 3n + 1$. □

Corollary 3.12. *For any friendship graph F_n $n \geq 2$, $\chi_d(T(F_n)) = \chi_d(K_{2n+1}) + n$.*

Theorem 3.13. *For any friendship graph F_n $n \geq 2$, $\chi_d(S(F_n)) = n + 3$.*

Proof. Let G be the subdivision graph of F_n. Here $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, ..., v_{2n+1}, e_1, e_2, ..., e_{3n}\}$. Let v_1 be the common vertex, v_{2i}, v_{2i+1} be the vertices of i^{th}(for $1 \leq i \leq n$) copy of C_3 in F_n, e_{3i+1}, $e_{3i+2}, 1 \leq i \leq n$ be the edges incident to v_1 and $e_{3i}, 1 \leq i \leq n$ be the edges which is not incident to v_1 in F_n. Define the dominator coloring

$f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_1) = 1, f(e_{3i+1}) = (e_{3i+2}) = 2$ for $0 \leq i \leq n-1$, $f(v_{2i}) = f(v_{2i+1}) = 3$ for $1 \leq i \leq n$ and $f(e_{3i}) = 3 + i$ for $1 \leq i \leq n$. Hence $\chi_d(S(F_n)) = n + 3$. □

Corollary 3.14. *For any friendship graph F_n $n \geq 2$, $\chi_d(S(F_n)) = \chi_d(K_n) + \chi_d(F_n)$.*

Theorem 3.15. *For any friendship graph F_n $n \geq 2$, $\chi_d(R(F_n)) = 4 + n$.*

Proof. Let G be the related subdivision graph of F_n. Here $V(G) = V(F_n) \cup E(F_n) = \{v_1, v_2, ..., v_{2n+1}, e_1, e_2, ..., e_{3n}\}$, hence $| V(G) |= 5n + 1$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_1) = 1, f(v_{2i}) = 2$, $f(v_{2i+1}) = 3$ for $1 \leq i \leq n$, $f(e_{3i+1}) = f(e_{3i+2}) = 4$ for $0 \leq i \leq n-1$ and $f(e_{3i}) = 4 + i$ for $1 \leq i \leq n$ Hence $\chi_d(R(F_n)) = 4 + n$. □

4. A Study on Dominator Coloring Number of Barbell Graphs

Theorem 4.1. *For the barbell graph B_n $n \geq 3$, $\chi_d(B_n) = n + 1$.*

Proof. Let $G = B_n$ be the barbell graph, which is obtained by connecting two copies of complete graphs K_n and K'_n by a bridge. Let $v_1, v_2, ..., v_n$ be the vertices of K_n, $v'_1, v'_2, ..., v'_n$ be the vertices of K'_n and the bridge $e = v_n v'_n$(say). Here $| V(G) |= 2n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v_i) = f(v'_i) = i$ for $i \leq n-3$. Since $v_n \sim v'_n$, $f(v_n) \neq f(v'_n)$, $f(v_n) = n-2$ and $f(v'_n) = n-1$. Hence, $f(v_{n-1}) = n-1$, $f(v'_{n-1}) = n-2$. $\Rightarrow v_n$ dominates the color class of $n-2$ and v'_n dominates the color class of $n-1$. If $f(v_{n-2}) = f(v'_{n-2}) = n$, then the vertices $v_1, v_2, ..., v_{n-2}$ and $v'_1, v'_2, ..., v'_{n-2}$ doesn't dominate any color class. Hence $f(v_{n-2}) = n$ and $f(v'_{n-2}) = n+1$, implies the vertices $v_1, v_2, ..., v_{n-2}$ dominates the color class of n and $v'_1, v'_2, ..., v'_{n-2}$ dominates the color class of $n+1$.
$\Rightarrow \chi_d(B_n) = n + 1$. □

Corollary 4.2. *For the barbell graph B_n $n \geq 3$, $\chi_d(B_n) = \chi(B_n) + 1$.*

Proof. For a proper coloring of B_n, each color can be used maximum twice, hence $\chi(B_n) = n$. Hence $\chi_d(B_n) = n + 1 = \chi(B_n) + 1$. □

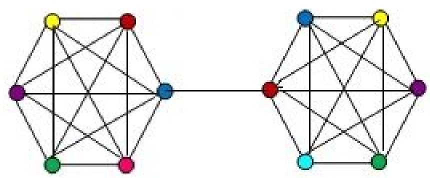

Figure 5. Dominator coloring of B_6

Theorem 4.3. *For any barbell graph B_n $n \geq 3$, $\chi_d(D_2(B_n)) = 2n - 1$.*

Proof. Let $G = D_2(B_n)$ be the shadow graph of the barbell graph B_n, and let $v'_1, v'_2, ..., v'_{2n}$ be the vertices of B'_n, the first copy of B_n and $v''_1, v''_2, ..., v''_{2n}$ be the vertices of B''_n, the second copy of B_n. Hence $| V(G) |= 4n$. Define the dominator coloring $f : V(G) \to \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(v'_1) = f(v''_1) = f(v'_n) = f(v''_n) = 1$ and $f(v'_i) = f(v''_i) = i$, if $2 \leq i \leq 2n-1$. Hence $\chi_d(D_2(B_n)) = 2n - 1$. □

