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Classification of a Nonlinear Fin Equation in Spherical Coordinates via Lie Symmetry Method

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Abstract: The nonlinear fin equation in Spherical coordinates is studied in this paper. Classification of temperature dependent thermal conductivity and radial variable heat transfer coefficient is performed via Lie symmetry analysis. Using these symmetries, nonlinear fin equation in spherical coordinates has been transformed into an ordinary differential equation. An Exact solution of this ODE is obtained whenever possible.

Keywords: Lie Symmetry Analysis, Fin Equation, Exact Solutions.

1. Introduction

Fins are extended surfaces used to increase the heat exchange from a hot or cold surface to surrounding area. There are multifarious uses of fins such as compressors, cooling of computer processor, air-cooled craft engine, in air conditioning etc. The heat transfer in fins of different shapes and profiles with variety of boundary conditions is described by mathematical models [8]. There have been studies using a number of techniques to discuss the heat transfer through fin of different shapes. For example, [15] discussed the problem

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \tag{1}$$

using separation of variables and a Newton-Raphson method to compute the temperature profiles and heat transfer per fin length. More recently, Ali, Bokhari, Zaman [22] have considered the fin equation in cylindrical coordinates in the form

$$\frac{1}{x}\frac{\partial}{\partial x}(xk(u)u_x)+\frac{1}{x}\frac{\partial}{\partial y}(\frac{1}{x}k(u)u_y)-N^2f(x)u=u_t. \tag{2}$$

They used Lie symmetry analysis to transform this equation into an ordinary differential equation and exact solution are obtained. Pakdemirli, Sahin [20, 21] studied the problem

$$\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial\theta}{\partial x}\right)-N^2f(x)\theta=\theta_t \tag{3}$$

by using the Lie symmetries of the governing partial differential equation. This method was introduced by Sophus Lie that has been applied to find exact solutions of a number of linear and nonlinear partial differential equations in engineering

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and mathematical physics. Bokhari, Kara, Zaman [5] studied the non-dimensional nonlinear fin equation in the case of temperature dependent thermal conductivity which has the form

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) - N^2 h(x)u = u_t \tag{4}$$

where u is the dimensionless temperature, k the thermal conductivity, h the heat transfer coefficient and N the fin parameter. They considered further group theoretic analysis that leads to an alternative set of exact solution. An extension of a series solution of the non-linear fin problem with temperature dependent thermal conductivity is performed by Sin Kim, and Cheng-Hung Huang [9] who considered the following problem:

$$A_c\frac{d}{dx}\left(k(u)\frac{du}{dx}\right) - ph(u-u_a) = 0, \quad 0 < x < L \tag{5}$$

where h is the heat transfer coefficient that may depend on the temperature and generally it can be expressed in power form, A_c is the constant cross-sectional area of fin, p and L are the perimeter and length of fin respectively. Moitshekia, Hayat and Malik [18] improved the result of Sin Kim, and Cheng-Hung Huang [9] by finding exact solution of problem (4). They used the Classical Lie symmetry techniques to construct the exact solutions which satisfy the realistic boundary conditions. Moitsheki [12] considered a radial one-dimensional fin with a profile area Ap in the following form:

$$\frac{A_p}{R}\frac{d}{dR}\left(Rf(R)k(u)\frac{du}{dR}\right) = ph(u)\left(u-u_a\right), \quad r_b < R < r_a. \tag{6}$$

where k and h are the non-uniform thermal conductivity and heat transfer coefficient, respectively, depending on the temperature. He constructed some exact solutions for thermal diffusion in a fin with a rectangular profile and another with a hyperbolic profile by employing classical Lie symmetry techniques. Further, Moitsheki [13] studied a heat transfer problem of a longitudinal fin with triangular and parabolic profiles by considering the following problem:

$$A_p\frac{d}{dx}\left(F(x)k(u)\frac{du}{dx}\right) = ph(u)\left(u-u_a\right), \quad 0 < x < L \tag{7}$$

where A_p is profile area, $F(x)$ is the function of fin profile, k and h are the non-uniform thermal conductivity and heat transfer coefficient depending on the temperature. He obtained exact solutions that satisfy the realistic boundary conditions. Transient heat transfer through a longitudinal fin of several profiles which has the following form

