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Fractional Differential Operator of Generalized Mittag-Leffler Function Using Jacobi Polynomial

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Abstract: The paper is devoted to the study of generalized fractional calculus of the generalized Mittag-Leffler function $E^{\delta}_{\upsilon,\rho}(z)$ which is an entire function of the form

$$E^{\delta}_{\upsilon,\rho}(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s\, z^s}{\Gamma(\upsilon s+\rho)\ \ s!}$$

Where $\upsilon>0$ and $\rho>0$. For $\delta=1$, it is reduces to Mittag-Leffler function $E_{\upsilon,\rho}(z)$. We have shown that the generalized fractional calculus operators transform such function with power multipliers in to generalized Wright function. Some elegant results obtained by Kilbas and Saigo [11], Saxena and Saigo [24] are the special cases of the result derived in this paper.

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1. Introduction

The function $E_{\upsilon}(z)$ is defined by the series representation

$$E_{\upsilon}(z)=\sum_{s=0}^{\infty}\frac{z^s}{\Gamma(\upsilon s+1)\ \ s!},\ \ \upsilon>0,\ z\in C \tag{1}$$

Mittag-Leffler [19, 20], Wimen [26, 27], Agarwal [1], Humbert and Agarwal [11], investigated the generalization of the above function $E_{\upsilon}(z)$ in the following manner; see [4]

$$E^{\delta}{}_{\upsilon,\rho}(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s z^s}{\Gamma(\upsilon n+\rho)\ \ s!}\qquad \upsilon>0,\ \rho>0,\ z\in C \tag{2}$$

Where C be the set of complex numbers. For a detailed study of various properties, generalizations and applications of this function we can refer to papers of Dzherashyan [2], Kilbas and Saigo [9, 10, 11, 12], Kilbas, Saigo and Saxena [15], Gorenflo and Mainardi [7], Gorenflo, Kilbas and Rogosin [5] and Gorenflo, Luchko and Rogosin [6]. A more generalized form of (2) is introduced by Prabhakar [21] as:

$$E^{\delta}_{\upsilon,\rho}(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s\, z^s}{\Gamma(\upsilon n+\rho)\ \ s!} \tag{3}$$

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Where $\upsilon, \rho, \delta \in C \;\; (Re\,(\upsilon) > 0)$ and $E^{\delta}{}_{\upsilon,\rho}\,(z)$ is an entire function of order $[Re\,(\upsilon)]^{-1}$, [21]. For various properties other detail of (3), see [14]. The generalized Wright function ${}_p\Psi_q\;(z)$ defined for $z \in C,\;\; a_i,\;\; b_j \in C$ and $\alpha_i,\;\; \beta_j \in R(\alpha_i,\;\; \beta_j \neq 0,\;\; i = 1, 2, \ldots, p,\;\; j = 1, 2, \ldots, q)$ is given by the series

$$ {}_p\Psi_q\;(z) = {}_p\Psi_q\left[\left(\begin{array}{c} (a_i, \alpha_i\;)_{(1.p)} \\ (b_j, \beta_j\;)_{(1.q)} \end{array} \middle| \; z \right)\right] = \sum_{s=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\,(a_i + \alpha_i s)\;\; z^s}{\prod_{j=1}^{q} \Gamma\,(b_j + \beta_j s)\;\, s!} \tag{4} $$

Where C is the set of complex number and $\Gamma\,(z)$ is the Euler gamma function [3] and the function (4) was introduces by Wright [29] and known as generalized Wright function. Condition for the existence of the generalized Wright function (4) together with its representation in terms of Mellins-Barnes integral and in terms of H-function were established in [13]. Some particular cases of generalized Wright function (4) were presented in [13]. Wright in [28, 31] investigated by "steepest descent" method, the asymptotic expansion of the function $\Phi\,(\alpha,\;\beta,\;z)$ for the large value of z in the cases $\alpha > 0$ and $-1 < \alpha < 0$, respectively. In [28] Wright indicated the application of the obtained results to the asymptotic theory of partitions. In [29, 30, 32] Wright extended the last result to the generalized Wright function ${}_p\Psi_q\;(z)$ for all values of the argument z under the condition

$$ \beta_n = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1 \tag{5} $$

For the detailed study of various properties, generalizations and applications of Wright function and generalized Wright function, we refer to papers of Wright [28, 29, 30, 31, 32], Luchko [16] and Kilbas [13].