Corollary 4.4. *For any barbell graph B_n $n \geq 3$, $\chi_d(D_2(B_n)) = \chi_d(B_n) + \chi(B_n) - i(B_n)$.*

Theorem 4.5. *For any barbell graph $B_n, n \geq 3$, $\chi_d(S(B_n)) = 2n + 1$.*

Proof. Let $G = S(B_n)$ be the subdivision graph of the barbell graph B_n. Here, $V(B_n) = \{v_1, v_2, .. ., v_{2n}\}$ and $E(B_n) = \{e_1, e_2, ..., e_{n(n-1)+1}\}$. Hence $V(G) = V(B_n) \cup E(B_n) = \{v_1, v_2, ..., v_{2n}, e_1, e_2, ..., e_{n(n-1)+1}\}$, where $e_1, e_2, ..., e_{n(n-1)+1}$ are the newly introduced vertices to obtain the subdivision of B_n. Define the dominator coloring $f : V(G) \rightarrow \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(e_i) = 1$ for $1 \leq i \leq n(n-1) + 1$ and $f(v_i) = i + 1$ for $1 \leq i \leq 2n$. Hence, $\chi_d(S(B_n)) = 2n + 1$. □

Corollary 4.6. *For any barbell graph $B_n, n \geq 3, \chi_d(S(B_n)) = \chi_d(B_n) + \chi(B_n)$.*

Theorem 4.7. *For any barbell graph $B_n, n \geq 3$, $\chi_d(R(B_n)) = 2n$.*

Proof. Let $G = R(B_n)$ be the related subdivision graph of the barbell graph B_n, Here, $V(B_n) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ such that $e = v_1v'_1$ be the bridge and $E(B_n) = \{e_1, e_2, ..., e_{n(n-1)+1}\}$. Hence $V(G) = V(B_n) \cup E(B_n) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n, e_1, e_2, ..., e_{n(n-1)+1}\}$, where $e_1, e_2, ..., e_{n(n-1)+1}$ are the newly introduced vertices to obtain the related subdivision of B_n. Define the dominator coloring $f : V(G) \rightarrow \{1, 2, 3, ..., \chi_d(G)\}$ as follows: $f(e_i) = 1$ for $1 \leq i \leq n(n-1)+1$, $f(v_i) = i+1$ for $1 \leq i \leq n$, $f(v'_1) = n+1$ and $f(v'_n) = n+i$ for $2 \leq i \leq n$. Hence $\chi_d(R(B_n)) = 2n$. □

Observation 4.8.

(1). $\chi_d(L(B_3)) = 4$.

(2). $\chi_d(L(B_4)) = 6$.

(3). $\chi_d(L(B_5)) = 9$.

Observation 4.9.

(1). $\chi_d(M(B_3)) = 6$.

(2). $\chi_d(M(B_4)) = 7$.

(3). $\chi_d(M(B_5)) = 11$.

Observation 4.10.

(1). $\chi_d(C(B_3)) = 5$.

(2). $\chi_d(C(B_4)) = 7$.

Observation 4.11.

(1). $\chi_d(T(B_3)) = 7$.

(2). $\chi_d(T(B_4)) = 9$.

5. Conclusion

In this paper, we obtain the dominator coloring number of the barbell graph B_n and friendship graph F_n and successfully derived expressions for different operations such as shadow, center, line, total, middle, subdivision and related subdivision. There are many interesting graph operations such as Corona, Cartesian and Strong Cartesian of graphs etc. to which this study can be extended.

References

[1] J. A. Bondy and U. S. R. Murthy, *Graph theory with applications*, North-Holland, New York, (1982).

[2] G. Chartrand and P. Zhang, *Chromatic graph theory*, CRC Press, (2009).

[3] G. Chartrand and L. Lesniak, *Graphs and digraphs*, CRC Press, Boca raton, (2000).

[4] E. Cockayne' and S. Hedetniemi, *Dominating partitions of graphs*, Tech. Rep., (1979), Unpublished manuscript.

[5] J. E. Fields, *Introduction to Graph Theory*, December 13, (2001).

[6] J. A. Gallian, *Dynamic Survey DS6: Graph Labeling*, Electronic J. Combinatorics, (2007), #DS6, 1-58.

[7] R. Gera, S. Horton and C. Rasmussen, *Dominator Colorings and Safe Clique Partitions*, Congressus Numerantium, (2006).

[8] Herbster and Pontil, *Prediction on a Graph with a Perception*, Neural Information Processing Systems Conference, (2006).

[9] F. Harary, *Graph theory*, Addison-Wesley Pub. Co. Inc., Philippines, (1969).

[10] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, (1998).

[11] T. R. Jensen and B. Toft, *Graph coloring problems*, John Wiley & Sons, (1995).

[12] M. Kubale, *Graph colorings*, American Mathematical Society, (2004).

[13] Marcel Dekker, *Fundamentals of Domination in Graphs*, New York, (1998).

[14] Ralluca Michelle Gera, *On dominator coloring in graphs*, Department of Applied Mathematics, Monterey, CA 93943, USA.

[15] N. Revathi, *Mean Labeling of Some Graphs*, International Journal of Science and Research, 4(10)(2015), 1921-1923

[16] K. Sutha, K. Thirusangu and S. Bala, *Some Graph Labelings on Middle Graph of Extended Duplicate Graph of a Path*, Annals of Pure and Applied Mathematics, 8(2)(2014), 169-174.