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Convex and Weakly Convex Subsets of a Pseudo Ordered Set

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Abstract: In this paper the notion of convex and weakly convex (w-convex) subsets of a pseudo ordered set is introduced and several characterizations are proved. It is proved that set of all convex subsets of a pseudo ordered set A forms a complete lattice. Notion of isomorphism of psosets is introduced and characterization for convex isomorphic psosets is obtained. It is proved that lattice of all w-convex subsets of a pseudo ordered set A denoted by $WCS(A)$ is lower semi modular. Also we have proved that for any two pseudo ordered sets A and A^1, w-convex homomorphism maps atoms of $WCS(A)$ to atoms of $WCS(A^1)$. Concept of path preserving mapping is introduced in a pseudo ordered set and it is proved that every mapping of a pseudo ordered set A to itself is path preserving if and only if A is a cycle.

MSC: 06B20, 06B10.

Keywords: Psoset, convex subset, w-convex subset, w-convex hull, lattice.

1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A, \trianglelefteq\rangle$ is called a *pseudo-ordered set* or a *psoset*.For $a, b \in A$ if $a \trianglelefteq b$ and $a \neq b$, then we write $a \triangleleft b$. For a subset B of A, the notions of a lower bound, an upper bound, the greatest lower bound (GLB or meet, denoted by $\wedge B$), the least upper bound (LUB or join, denoted by $\vee B$) are defined analogous to the corresponding notions in a poset [1]. It is shown in [3] that any psoset can be regarded as a digraph (possibly infinite) in which for any pair of distinct elements u and v there is no directed line between u and v or if there is a directed line from u to v, there is no directed line from v to u. Define a relation $\sqsubseteq_B$ on a subset B of a psoset $\langle A, \trianglelefteq\rangle$ by setting $b \sqsubseteq_B b^1$for two elements b and $b^1$of B if and only if there is a directed path in B from b to b^1 say $b = b_0 \trianglelefteq b_1 \trianglelefteq \cdots \trianglelefteq b_n = b^1$ for some $n \geq 0$. The relation $\sqsupseteq_B$ is defined dually.

If for each pair of elements b and b^1 of B at least one of the relations $b \sqsubseteq_B b^1$ or $b^1 \sqsubseteq_B b$ holds, then B will be called a *pseudo chain* or a *p-chain*. If for each pair of elements b and b^1 of B both the relations $b \sqsubseteq_B b^1$ and $b^1 \sqsubseteq_B b$ hold, then B will be called a *cycle*. The empty set and a single element set in a psoset are cycles. A non-trivial cycle contains at least three elements. A psoset is said to be *acyclic* if it does not contain any non-trivial cycle.

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2. Convex Subsets of a Psoset

Definition 2.1. *A subset S of a pseudo ordered set A is said to be a convex subset of A whenever $a, b \in S$ and $c \in A$ such that $a \trianglelefteq c \trianglelefteq b$ then $c \in S$.*

Set of all convex subsets of a psoset A is denoted by $CS(A)$. Clearly the empty set ϕ and the psoset A are convex subsets. Any singleton is a convex subset of A. Obviously $\langle CS(A), \subseteq\rangle$ is a poset where $\subseteq$ is the set inclusion relation defined on S. For $K_1, K_2 \in CS(A)$, define $K_1 \wedge K_2 = K_1 \cap K_2$ and $K_1 \vee K_2 =$ smallest convex subset of A containing $K_1 \cup K_2$. Then $\langle CS(A), \subseteq\rangle$ is a complete lattice with smallest element ϕ and greatest element A. The lattice $\langle CS(A), \subseteq\rangle$ is atomistic. One element subsets of A are atoms of $CS(A)$ and each element of $CS(A)$ different from ϕ is a join of some atoms.