2. Fractional Calculus Operators and Generalized Fractional Calculus Operators

The left and right-sided Riemann-Liouville fractional calculus operators are defined by Samko, Kilbas and Marichev [23], for $\alpha \in C,\;\; Re\,(\alpha) > 0$

$$ \left(D_{0+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^{[Re(\alpha)]+1} \left[I_{0+}^{1-\alpha+[Re(\alpha)]} f\right](x) \tag{6} $$

$$ = \left(\frac{d}{dx}\right)^{[Re(\alpha)]+1} \frac{1}{\Gamma(1 - \alpha + [Re\,(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha - [Re(\alpha)]}}\, dt; (x > 0) \tag{7} $$

$$ \left(D_{0-}^{\alpha} f\right)(x) = \left(-\frac{d}{dx}\right)^{[Re(\alpha)]+1} \left[I_{-}^{1-\alpha+[Re(\alpha)]} f\right](x) \tag{8} $$

$$ = \left(-\frac{d}{dx}\right)^{[Re(\alpha)]+1} \frac{1}{\Gamma(1 - \alpha + [Re\,(\alpha)])} \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha - [Re(\alpha)]}}\, dt; (x > 0) \tag{9} $$

Where $[Re\,(\alpha)]$ is the integral of Re(α).

An interesting and useful generalization of the Riemann-Liouville and Eardly-Kober fractional derivative operator has been introduced by Saigo [22] in terms of Gauss hypergeometric function as given below. Let $\alpha,\;\beta,\;\gamma \in C$ and $x \in R_+$, then the generalized fractional integration operators associated with Gauss hypergeometric function are defined as follows:

$$ \left(I_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma\,(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2\mathrm{F}_1\left(\alpha+\beta,\;\, -\gamma\;; \alpha;\;\, 1 - \frac{t}{x}\right) f\,(t)\, dt;\;\; (R(\alpha) > 0) \tag{10} $$

$$ \left(I_{-}^{\alpha,\beta,\gamma} f\right)(x) = \frac{1}{\Gamma\,(\alpha)} \int_x^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\beta}\, {}_2\mathrm{F}_1\left(\alpha+\beta,\;\, -\gamma\;; \alpha;\;\, 1 - \frac{t}{x}\right) f\,(t)\, dt;\;\; (R(\alpha) > 0) \tag{11} $$

$$ \left(D_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0+}^{-\alpha-\beta,\alpha+\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^{k} \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} f\right)(x)\,;\;\; (Re\,(\alpha)) > 0; k = [Re\,(\alpha)] + 1 \tag{12} $$

$$ \left(D_{-}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{-\alpha-\beta,\alpha+\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^{k} \left(I_{-\alpha+k,-\beta-k,\alpha+\gamma} f\right)(x)\,;\;\; (Re\,(\alpha) > 0)\,; k = [Re\,(\alpha)] + 1. \tag{13} $$

3. Left-sided Generalized Fractional Differentiation of the Mittag-Leffler Function Using Jacobi Polynomial

In this section we consider the left-sided generalized fractional Differentiation formula of the generalized Mittag-Leffler function using Jacobi Polynomial. We are introducing here the Jacobi Polynomial which is defined via the hypergeometric function as

$$P_n{}^{(\alpha,\ \beta)}(z) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left[-n,\ \alpha+\beta+n+1; \alpha+1;\ \frac{1}{2}\ (1-z)\right] \tag{14}$$