$$\rho c_v\frac{\partial u}{\partial t} = A_p\frac{\partial}{\partial x}\left(F(x)k(u)\frac{\partial u}{\partial x}\right) - p\delta_b H(u)\left(u-u_a\right), \quad 0 < x < L \tag{8}$$

where k and H are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature, A_p profile area, p perimeter of the fin profile, $F(x)$ function of fin profiles and δ_b is the thickness of the fin at the base, is studied by Moitsheki and Harley [16]. They applied classical point symmetry method and performed some reductions. For heat transfer in a two-dimensional rectangular fin, Moitsheki and Rowjee [19] considered the problem:

$$\frac{\partial}{\partial y_1}\left(k(u)\frac{\partial u}{\partial y_1}\right) + \frac{\partial}{\partial x_1}\left(k(u)\frac{\partial u}{\partial x_1}\right) = s(u) \tag{9}$$

Here, u is the dimensionless temperature, x_1 is the longitudinal coordinate, y_1 is the transverse coordinate, s is the internal heat generation function, and k is the thermal conductivity. They constructed exact solution for the resulting linear equation and used symmetry analysis to classify the internal heat generating function and some reduction are executed. Many authors have considered the two-dimensional problem with $s = 0$ in equation (9) and thermal conductivity being a constant (see, e.g. [11, 15]) and the case $s = 0$ with a temperature-dependent thermal conductivity [7]. Moitsheki and Harley [17] also considered a two-dimensional pin fin with length L and radius R that has the following form.

$$\frac{1}{R}\frac{\partial}{\partial R}\left(Rk(u)\frac{\partial u}{\partial R}\right)+\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)=s(u) \tag{10}$$

They employed symmetry techniques to determine forms of the source or sink term for which the extra Lie point symmetries are admitted. Method of separation of variables is used to construct exact solutions when the governing equation is linear. Symmetry reductions result in reduced ordinary differential equations when the problem is nonlinear and some invariant solution for the linear case. In this paper, we intend to study the nonlinear $(2+1)$ fin equation by considering spherical fins with nonlinear thermal conductivity and variable heat transfer coefficient. The Lie symmetry method will be used to obtain exact solutions to this problem. The governing equation in this case is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k(u)\frac{\partial u}{\partial r}\right)+\frac{1}{rsin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}sin\theta k(u)\frac{\partial u}{\partial\theta}\right)\right]-N^2f(r)u=u_t \tag{11}$$

The above equation can be rewritten as

$$r^2k(u)u_{rr}+r^2k_uu_r^2+2rk(u)u_r+k(u)u_{\theta\theta}+k_uu_\theta^2+\frac{\cos\theta}{\sin\theta}k(u)u_\theta-r^2N^2f(r)u-r^2u_t=0 \tag{12}$$

Using the substitution $r = x$, and $\cos\theta = y$, the above equation is transformed to

$$x^2k(u)u_{xx}+x^2k_uu_x^2+2xk(u)u_x+k(u)(1-y^2)u_{yy}+k_u(1-y^2)u_y^2-2yk(u)u_y-x^2N^2f(x)u-x^2u_t=0 \tag{13}$$

This paper is organized as follows. In section 2, symmetry analysis of the given problem is performed via Lie symmetry and in section 3 complete classifications of solutions of equation (13) is presented. In section 4 symmetry generators are listed. Reduction of the problem to an ODE are shown in section 5. Finally, we conclude some results about this problem.

2. Symmetry Analysis of the Fin Equation

In this section, we perform the symmetry analysis of (13). To this end, we use the Lie symmetry method. The symmetry generator associated with (13) is given by

$$X=\xi(x,y,t,u)\frac{\partial}{\partial x}+\eta(x,y,t,u)\frac{\partial}{\partial y}+\tau(x,y,t,u)\frac{\partial}{\partial t}+\phi(x,y,t,u)\frac{\partial}{\partial u},$$

Requiring invariance of (13) with respect to the prolonged symmetry generator yields,

$$X^{(2)}=X+\phi^x\frac{\partial}{\partial u_x}+\phi^y\frac{\partial}{\partial u_y}+\phi^t\frac{\partial}{\partial u_t}+\phi^{xy}\frac{\partial}{\partial u_{xy}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}+\phi^{yt}\frac{\partial}{\partial u_{yt}}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{yy}\frac{\partial}{\partial u_{yy}}+\phi^{tt}\frac{\partial}{\partial u_{tt}}. \tag{14}$$