Example 2.2. *Consider the psoset $\langle A, \trianglelefteq\rangle$ represented in Figure 1. $CS(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c, d\}, A\}$ and $\langle CS(A), \subseteq\rangle$ is a lattice which is represented in Figure 2 Where $P = \phi, Q = \{a\}, R = \{b\}, S = \{c\}, T = \{d\}, U = \{a, b\}, V = \{b, c, d\}$ and $W = A$.*

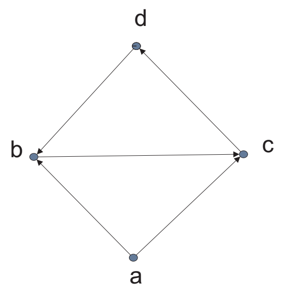

Figure 1:

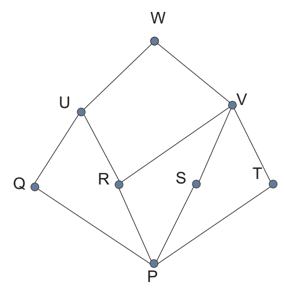

Figure 2:

Remark 2.3. *$\langle CS(A), \subseteq\rangle$ of Figure 2 is a non modular lattice. But $CS(A)$ is both lower semimodular and upper semimodular.*

Definition 2.4. *Let $\{X_i : i \in I\}$ be an arbitrary collection of subsets of a pseudo ordered set A. The set of all $Z \in CS(A)$ such that $X_i \subset Z$ for each $i \in I$ will be denoted by $CS_A(X_i : i \in I)$. If $\{X_i : i \in I\} = \{X, Y\}$, we denote $CS_A(X_i : i \in I) = CS_A(X, Y)$ and for $X = \{a\}$ and $Y = \{b\}$, we denote it by $CS_A(a, b)$.*

Definition 2.5. *Let A be any psoset and $M \subseteq A$. Let $CS_A(M) = \cap\{K_i : i \in I\}$ where K_i runs over all convex subsets of A containing M.*

We write $CS_A(a, b)$ for $CS_A(\{a, b\})$.

Definition 2.6. *We say that two psosets A and A^1 are convex isomorphic if and only if $\langle CS(A), \subseteq\rangle$ and $\langle CS(A^1), \subseteq\rangle$ are isomorphic.*

Let F be a mapping from A into B and $\phi \neq C \subseteq A$. Denote by F/C, the restriction of F onto the subset C. That is $F/C = F \cap (C \times B)$. All one element subsets of A are atoms in the lattice $\langle CS(A), \subseteq\rangle$. If F is an isomorphism between $\langle CS(A), \subseteq\rangle$ and $\langle CS(A^1), \subseteq\rangle$, then as every isomorphism of atomic lattices maps atoms onto atoms, we get $F(\{a\}) = \{a^1\} \in CS(A^1)$ where $a^1 \in A^1$.

Definition 2.7. *Let F be an isomorphism of lattices $\langle CS(A), \subseteq\rangle$ and $\langle CS(A^1), \subseteq\rangle$. Let f be a mapping from A to A^1 such that $\{f(a)\} = F(\{a\})$ for each $a \in A$. Then we say that the mapping f is associated with the isomorphism F.*

Denote $f(S) = \{f(x)|x \in S\}$ for a subset S of A.

Lemma 2.8. *$F(S) = f(S)$ for any $S \in CS(A)$.*

Proof. If $a \in S$ then $\{a\} \subseteq S$ and $F(\{a\}) = \{f(a)\} \subseteq F(S)$ as F is an isomorphism. Then $f(a) \in F(S)$ and thus $f(S) \subseteq F(S)$.
Conversely, if $a^1 \in F(S)$ then $\{a^1\} \subseteq F(S)$ and $F^{-1}(\{a^1\}) = \{f^{-1}(a^1)\} \subseteq S$ as F^{-1} is also an isomorphism. Then $f^{-1}(a^1) \in S$ and $a^1 \in f(S)$ so that $F(S) \subseteq f(S)$ proving that $F(S) = f(S)$. □

Theorem 2.9. *Let f be associated with an isomorphism F of the lattices $\langle CS(A), \subseteq\rangle$ and $\langle CS(A^1), \subseteq\rangle$. Then $f(CS_A(M)) = CS_{A^1}(f(M))$ for any subset $M \subseteq A$.*