Where $(\alpha+1)_n$ is Pochhammer's symbol, on taking $z=\frac{t}{x}$ and solved. We are replacing the Gauss hypergeometric function by the Jacobi Polynomial in equation (12) it becomes

$$\left(D_{0+}^{\alpha+1,-\alpha-1-n,\ -(\alpha+\beta+n+1)}\frac{(\alpha+1)_n}{n!2^n}f\right)(x) = \frac{(\alpha+1)_n\ 2^{-n}}{n!}\ \frac{x^n}{\Gamma(-\alpha-1+k)}$$
$$\times\int_0^x (x-t)^{-\alpha-1+k-1}{}_2F_1\left[n,\ \beta+n+k; -\alpha-1+k;\ \frac{1}{2}\ \left(1-\frac{t}{x}\right)\right] f(t)\ dt \tag{15}$$

Theorem 3.1. *Let $\alpha,\beta,\gamma,\rho,\delta\in C$ be complex numbers such that $R(\alpha)>0$, $R(\rho+\beta+\gamma)>0$, $\upsilon>0$ and $a\in R$. If the condition (5) satisfied and $D_{0+}^{\alpha,\beta,\gamma}$ by equation (12) be the left-side operator of the generalized fractional Differentiation associated with Jacobi Polynomial, then there holds the following relationship*

$$\left(D_{0+}^{\alpha+1,-\alpha-1-n,-\ \ (\alpha+\beta+n+1)}\left(\frac{(\alpha+1)_n}{n!}2^n t^{\rho-1}E_{\upsilon,\rho}^{\delta}\left[at^{\upsilon}\right]\right)\right)(x)$$
$$= \frac{x^{n-1-\alpha+\rho}(\alpha+1)_n}{\mathrm{n!}\Gamma(\delta)}\ {}_2\psi_2\left\{\begin{matrix}\left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right) & (\delta,1) \\ (\rho-\alpha+\mathrm{n}-1,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{5n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right)\end{matrix}\ \middle|\ ax^{\upsilon}\right\} \tag{16}$$

Provided each member of the equation (16) exists.

Proof. By using the definition of generalized Mittag-Leffler function (3) and fractional Derivative formula (12), we have

$$\Theta = D_{0,+}^{\alpha+1,-\alpha-1-n,-(\alpha+\beta+n+1)}\left(\frac{(\alpha+1)_n}{2^n n!}t^{\rho-1}E_{\upsilon,\rho}^{\delta}\left[at^{\upsilon}\right]\right)(x)$$
$$= \left(\frac{d}{dx}\right)^k\left[\mathrm{I}_{0+}^{-\alpha-1+k,\alpha+1+n-k,-\beta-n-k}\frac{(\alpha+1)_n}{2^n n!}\left[t^{\rho-1}E_{\rho,\nu}^{\delta}\left(at^{\nu}\right)\right]\right](x)$$
$$= \left(\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{2^n n!}\frac{x^n}{\Gamma(-\alpha-1+k)}\times\int_0^{\infty}(x-t)^{-\alpha-1+k-1}{}_2F_1\left[n,\beta+n+k;-\alpha-1+k;\frac{1}{2}\left(1-\frac{t}{x}\right)\right]\left(t^{\rho-1}E_{\upsilon,\rho}^{\delta}\left[at^{\upsilon}\right]\right)dt$$
$$= \left(\frac{d}{dx}\right)^k\frac{x^n(\alpha+1)_n}{2^n n!\Gamma(-\alpha-1+k)}\int_0^{\infty}(x-t)^{-\alpha-1+k-1}\sum_{m=0}^{\infty}\left[\frac{(n)_m(\beta+n+k)_m}{(-\alpha+k-1)_m 2^m m!}\times\left(1-\frac{t}{x}\right)^m t^{\rho-1}\sum_{s=0}^{\infty}\frac{(\delta)_s(at^{\nu})^s}{\Gamma(\rho+\nu s)\,s!}dt\right]$$

Now by changing the order of summation and multiplying and dividing by $x^{\rho+\nu s-1}$ we have

$$= \left(\frac{d}{dx}\right)^k\frac{x^n(\alpha+1)_n}{2^n n!\Gamma(-\alpha-1+k)}\sum_{s=0}^{\infty}\frac{(\delta)_s x^{\rho+\nu s-1-\alpha+k+m-1}}{\Gamma(\rho+\nu s)\,s!}\sum_{m=0}^{\infty}\left[\frac{(n)_m(\beta+n+k)_m}{(-\alpha+k-1)_m 2^m m!}\times\int_0^{\infty}\left(1-\frac{t}{x}\right)^{-\alpha+k+m-1-1}\left(\frac{t}{x}\right)^{\rho+\nu s-1}dt\right]$$