In the above expression, the coefficients of the prolonged generator are functions of (x, y, t, u) and can be determined by the formulae

$$\phi^i=D_i(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{x,i}+\eta u_{y,i}+\tau u_{t,i}\,,$$

$$\phi^{ij}=D_iD_j(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{x,ij}+\eta u_{y,ij}+\tau u_{t,ij},$$

where D_i represents total derivative and subscripts of u partial derivative with respect to the respective coordinates. At this stage we use the Lie symmetry criterion that the PDE (13) is invariant under the prolonged symmetry generator (14) modulu the PDE, namely,

$$X^{(2)}[x^2k(u)u_{xx} + x^2k(u)_u u_x^2 + xk(u)u_x + k(u)_u u_y^2 + k(u)u_{yy} - x^2N^2f(x)u - x^2u_t]\Big|_{PDE(13)} = 0. \tag{15}$$

whenever $u_t = \frac{1}{x^2}[x^2k(u)u_{xx} + x^2k_u u_x^2 + 2xk(u)u_x + k(u)(1-y^2)u_{yy} + k_u(1-y^2)u_y^2 - 2yk(u)u_y - x^2N^2f(x)u]$. Using results from (15) and comparing terms involving derivatives of the dependent function u, leads to the following over determined system of linear PDEs in ξ, η, τ and ϕ:

$$\xi_u = 0 = \eta_u = \tau_u = \tau_x = \tau_y = \phi_{uu}, \tag{16}$$

$$k_u\phi - 2k\xi_x + k\tau_t = 0, \tag{17}$$

$$x^2\eta_x + (1-y^2)\xi_y = 0, \tag{18}$$

$$-2k\xi + 2xk_u\phi + x^2\xi_t - 2kx\xi_x + 2ky\xi_y + 2xk\tau_t - kx^2\xi_{xx} - k\xi_{yy} + ky^2\xi_{yy} + 2kx^2\phi_{xu} = 0, \tag{19}$$

$$-2xyk\eta - 2k\xi + 2ky^2\xi + x\phi k_u - xy^2\phi k_u - 2xk\eta_y + 2xky^2\eta_y + xk\tau_t - xky^2\tau_t = 0, \tag{20}$$

$$-2xk\eta + 4ky\xi - 2xyk_u\phi + x^3\eta_t - 2kx^2\eta_x + 2xyk\eta_y - 2xyk\tau_t - x^3k\eta_{xx} - xk\eta_{yy} + xy^2k\eta_{yy} + 2xk\phi_{yu} - 2xy^2k\phi_{yu} = 0, \tag{21}$$

$$-N^2x^2f_xu\xi - N^2x^2f\phi + 2xk\phi_x - x^2\phi_t + x^2k\phi_{xx} + k\phi_{yy} - y^2k\phi_{yy} + N^2x^2fu\phi_u - N^2x^2fu\phi_u - n^2x^2fu\tau_t - 2yk\phi_y = 0. \tag{22}$$

To determine the unknown functions ξ, τ, η and ϕ, we solve the above system starting by first considering (17), we have

$$\phi = \frac{k}{k_u}(2\xi_x - \tau_t) \tag{23}$$

Differentiating (17) with respect to u twice yields

$$\phi_{uu} = (\frac{k}{k_u})_{uu}(2\xi_x - \tau_t) \tag{24}$$

Using (16) into (24) leads to

$$(\frac{k}{k_u})_{uu}(2\xi_x - \tau_t) = 0. \tag{25}$$

In what follows, we consider the above equation to perform a complete classification of both k and f.

3. Classification

In this section, we provide a complete classification of solutions of (13). Firstly, we notice that the following three cases arise from (25):

(I) $(\frac{k}{k_u})_{uu} = 0$,

(II) $2\xi_x - \tau_t = 0$,

(III) $2\xi_x - \tau_t = 0 = (\frac{k}{k_u})_{uu}$.

For complete classification, we consider all the three cases one by one.