Proof. As $M \subseteq \bigcap CS_A(M)$, we have $f(M) \subseteq f(CS_A(M))$. Now, by lemma 2.8 $f(\bigcap CS_A(M)) = F(\bigcap CS_A(M)) \in CS(A^1)$ and therefore $CS_{A^1}(f(M)) \subseteq f(CS_A(M))$. On the other hand, let $Z \in CS(A^1)$ be such that $f(M) \subseteq Z$. Since F is surjective, there exists $W \in CS(A)$ with $F(W) = f(W) = Z$. It follows that $M \subseteq W$ and therefore $\bigcap CS_A(M) \subseteq W$, consequently $f(\bigcap CS_A(M)) \subseteq Z$ and $f(CS_A(M)) \subseteq CS_{A^1}(f(M))$. □

Theorem 2.10. *The following three conditions are equivalent for two psosets A and A^1.*

(i). The psosets A and A^1 are convex isomorphic.

(ii). There exists a bijection $f : A \longrightarrow A^1$ such that $f(CS_A(M)) = CS_{A^1}(f(M))$ for $M \subseteq A$.

(iii). There exists a bijection $f : A \longrightarrow A^1$ such that $f(CS_A((a, b))) = CS_{A^1}((f(a), f(b)))$ for each $a, b \in A$.

Proof. $(i) \Rightarrow (ii)$: follows from Theorem 2.9.
$(ii) \Rightarrow (iii)$: Follows directly.
$(iii) \Rightarrow (i)$: Let f be a bijection satisfying (iii). Denote by $P(A)$ the power set of A and define a mapping $F : P(A) \to P(A^1)$ such that $F(S) = f(S)$ for each $S \in P(A)$. Now, we prove that for any convex set S, its image $F(S)$ is also convex. Clearly $f(a), f(b) \in f(S) = F(S)$ for each $a, b \in S$. If $S \in CS(A)$ then $CS_A(a, b) \subseteq S$ for arbitrary $a, b \in S$ and so by (iii), we have $CS_{A^1}(f(a), f(b)) = f(CS_A(a, b)) \subseteq f(S) = F(S)$. This implies that the mapping F maps convex subsets of A onto convex subsets of A^1 and F is a bijection as f is a bijection. Therefore the restriction of the mapping $F/CS(A) : CS(A) \longrightarrow CS(A^1)$ is also a bijection. Since $S \subseteq T$ if and only if $F(S) \subseteq F(T)$ for each $S, T \in CS(A)$, the mapping $F/CS(A)$ is an isomorphism of lattices $\langle CS(A), \subseteq\rangle$ and $\langle CS(A^1), \subseteq\rangle$. Therefore psosets A and A^1 are convex isomorphic. □

3. w-convex Subsets of a Psoset

Definition 3.1. *A subset S of a psoset A is said to be a w-convex subset (weakly convex subset) of A whenever $a, b \in S$ and $c \in A$ such that $a \sqsubseteq_A c \sqsubseteq_A b$ then $c \in S$.*

Set of all w-convex subsets of a psoset A is denoted by $WCS(A)$ and it forms a lattice with respect to the relation $\subseteq$.

Remark 3.2.

(1). For $H_1, H_2 \in WCS(A)$, define $H_1 \wedge H_2 = H_1 \cap H_2$ and $H_1 \vee H_2 =$ the smallest w-convex subset of A containing $H_1 \cup H_2$.

(2). $\langle WCS(A), \subseteq\rangle$ is a complete lattice as φ is the least element and A is the greatest element of $WCS(A)$.

Example 3.3. *A psoset $\langle A, \trianglelefteq\rangle$ where $A = \{a, b, c, d\}$ and the lattice of all its w-convex subsetsare shown in Figure 3.*

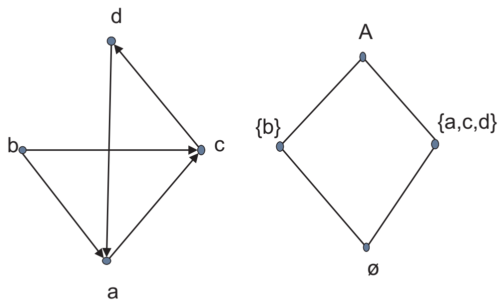

Figure 3:

Definition 3.4. *Let S be a subset of a psoset A. The w-convex hull of S denoted by $wch(S)$ is defined to be the smallest w-convex subset of A containing S.*

Theorem 3.5. *Let S be a subset of a psoset A. Then $wch(S) = \{q \in A | p_1 \sqsubseteq_A q \sqsubseteq_A p_2$ for some $p_1, p_2 \in S\}$ where p_1, p_2 need not be distinct.*

Proof. Let $Q = \{q \in A | p_1 \sqsubseteq_A q \sqsubseteq_A p_2$ for some $p_1, p_2 \in S\}$. Clearly Q is a subset of any w-convex subset of A containing S. Then $Q \subseteq wch(S)$. Let us prove that Q itself is a w-convex subset of A. Let $q_1, q_2 \in Q$ such that $q_1 \sqsubseteq_A r \sqsubseteq_A q_2$ for some $r \in A$. Now $q_1 \in Q$ implies there exist some $p_1, p_2 \in S$ such that $p_1 \sqsubseteq_A q_1 \sqsubseteq_A p_2$. Also $q_2 \in Q$ implies there exist $p_1^1, p_2^1 \in S$ such that $p_1^1 \sqsubseteq_A q_2 \sqsubseteq_A p_2^1$. Then $p_1 \sqsubseteq_A q_1 \sqsubseteq_A r \sqsubseteq_A q_2 \sqsubseteq_A p_2^1$ which implies $r \in Q$. Therefore $Q = wch(S)$. □

Corollary 3.6. *For any element a in a cycle C, $wch(\{a\}) = C$.*

A lattice L is said to be *lower semi modular* if $x \vee y$ covers x and y imply x and y cover $x \wedge y$.

Theorem 3.7. *Lattice of all w-convex subsets of a psoset A is lower semi modular.*

Proof. Let $S_1, S_2 \in WCS(A)$, lattice of all w-convex subsets of a psoset A. Let $P = S_1 \vee S_2$ and $Q = S_1 \wedge S_2$. Let P cover both S_1 and S_2. It suffices to prove that S_1 covers Q. Suppose there exists a w-convex subset S^1 of A such that $Q \subsetneq S^1 \subsetneq S_1$. Let $s_0 \in S^1 - Q$. Then $s_0 \in S_1$ and $s_0 \notin S_2$. Let $s_1 \in S_1 - S^1$. Then $s_1 \in S_1$, $s_1 \notin S_2$ and $s_1 \neq s_0$. Now $S_2 \subset S_2 \cup \{s_0\} \subset P$, as P is the smallest w-convex subset of A containing $S_1 \cup S_2$. But P covers S_2 imply $S_2 \cup \{s_0\}$ is not a w-convex subset of A. Therefore there exists a path between s_0 and an element $k_0 \in S_2$ which does not lie completely in $S_2 \cup \{s_0\}$. Assume the path in the form $k_0 \sqsubseteq_A s_0$. (similar argument holds if the path is of the form $s_0 \sqsubseteq_A k_0$).
Let $X = \{s \in S^1 - Q | k \sqsubseteq_A s$ for some $k \in S_2\}$. X is non empty as $s_0 \in X$ and $S_2 \subset S_2 \cup X$. Further as $s_1 \notin X$ (in fact $s_1 \notin S^1$), we have $S_2 \cup X \subset P$. As P covers S_2, $S_2 \cup X \notin WCS(A)$. Therefore there exists a path $m \sqsubseteq_A n$ between two elements m, n of $S_2 \cup X$ which is not contained in $S_2 \cup X$. This implies $m \sqsubseteq_A t \sqsubseteq_A n$ but $t \notin S_2 \cup X$. We can assume that $t \in P$ as $S_2 \cup X$ is not a w-convex subset of P. In the following cases either we get a contradiction to the w-convexity of S_2 or S^1 itself is not w-convex, proving S_1 covers Q.
Case (1): Let $m, n \in S_2$. This is a contradiction to the w-convexity of S_2.
Case (2): Let $m \in X$ and $n \in S_2$. As $m \in X$, $m \notin S_2$. By the definition of X there exists a $k \in S_2$ such that $k \sqsubseteq_A m$. Thus $k \sqsubseteq_A m$ and $m \sqsubseteq_A t \sqsubseteq_A n$ imply $k \sqsubseteq_A t \sqsubseteq_A n$, which contradicts the w-convexity of S_2.
Case (3): Let $m, n \in X$. As $m \in X$, there exists a path $k \sqsubseteq_A m$ for some $k \in S_2$. But we have a path $m \sqsubseteq_A t$ which implies there is a path $k \sqsubseteq_A t$. But $t \notin X$, ie $t \notin S^1 - Q$. Since $t \notin S_2$, it can not be in Q. So $t \notin S^1$. But $m, n \in X \subseteq S^1$ shows that S^1 is not a w-convex subset of A.
Case (4): Let $m \in S_2$ and $n \in X$. As we have a path from $m \sqsubseteq_A t$, $t \notin X = S^1 - Q$ and since $t \notin S_2$ imply $t \notin S^1$. Now P