Let us assume that $\frac{t}{x}=u$ so $dt=xdu$ and limit varies from 0 to 1 we have

$$= \left(\frac{d}{dx}\right)^k\frac{x^{n+\rho+k-\alpha-1}(\alpha+1)_n}{2^n n!\Gamma(-\alpha+k-1)}\sum_{s=0}^{\infty}\frac{(\delta)_s(ax^{\nu})^s}{\Gamma(\rho+\nu s)\,s!}\sum_{m=0}^{\infty}\left[\frac{(n)_m(\beta+n+k)_m}{(-\alpha+k-1)_m 2^m m!}\ \times\int_0^1(1-u)^{-\alpha+k+m-1-1}u^{\rho+\nu s-1}du\right]$$

Now by using beta functions

$$\mathrm{B}(m,n) = \int_0^1(1-x)^{m-1}(x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

we have

$$= \left(\frac{d}{dx}\right)^k \frac{x^{n-\alpha+k-1+\rho}(\alpha+1)_n}{2^n n\Gamma(-\alpha+k-1)} \sum_{s=0}^{\infty} \frac{(\delta)_s (ax^\nu)^s}{\Gamma(\rho+\nu s)\, s!} \sum_{m=0}^{\infty} \left[\frac{(n)_m(\beta+n+k)_m}{(-\alpha+k-1)_m 2^m m!} \times \frac{\Gamma(-\alpha+k+m-1)\Gamma(\rho+\nu s)}{\Gamma(-\alpha+k-1+m+\rho+\nu s)}\right]$$

Multiplying and dividing by $\Gamma(\rho+\nu s-\alpha+k-1)$ we have

$$= \left(\frac{d}{dx}\right)^k \frac{x^{n-\alpha+k-1+\rho}(\alpha+1)_n}{2^n \mathrm{n}!} \times \sum_{s=0}^{\infty} \frac{(\delta)_s (ax^\nu)^s}{\Gamma(\rho+\nu s)\Gamma(\rho+\nu s-\alpha+k-1)} {}_2F_1\left(n, \beta+n+k; \rho+\nu s-\alpha+k-1; \frac{1}{2}\right)$$

Now by using Kummer's special case

$${}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{2}\right) = \frac{2^\alpha \Gamma(\gamma)\Gamma\left(1+\frac{\gamma-\beta}{2}\right)}{\Gamma(\gamma+\alpha)\Gamma\left(1+\frac{\gamma-\beta}{2}-\alpha\right)}$$

We have

$$= \left(\frac{d}{dx}\right)^k \frac{(\alpha+1)_n}{n!} \sum_{s=0}^{\infty} \left[\frac{(\delta)_s a^s x^{n-\alpha+k-1+\rho+\nu s}}{\Gamma(\rho+\nu s)\, s!\Gamma(\rho+\nu s-\alpha+k-1)} \times \frac{\Gamma(\rho+\nu s-\alpha+k-1)\Gamma\left(1+\frac{\rho+\nu s-\alpha-1-\beta-n}{2}\right)}{\Gamma(\rho+\nu s-\alpha+k-1+n)\Gamma\left(1+\frac{\rho+\nu s-\alpha-1-\beta-n}{2}-2n\right)}\right]$$

By the use of Jacobi Polynomial $P_n{}^{(\alpha,\ \beta)}(z)$ (14), series form of generalized Mittag-Leffler function (3), interchanging the order of integration and summation and evaluating the inner integral by the use of known formula of Beta integral. Finally by the virtue of Gauss summation theorem, we have,

$$\Theta = \frac{x^{\rho+n-\alpha-1}(a+1)_n}{n!G(\delta)} \sum_{s=0}^{\infty} \left[\frac{G(\delta+s)G\left(\frac{\upsilon}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2}+\frac{\upsilon}{2}s\right)}{G(\upsilon-\alpha-1+n+\upsilon s)G\left(\frac{\upsilon}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{5n}{2}+\frac{1}{2}+\frac{\upsilon}{2}s\right)} \frac{(ax^\upsilon)^s}{s!}\right]$$

or

$$\Theta = \frac{x^{\rho+n-\alpha-1}(\alpha+1)_n}{n!\Gamma(\delta)} {}_2\psi_2 \left[\left(\begin{array}{cc|c} \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right) & (\delta,1) & \\ (\rho-\alpha+n-1,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{5n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right) & ax^\upsilon \end{array}\right)\right]$$