3.1. Case I

Solving the differential equation $(\frac{k}{k_u})_{uu} = 0$, we determine $k(u)$ as,

$$k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}, \tag{26}$$

where γ, α and β are some integration constants. Using (26) into (23), instantly gives

$$\phi = (\alpha u + \beta)(2\xi_x - \tau_t). \tag{27}$$

Using (26) and (27) into (19), yields

$$-2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi + 2x\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_x + x^2\xi_t + 2y\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_y + (4\alpha-1)x^2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_{xx} - (1-y^2)\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_{yy} = 0. \tag{28}$$

Differentiating (28) with respect to u gives

$$-2\gamma(\alpha u + \beta)^{\frac{1}{\alpha}-1}\left[\xi - x\xi_x - y\xi_y - (4\alpha - 1)x^2\xi_{xx} + (1 - y^2)\xi_{yy}\right] = 0. \tag{29}$$

All constants involved in the above Eqs. are non-zero. Thus this is satisfied only when $\xi - x\xi_x - y\xi_y - (4\alpha - 1)x^2\xi_{xx} + (1 - y^2)\xi_{yy} = 0$ (the case $\alpha = 1$ not be considered as it becomes a special case of (I.b) that is dealt with later. The Ansatz solution of the above equation is

$$\xi = \lambda_1(t)x + \lambda_2(t)y. \tag{30}$$

Using (30) into (28) yields

$$\xi = c_1x + c_2y. \tag{31}$$

Using (31) into (18) yields

$$\eta = \frac{1 - y^2}{x}c_2 + \gamma(y, t). \tag{32}$$

To determine $\gamma(y, t)$ we use (32) into (20) to find that,

$$\gamma(y, t) = \sqrt{1 - y^2}\beta(t). \tag{33}$$

Therefore, (32) becomes

$$\eta = \frac{1 - y^2}{x}c_2 + \sqrt{1 - y^2}\delta(t). \tag{34}$$

Again, to determine $\delta(t)$, we use (34) into (21), to infer that,

$$\left[-2kx\sqrt{1-y^2} - \frac{2kxy^2}{\sqrt{1-y^2}} + kx(1-y^2)^{\frac{-3}{2}} - kxy^2(1-y^2)^{\frac{-3}{2}}\right]\delta(t) + x^3\sqrt{1-y^2}\delta_t(t) = 0. \tag{35}$$

The solution of the above equation is the trivial solution which is $\delta(t) = 0$. Consequently, (34) becomes

$$\eta = \frac{1 - y^2}{x}c_2. \tag{36}$$

Using (31) and (36) into (22) yields

$$-N^3x^3f_xuc_1 - N^2x^2yf_xuc_2 - \beta N^2x^2f(2c_1 - \tau_t) + x^2(\alpha u + \beta)\tau_{tt} - N^2x^2fu\tau_t = 0. \tag{37}$$

Differentiating (37) with respect to t gives

$$\beta N^2 x^2 f \tau_{tt} + x^2(\alpha u + \beta)\tau_{ttt} - N^2 x^2 f u \tau_{tt} = 0. \tag{38}$$

Again, differentiating (38) with respect to t, we obtain

$$\frac{\tau_{ttt}}{\tau_{tt}} = \frac{N^2 f}{\alpha}. \tag{39}$$

This implies that $f(x) = c$ and hence

$$\tau(t) = \frac{c_3\alpha^2}{N^4 c^2} e^{\frac{N^2 c}{\alpha}} + c_4 t + c_5. \tag{40}$$

Using (40) with $f(x) = c$ into (38), yields

$$\beta c_3 (1 + \frac{1}{\alpha}) = 0. \tag{41}$$

From (41) four cases arise:

(I.a) $\beta = 0, c_3 \neq 0$ and $\alpha > 0$,

(I.b) $\beta \neq 0, c_3 = 0$ and $\alpha > 0$,

(I.c) $\beta \neq 0, c_3 \neq 0$ and $\alpha = -1$,

(I.d) $\beta = 0, c_3 \neq 0$ and $\alpha = -1$.

We first consider I.a.