covers S_2 and $t \notin S_2$, we must have $wch(S_2 \cup \{t\}) = P$. Thus every element of P is in S_2 or else lies on some path between t and an element of S_2. In particular consider some $s_0 \in X$, we have $k_0 \sqsubseteq_A s_0$ and $t \sqsubseteq_A n$. If $k \sqsubseteq_A s_0 \sqsubseteq_A t$ where $k \in S_2$, then we have $s_0 \sqsubseteq_A t \sqsubseteq_A n$ which proves that S^1 is not w-convex. On the other hand if $t \sqsubseteq_A s_0 \sqsubseteq_A k$ then $k_0 \sqsubseteq_A s_0 \sqsubseteq_A k$, contradicting the w-convexity of S_2. □

Theorem 3.8. *If S covers S^1 in $WCS(A)$ and p, q belong to $S - S^1$ then $wch(\{p\}) = wch(\{q\})$.*

Proof. As S covers S^1, $wch(S^1 \cup \{p\}) = wch(S^1 \cup \{q\}) = S$. Then p lies in a path from q to r where $r \in S^1$ and q lies in a path from p to s where $s \in S^1$. If there exist paths $p \sqsubseteq_A q$ and $q \sqsubseteq_A p$ then the proof is done. But if both paths have the same direction say $p \sqsubseteq_A q$, we have paths $p \sqsubseteq_A r$ and $s \sqsubseteq_A p$ with $r, s \in S^1$, contradicting the w-convexity of S^1. □

Definition 3.9. *Let $\langle A, \trianglelefteq \rangle$ and $\langle A^1, \trianglelefteq^1 \rangle$ be any two psosets. A mapping $f : A \longrightarrow A^1$ is called*

(1). order preserving if for $a, b \in A$, $a \trianglelefteq b$ implies $f(a) \trianglelefteq^1 f(b)$.

(2). path preserving if for $a, b \in A$, $a \sqsubseteq_A b$ implies $f(a) \sqsubseteq_{A^1} f(b)$.

Remark 3.10. *Any order preserving mapping f is path preserving. The converse is not true. For example, in the psoset A of Figure 4, define a mapping $f : A \longrightarrow A$ by $f(a) = b, f(b) = a, f(c) = c$. Clearly f is path preserving but not order preserving.*

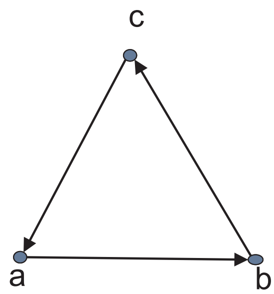

Figure 4:

Theorem 3.11. *Every mapping of a psoset A to itself is path preserving if and only if A is a cycle.*

Proof. If A is a cycle, then for any two elements a, b of A, both $a \sqsubseteq_A b$ and $b \sqsubseteq_A a$ hold. Therefore every mapping of A to itself is path preserving. Conversely, let us assume that A is not a cycle. Then there exists at least one pair of elements say (a, b) in A such that $a \sqsubseteq_A b$ holds but $b \sqsubseteq_A a$ does not hold. Define $f : A \longrightarrow A$ by $f(b) = a$ and $f(c) = b$ for all $c \neq b$. Then f is not path preserving as $a \sqsubseteq_A b$ but $f(a) \sqsubseteq_A f(b)$ does not hold. □

One can easily prove the following theorem.