Interchanging the order of integration and summations, which is permissible under the conditions, stated with the theorem due to convergence of the integrals involved in the process. This completes the proof of the theorem. □

Corollary 3.2. *For $Re(\alpha) > 0$, $Re(\rho+\beta+\gamma) > 0$, $\nu > 0$ and $a \in R$. If the condition (5) is satisfied, then there holds the formula*

$$\left(D_{0+}^{\alpha+1,-\alpha-1-n,(\alpha+\beta+n+1)}\left(\frac{(\alpha+1)_n}{n!2^n}t^{\rho-1}E_{\upsilon,\rho}\left[at^\upsilon\right]\right)\right)(x)$$
$$= \frac{x^{\rho+n-\alpha-1}(\alpha+1)_n}{n!} {}_2\psi_2 \left\{\begin{array}{cc|c} \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right) & (1,1) & \\ (\rho-\alpha+n-1,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{5n}{2}+\frac{1}{2},\frac{\upsilon}{2}\right) & ax^\upsilon \end{array}\right\} \quad (17)$$

Provided that each member of equation (17) makes sense.

4. Right-sided Generalized Fractional Differentiation of the Mittag-Leffler Function Using Jacobi Polynomial

In this section we have discussed the right-sided generalized fractional integral formula of the generalized Mittag-Leffler function using Jacobi Polynomial.

Theorem 4.1. *Let $\alpha, \beta, \gamma, \rho, \delta \in C$ be complex numbers such that $Re(\alpha) > 0$, $Re(\rho) > \max[Re(\alpha+\beta)+k, -Re(\gamma)]$, $\nu > 0$ and $a\ \epsilon R$ with the condition $Re(\alpha+\beta+\gamma)+k \neq 0$ where $k = ([Re(\alpha)]+1)$. If the condition (5) satisfied and $D_{0-}^{\alpha,\beta,\gamma}$ be the Right-side operator of the generalized fractional integration associated with Jacobi Polynomial, then there holds the following relationship*

$$\left(D_{0,-}^{\alpha+1,-\alpha-1-n,-(\alpha+\beta+n+1)}\left(\frac{(\alpha+1)_n}{n!2^n}t^{\alpha+1-\rho}E_{\upsilon,\rho}^{\delta}\left[at^{-\upsilon}\right]\right)\right)(x)$$
$$= \frac{(\alpha+1)_n}{n!}\frac{x^{-n-\rho}}{\Gamma(\delta)}\,{}_3\Psi_3\left[\left(\begin{array}{ccc|c} \left(\frac{\rho}{2}-\frac{\beta}{2}-\frac{\alpha}{2}-\frac{1}{2},\frac{\upsilon}{2}\right) & (\rho+n,\upsilon) & (\delta,1) & \\ (\rho+2n-\alpha-1,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-1,\frac{\upsilon}{2}\right) & (\rho,\upsilon) & ax^{-\upsilon}\end{array}\right)\right] \tag{18}$$

Provided both the sides of (18) exists.