3.1.1. Subcase (I.a.) $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$ and $f(x) = c$.

Using (40) into (37), leads to $c_4 = 0$. Therefore, the expression for the infinitesimal symmetry generators ξ, η, τ and ϕ take the form,

$$\begin{aligned} &\xi = c_1 x + c_2 y, \quad \eta = \frac{(1-y^2)}{x} c_2, \quad \tau = \frac{c_3 \alpha}{c N^2} \exp\left(\frac{N^2 c}{\alpha} t\right) + c_5, \\ &\phi = \alpha u \left(2c_1 - c_3 c N^2 \exp\left(\frac{N^2 c}{\alpha} t\right)\right). \end{aligned} \tag{42}$$

The *four* symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} + 2\alpha u \frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \\ &X_3 = \frac{\alpha}{c N^2} e^{\frac{N^2 c}{\alpha} t} \frac{\partial}{\partial t} - \alpha u e^{\frac{N^2 c}{\alpha} t} \frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{43}$$

3.1.2. Subcase (I.b.) $k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$ and $f(x) = c$.

Using (40) into (37) with the above values of k and f, lead to $c_4 = 0 = c_1$. Therefore, the expression for the infinitesimal symmetry generators ξ, η, τ and ϕ take the form,

$$\xi = c_2 y, \quad \eta = \frac{(1-y^2)}{x} c_2, \quad \tau = c_5, \quad \phi = 0. \tag{44}$$

The *two* symmetry generators associated with above infinitesimals are given by,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{45}$$

3.1.3. Subcase (I.c.) $k(u) = \frac{\gamma}{\beta - u}$ and $f(x) = c$.

Using (40) into (37) with the above values of k and f, lead to $c_4 = 0 = c_1 = c_3$. Therefore, the expression for the infinitesimal symmetry generators $\xi, \eta, \tau,$ and ϕ take the form,

$$\xi = c_2 y, \quad \eta = \frac{(1-y^2)}{x} c_2, \quad \tau = c_5, \quad \phi = 0. \tag{46}$$

The *two* symmetry generators associated with above infinitesimals are given by,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{47}$$

3.1.4. Subcase (I.d.) $k(u) = -\frac{\gamma}{u}$ and $f(x) = c$.

Using (40) into (37) with the above values of k and f, lead to $c_4 = 0$. Therefore, the expression for the infinitesimal symmetry generators ξ, η, τ and ϕ take the form,

$$\begin{aligned} &\xi = c_1 x + c_2 y, \quad \eta = \frac{(1-y^2)}{x} c_2, \\ &\tau = \frac{-c_6}{cN^2} e^{-cN^2 t} + c_5, \quad \phi = -u(2c_1 - c_3 e^{-cN^2 t}). \end{aligned} \tag{48}$$

The *four* symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \\ &X_3 = -\frac{1}{cN^2} e^{-N^2 ct}\frac{\partial}{\partial t} + u e^{-N^2 ct}\frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{49}$$

3.2. Case II

In accordance with $2\xi_x - \tau_t = 0$, the system ((16)-(22)) becomes

$$\xi_u = 0 = \eta_u = \tau_u = \phi = \tau_y = \tau_x, \tag{50}$$

$$x^2\eta_x + (1-y^2)\xi_y = 0, \tag{51}$$

$$-2k\xi + x^2\xi_t + 2kx\xi_x + 2ky\xi_y - kx^2\xi_{xx} - k\xi_{yy} + ky^2\xi_{yy} = 0, \tag{52}$$

$$-2xyk\eta - 2k\xi + 2ky^2\xi - 2xk\eta_y + 2xky^2\eta_y + 2xk\xi_x - 2xky^2\xi_x = 0, \tag{53}$$

$$-2xk\eta + 4ky\xi - 2xyk_u\phi + x^3\eta_t - 2kx^2\eta_x + 2xyk\eta_y - 4xyk\xi_x - x^3k\eta_{xx} - xk\eta_{yy} + xy^2k\eta_{yy} = 0, \tag{54}$$

$$-N^2x^2f_xu\xi - 2N^2x^2fu\xi_x = 0. \tag{55}$$

From (51), we have

$$\eta_x = -\frac{(1-y^2)}{x^2}\xi_y. \tag{56}$$

Differentiation (52) with respect to u yields

$$-2k_u\xi + 2k_ux\xi_x + 2k_uy\xi_y - k_ux^2\xi_{xx} - k_u\xi_{yy} + k_uy^2\xi_{yy} = 0, \tag{57}$$

then,

$$k_u\left(-2\xi + 2\xi_x + 2y\xi_y - x^2\xi_{xx} - \xi_{yy} + y^2\xi_{yy}\right) = 0. \tag{58}$$

This leads to

$$\left(-2\xi + 2\xi_x + 2y\xi_y - x^2\xi_{xx} - \xi_{yy} + y^2\xi_{yy}\right) = 0, \tag{59}$$