Theorem 3.12. *Let $f : A \longrightarrow A^1$ be path preserving. If S is a w-convex subset in A then $f(S)$ is a w-convex subset in A^1.*

Definition 3.13. *Let $\langle A, \trianglelefteq \rangle$ and $\langle A^1, \trianglelefteq^1 \rangle$ be any two psosets. A mapping $f : A \to A^1$ is called a homomorphism if*

(1). f is order preserving.

(2). $a^1 \trianglelefteq^1 b^1$ in A^1 implies there exists $a \in f^{-1}(a^1)$ and $b \in f^{-1}(b^1)$ such that $a \trianglelefteq b$.

Theorem 3.14. *Let $f : A \to A^1$ be a homomorphism. If S^1 is a w-convex subset of A^1 then $f^{-1}(S^1)$ is a w-convex subset of A.*

Proof. Let $a, b \in f^{-1}(S^1)$ such that $a \sqsubseteq_A c \sqsubseteq_A b$ for some $c \in A$. If $c \notin f^{-1}(S^1)$ then $f(c) \notin S^1$. Now $f(a) \sqsubseteq_{A^1} f(c) \sqsubseteq_{A^1} f(b)$ and $f(c) \notin S^1$, a contradiction to the w-convexity of S^1. Hence $c \in f^{-1}(S^1)$ and $f^{-1}(S^1)$ is w-convex. □

Remark 3.15. *If $f : A \longrightarrow A^1$ is a homomorphism between two psosets A and A^1 and if S is a w-convex subset of A then $f(S)$ need not be a w-convex subset of A^1. For, in Figure 5 define a map $f : A \longrightarrow A^1$ by $f(a) = w, f(b) = y, f(c) = x, f(d) = z$. Clearly f is a homomorphism. Observe that $\{b\}$ is w-convex in A where as $f(\{b\}) = \{y\}$ is not w-convex in A^1.*

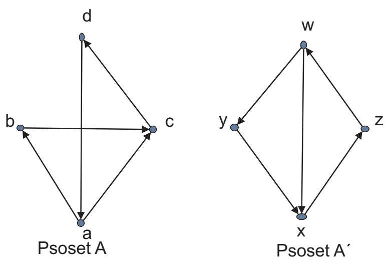

Figure 5:

Definition 3.16. *A homomorphism between two psosets A and A^1 is called a w-convex homomorphism if it takes w-convex subsets of A onto w-convex subsets of A^1.*

An element a of a lattice L is said to be an *atom* if for any $b \in L, 0 \leq b \leq a$ imply either $b = 0$ or $b = a$ where 0 is the least element of L.

Remark 3.17.

(1). A cycle in a psoset A is always an atom of $WCS(A)$

(2). w-convex hull of a single element in a psoset A is an atom in $WCS(A)$.

Theorem 3.18. *Let $f : A \rightarrow A^1$ be a w-convex homomorphism. If S is an atom in $WCS(A)$ then $f(S)$ is an atom in $WCS(A^1)$. Conversely if S^1 is an atom in $WCS(A^1)$ then there exists an atom S in $WCS(A)$ such that $f(S) = S^1$.*

Proof. If $f(S)$ is not an atom in $WCS(A^1)$ then there exists a w-convex subset S^1 in $WCS(A^1)$ such that $\varphi \subset S^1 \subset f(S)$. But then $\varphi \subset f^{-1}(S^1) \cap S \subset S$, where $f^{-1}(S^1) \cap S$ is also a w-convex subset of A, contradicting the fact that S is an atom. Conversely, let S^1 be an atom in $WCS(A^1)$ and $S \subseteq f^{-1}(S^1)$ be an atom in $WCS(A)$. Then $\varphi \subseteq f(S) \subseteq S^1$ and since S^1 is an atom in $WCS(A^1)$, we have $f(S) = S^1$. □

Corollary 3.19. *Any w-convex homomorphism maps acyclic psosets into acyclic psosets.*

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