Proof. By using the definition of generalized Mittag-Leffler function (3) and generalized fractional derivative formula (13) and integral formula (11), we have

$$\Lambda = (D_{0,-}^{\alpha+1,-\alpha-1-n,\ -(\alpha+\beta+n+1)}\left(\frac{(\alpha+1)_n}{n!}2^{-n}t^{\alpha+1-\rho}E_{\upsilon,\rho}^{\delta}\left[at^{-\upsilon}\right]\right)(x))$$
$$= \left(-\frac{d}{dx}\right)^k\left(I_{0,-}^{-\alpha-1+k,\alpha+1+n-k,-\beta-n}\frac{(\alpha+1)_n}{2^n n!}t^{\alpha+1-\rho}E_{\rho,\nu}^{\delta}\left(at^{-\nu}\right)\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{n!\Gamma(-\alpha-1+k)}2^{-n}\int_x^\infty(t-x)^{-\alpha-1+k-1}t^{-n}\times{}_2F_1\left[n,\beta+n;-\alpha-1+k;\frac{1}{2}\left(1-\frac{t}{x}\right)\right]\left(t^{\alpha-\rho+1}E_{\upsilon,\rho}^{\delta}\left[at^{-\upsilon}\right]\right)dt$$

Multiplying and dividing by $x^{\nu s}$ by changing the order of summation, we have

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{\Gamma(-\alpha-1+k)(2)^n n!}$$
$$\sum_{s=0}^\infty\frac{(\delta)_s\left(ax^{-\nu}\right)^s}{\Gamma(\rho+\nu s)s!}\sum_{m=0}^\infty\left[\frac{(n)_m(\beta+n)_m}{(-\alpha-1+k)_m m!2^m}\times\int_x^\infty\left(1-\frac{x}{t}\right)^{-\alpha-1+k-1+m}t^{-\alpha-1+k-1-n+\alpha+1-\rho-\nu s}x^{\nu s}dt\right]$$

Multiplying and dividing by $x^{-n-\rho+k}$, we have

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{n!2^n\Gamma(-\alpha-1+k)}$$
$$\sum_{s=0}^\infty\frac{(\delta)_s\left(ax^{-\upsilon}\right)^s}{\Gamma(\rho+\nu s)s!}\sum_{m=0}^\infty\left[\frac{(n)_m(\beta+n)_m x^{-n-\rho+k}}{(-\alpha-1+k)_m 2^m m!}\times\int_x^\infty\left(1-\frac{x}{t}\right)^{-\alpha-1+k+m-1}\left(\frac{x}{t}\right)^{\rho+\nu s+n-k}\left(\frac{dt}{t}\right)\right]$$

Let us assume that $\frac{x}{t} = u$ then $0 = udt + tdu$ then limit varies from 1 to 0

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{2^{-n}n!}\sum_{s=0}^\infty\frac{(\delta)_s\left(ax^{-\nu}\right)^s}{\Gamma(\rho+\nu s)s!}\sum_{m=0}^\infty\left[\frac{(n)_m(\beta+n)_m}{(-\alpha-1+k)_m 2^m m!}\times\int_0^1(1-u)^{-\alpha-1+k+m-1}u^{\rho+\nu s+n-k-1}du\right]$$

By using beta function we have

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n x^{-n-\rho+k}}{2^n n!\Gamma(-\alpha-1+k)}\sum_{s=0}^\infty\frac{(\delta)_s\left(ax^{-\nu}\right)^s}{\Gamma(\rho+\nu s)s!}\sum_{m=0}^\infty\left[\frac{(n)_m(\beta+n)_m}{(-\alpha-1+k)_m 2^m m!}\times\frac{\Gamma(-\alpha-1+k+m)\Gamma(\rho+\nu s+n-k)}{\Gamma(-\alpha-1+k+m+\rho+\nu s+n-k)}\right]$$

Now multiplying and dividing by $\Gamma(-\alpha-1+\rho+\nu s+n)$

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{2^n n!}\sum_{s=0}^\infty\left[\frac{(\delta)_s(a)^s\Gamma(\rho+\nu s+n-k)x^{-\rho-\nu s-n+k}}{\Gamma(\rho+\nu s)\Gamma(-\alpha-1+\rho+\nu s+n)s!}\times{}_2F_1\left(n,\beta+n;n-\alpha-1+\rho+\nu s;\frac{1}{2}\right)\right]$$