From previous case, the solution of (59) is given by

$$\xi = c_1x + c_2y. \tag{60}$$

We follow the same procedure followed in the previous case, we end up with the following expressions of ξ, η, τ and ϕ, namely,

$$\xi = c_1x + c_2y, \quad \eta = \frac{(1-y^2)}{x}c_2, \quad \tau = 2c_1, \quad \phi = 0. \tag{61}$$

The above values of ξ, η, τ and ϕ satisfy the system ((50)-(54)). At this stage, we use (61) in (55) to get,

$$-f_x(c_1x + c_2y) - 2c_1f = 0 \tag{62}$$

Differentiating (62) with respect to y, we obtain

$$-c_2f_x = 0. \tag{63}$$

From (63), two cases arise:

(II.a) $c_2 = 0, and\ f_x \neq 0,$

(II.b) $c_2 \neq 0, and\ f_x = 0.$

First, we consider (II.a).

3.2.1 Case II.a

Using theses conditions arising in this case into (62), gives

$$-c_1[xf_x + 2f] = 0 \tag{64}$$

From (64), two cases arise:

(II.a.1) $c_1 = 0$ and $xf_x + 2f \neq 0$,

(II.a.2) $c_1 \neq 0$ and $xf_x + 2f = 0$.

Considering first (II.a.1).

3.2.1.1 Case II.a.1

In the light of the conditions of this case, the $k(u)$, and $f(x)$ are arbitrary functions and the general expressions of ξ, η, τ and ϕ, have the following form:

$$\xi = \eta = \phi = 0, \quad \tau = c_3. \tag{65}$$

the only one generator corresponding to this case is $\mathbf{X} = \frac{\partial}{\partial \mathbf{t}}$.

3.2.1.1 Case II.a.2

In accordance with these conditions of this case, the $k(u)$, is arbitrary function, $f(x) = \frac{c}{x^2}$ and the general expressions of ξ, η, τ and ϕ, have the following form:

$$\xi = c_1x, \quad \eta = 0, \quad \tau = 2c_1t + c_3, \quad \phi = 0. \tag{66}$$

and the generators in this case are

$$X_1 = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}. \tag{67}$$

3.2.2 Case II.b ($c_2 \neq 0$ and $f(x) = c$.)

Using theses conditions arising in this case into ((62),gives $c_1 = 0$. Thus, we infer that $f(x) = c$ and $k(u)$ is arbitrary, and hence the general expression of ξ, η, τ and ϕ are

$$\xi = c_2 y, \quad \eta = \frac{(1-y^2)}{x}c_2, \quad \tau = c_3, \quad \phi = 0. \tag{68}$$

The symmetry generators in this case are,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{69}$$

4. Symmetry Generators

In this section, we list the Lie symmetry generators obtained above for different values of $k(u)$ and $f(x)$.

1-$f(x) = c$

a- $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$. In this case the symmetry generators are

$$X_1 = x\frac{\partial}{\partial x} + 2\alpha u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t},$$
$$X_4 = \frac{\alpha}{N^2 c}\exp\left(\frac{N^2 c}{\alpha}t\right)\frac{\partial}{\partial t} - \alpha u \exp\left(\frac{N^2 c}{\alpha}t\right)\frac{\partial}{\partial u}.$$

The commutation relation for these generators are given in the following table.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	X_2
X_2	X_2	0	0	X_1
X_3	0	0	0	$\frac{cN^2}{\alpha}X_4$
X_4	0	0	$-\frac{cN^2}{\alpha}X_4$	0

Table 1. Commutator table of the fin equation

b- $k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$. In this case the symmetry generators are

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table.

$[X_i, X_j]$	X_1	X_2
X_1	0	0
X_2	0	0

c- $k(u) = \frac{\gamma}{(\beta - u)}$. In this case the symmetry generators are

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table.

$[X_i, X_j]$	X_1	X_2
X_1	0	0
X_2	0	0

Table 2. **Commutator table of the fin equation**

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	X_2
X_2	X_2	0	0	X_1
X_3	0	0	0	$-cN^2X_4$
X_4	0	0	cN^2X_4	0

Table 3. **Commutator table of the fin equation**

d- $k(u) = -\frac{\gamma}{u}$. In this case the symmetry generators are

$$X_1 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = -\frac{1}{N^2c}\exp(-N^2ct)\frac{\partial}{\partial t} + u\exp(-N^2ct)\frac{\partial}{\partial u}.$$

The commutation relation for these generators are given in the following table.

2- $f(x)$ and $k(u)$ are arbitrary functions. In this case, we have only one generator which is $X = \frac{\partial}{\partial t}$.

3- $f(x) = \dfrac{c}{x^2}$ and $k(u)$is arbitrary. In this case, we have only two generators which are

$$X_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table.

$[X_i, X_j]$	X_1	X_2
X_1	0	$-2X_2$
X_2	$2X_2$	0

Table 4. **Commutator table of the fin equation**

5. Reduction Under two Dimensional Subalgebra

In what follows, we will show the reduction of the given problem to an ODE using two dimensional subalgebra.

5.1. Case 1

In this subsection, we present solutions of (13) via reductions. These reductions are obtained by the similarity variables obtained through symmetry generators. To perform reductions of (13), we first consider two symmetry generators, from Table (1) (In this case $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$ and $f(x) = c$). Here X_1, and X_3 span an abelian subalgebra. To start reduction, we first consider X_3. The characteristic equation corresponding to this generator,

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}. \tag{70}$$

Solving the above equation it is straight forward [1] to find that it yields the similarity variables, $r = x$ and $s = y$ with $w(r,s) = u$. Replacing u in (13) in terms of new variables becomes,

$$r^2k(w)w_{rr} + r^2k_w w_r^2 + 2rk(w)w_r + k(w)(1-s^2)w_{ss} + (1-s^2)k_w w_s^2 - 2sk(w)w_s - N^2cr^2w = 0. \tag{71}$$

To proceed further, we first transform X_1 in terms of new variables r, s and w. Thus, $\widehat{X}_1 = r\frac{\partial}{\partial r} + 2\alpha w\frac{\partial}{\partial w}$. The similarities corresponding to this generator are $z = s$ and $v(z) = r^{2\alpha}w$. This reduces (71) to a second-order differential equation given by,

$$2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha-1)v^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}v^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1-z^2)v^{\frac{1}{\alpha}}v_{zz} + \gamma\alpha^{\frac{1}{\alpha}-1}(1-z^2)v^{\frac{1}{\alpha}-1}v_z^2 - 2\gamma\alpha^{\frac{1}{\alpha}}zv^{\frac{1}{\alpha}}v_z - N^2cv = 0. \tag{72}$$

5.2. Case 2

From Table 1, $[X_1, X_4] = 0$ are commutative. Thus, the reduction can be started either by X_1 or X_4. To this end, we first consider X_1. The characteristic equation corresponding to this generator is

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2\alpha u} \tag{73}$$

The similarity variables corresponding to above equation become $r = y$, $s = t$ and $u = x^{2\alpha}w$. These variables reduce (13) to a PDE of the form,

$$2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha-1)w^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}w^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1-r^2)w^{\frac{1}{\alpha}}w_{rr} + \gamma\alpha^{\frac{1}{\alpha}-1}(1-r^2)w^{\frac{1}{\alpha}-1}w_r^2 - 2\gamma\alpha^{\frac{1}{\alpha}}rw^{\frac{1}{\alpha}}w_r - N^2cw - w_s = 0. \tag{74}$$

Using similarity variables transformation obtained from X_4, transforms $\widehat{X}_4 = \frac{\alpha}{N^2c}\exp\left(\frac{N^2c}{\alpha}s\right)\frac{\partial}{\partial s} - \alpha w\exp\left(\frac{N^2c}{\alpha}s\right)\frac{\partial}{\partial w}$. This leads to the new coordinates $r = z$, $v(z) = \exp(-N^2cs)w$. In the light of these similarities, (74) transforms to,

$$(4\alpha^2 - 6\alpha)v + (1-z^2)v_{zz} + \frac{1}{\alpha}(1-z^2)\frac{1}{v}v_z^2 - 2zv_z = 0. \tag{75}$$

Choosing $\alpha = \frac{3}{2}$, the above equation takes the form,

$$(1-z^2)v_{zz} + \frac{2}{3}(1-z^2)\frac{1}{v}v_z^2 - 2zv_z = 0. \tag{76}$$

giving exact solution

$$u(x,y,t) = c_2x^3\exp(-N^2ct)\left(6c_1 + 5\ln(y-1) - 5\ln(y+1)\right)^{\frac{3}{5}}. \tag{77}$$

The graph of this solution is plotted in Figure 1 and Figure 2.