By using Kummer's special rule, we have

$$= \left(-\frac{d}{dx}\right)^k\frac{(\alpha+1)_n}{2^n n!}\sum_{s=0}^\infty\left[\frac{(\delta)_s a^s x^{-\rho-\nu s-n+k}\Gamma(\rho+\nu s+n-k)}{\Gamma(\rho+\nu s)\Gamma(-\alpha-1+\rho+\nu s+n)}\times 2^n\frac{\Gamma(-\alpha-1+\rho+\nu s+n)\Gamma\left(\frac{\rho+\nu s-\beta-\alpha+1}{2}\right)}{\Gamma(2n-\alpha-1+\rho+\nu s)\Gamma\left(1+\frac{\rho+\nu s-\alpha-\beta-1}{2}-n\right)}\right]$$

$$= \left(-\frac{d}{dx}\right)^k \frac{(\alpha+1)_n}{n} \sum_{s=0}^{\infty} \left[\frac{(\delta)_s a^s x^{-\rho-\nu s-n+k} \Gamma\left(\frac{1-\alpha-\beta+\rho+\nu s}{2}\right) \Gamma(\rho+n-k+\nu s)}{\Gamma(\rho+\nu s)\Gamma(2n-\alpha-1+\rho+\nu s)\Gamma\left(1+\frac{\rho-\alpha-\beta-1-2n+\nu s}{2}\right)} \right]$$

$$= \frac{(\alpha+1)_n}{n!} \frac{x^{-n-\rho}}{\Gamma(\delta)} \sum_{s=0}^{\infty} \left[\frac{\Gamma\left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}+\frac{\upsilon s}{2}\right)\Gamma((\rho+n+\upsilon s))\Gamma(\delta+s)}{\Gamma(\rho+2n-\alpha-1+\upsilon s)\Gamma\left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-n+\frac{\upsilon s}{2}\right)\Gamma(\rho+\upsilon s)} \frac{(ax^{-\upsilon})^s}{s!} \right]$$

$$= \frac{(\alpha+1)_n}{n!} \frac{x^{-n-\rho}}{\Gamma(\delta)} {}_3\Psi_3 \left[\left(\begin{array}{ccc|c} \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}, \frac{\upsilon}{2}\right) & (\rho+n,\upsilon) & (\delta,1) & \\ (2n-\alpha+\rho-1,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-n, \frac{\upsilon}{2}\right) & (\rho,\upsilon) & ax^{-\upsilon} \end{array} \right) \right] \tag{19}$$

Provided each member of the equation (19) makes sense. □

Corollary 4.2. *For $Re(\alpha+1) > 0$, $Re(\rho+\beta+n) > 0$, $\vartheta > 0$ and $a \in R$. If the condition (5) is satisfied, then there holds the formula*

$$\left(D_{0,-}^{\alpha+1,-\alpha-1-n,-(\alpha+\beta+n+1)} \left(\frac{(\alpha+1)_n}{n!} 2^{-n} t^{\alpha+1-\rho} E_{\upsilon,\rho}\left[at^{-\upsilon}\right]\right)\right)(x)$$

$$= \frac{(\alpha+1)_n}{n!} x^{-n-\rho} {}_3\Psi_3 \left[\left(\begin{array}{ccc|c} \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}, \frac{\upsilon}{2}\right) & (\rho+n,\upsilon) & (1,1) & \\ (2\mathrm{n}-\alpha-1+\rho,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-n, \frac{\upsilon}{2}\right) & (\rho,\upsilon) & ax^{-\upsilon} \end{array} \right) \right]$$

$$= \frac{(\alpha+1)_n}{n!} x^{-n-\rho} {}_3\Psi_3 \left[\left(\begin{array}{ccc|c} \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}, \frac{\upsilon}{2}\right) & (\rho+n,\upsilon) & (1,1) & \\ (2\mathrm{n}-\alpha-1+\rho,\upsilon) & \left(\frac{\rho}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-n, \frac{\upsilon}{2}\right) & (\rho,\upsilon) & ax^{-\upsilon} \end{array} \right) \right] \tag{20}$$

provided that each member of equation (20) makes sense.

5. Concluding Remarks

We conclude this paper with the remarks that the results proved in this paper are new and most likely to find some applications to the solutions of certain fractional differential and integral equations. So that our paper is concluded that, the function introduced and reported results are significant and can lead to yield number of other differential formulas involving various Mittag-Leffler type functions using Jacobi polynomial.

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