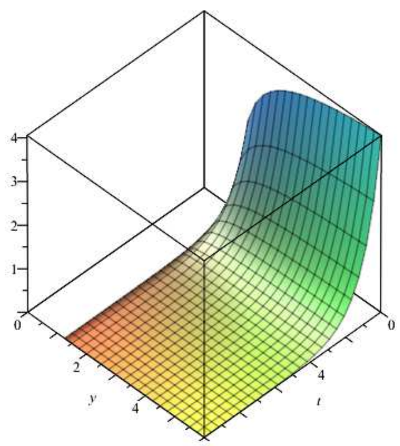

Figure 1. **Plot of solution given by (77) with** $c_1 = 2,\ N^2c = 1,\ c_2 = 1$ **and** $x = constant$.

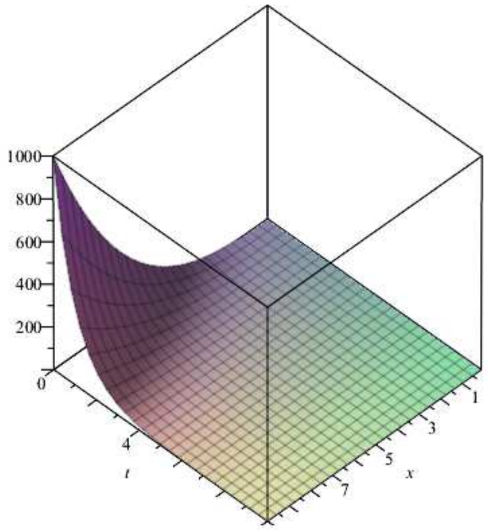

Figure 2. **Plot of solution given by (77) with $c_1 = 0,\ N^2c = 1\ c_2 = 1$, and $y = constant$.**

5.3. Case 3

In this case we execute reduction using table (3). Here $f(x) = \dfrac{c}{x^2}$ and $k(u)$ is arbitrary. We consider the symmetry generators X_1, X_2 which satisfy a commutative relationship $[X_1, X_2] = -2X_2$ as shown in table(4). First considering X_2, and follow the procedure in the previous cases, the generator X_2 reduces (13) to

$$r^2k(w)w_{rr} + r^2k_w w_r^2 + 2rk(w)w_r + k(w)(1-s^2)w_{ss} + (1-s^2)k_w w_s^2 - 2sk(w)w_s - N^2cw = 0. \tag{78}$$

In the light of X_2, the X_1 transforms to $\widehat{X}_1 = r\frac{\partial}{\partial r}$ which gives $z = s$ with $w = v(z)$. In the light of these similarity variables, (78) reduces to the following ODE:

$$k(v)(1-z^2)v_{zz} + (1-z^2)k_v v_z^2 - 2zk(v)v_z - N^2cv = 0. \tag{79}$$

The reductions performed above are given in the tabular form in the following:

Case#	Algebra	Reduction	z	v
Case 1	$[X_1, X_3] = 0$	$2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha-1)v^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}v^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1-z^2)v^{\frac{1}{\alpha}}v_{zz}$ $+\gamma\alpha^{\frac{1}{\alpha}-1}(1-z^2)v^{\frac{1}{\alpha}-1}v_z^2 - 2\gamma\alpha^{\frac{1}{\alpha}}zv^{\frac{1}{\alpha}}v_z - N^2cv = 0$	y	$x^{2\alpha}u$
Case 2	$[X_1, X_4] = 0$	$(4\alpha^2 - 6\alpha)v + (1-z^2)v_{zz} + \frac{1}{\alpha}(1-z^2)\frac{1}{v}v_z^2 - 2zv_z = 0$	y	$e^{-N^2ct}x^{2\alpha}u$
Case 3	$[X_1, X_2] = -2X_2$	$k(v)(1-z^2)v_{zz} + (1-z^2)k_v v_z^2 - 2zk(v)v_z - N^2cv = 0$	y	u

Table 5. **Reduction**

6. Conclusion

As a consequence of the results obtaining in this paper, we notice that the reduction of the given equation to ODE may lead to find its exact solution. Some of these ODEs can not be solved readily. However, the reduced form is generally simpler than the original non-linear PDE and we may use symmetry or other methods to solve them.

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