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Some Properties of $sg\alpha$-continuous Functions

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Abstract: In [7] the authors, introduced the notion of $sg\alpha$-continuity and investigated its fundamental properties. In this paper, we investigate some more properties of this type of continuity.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, seperation axioms etc. by utiliaing generalized open sets (See [1–3]). One of the most well known notions and also an inspiration source is the notion of α-open [5] sets introduced by Njastad in 1965. Quite recently, as generalization of closed sets called $sg\alpha$-closed sets were introduced and studied by the present authors in [6]. In [7] the authors, introduced the notion of $sg\alpha$-continuity and investigated its fundamental properties. In this paper, we investigate some more properties of this type of continuity.

2. Preliminaries

Throughout this paper (X,τ) and (Y,σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively.

Definition 2.1. *A subset A of a space X is called semi-open [4] (respectively α-open [5]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (respectively $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). The complement of α-open set is called α-closed.*

The α-closure of a subset A of X, denoted by $\alpha\,\mathrm{Cl}(A)$ is defined to be the intersection of all α-closed sets containing A in X.

Definition 2.2. *A subset A of a space X is called $sg\alpha$-closed [6] if $\alpha\,\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in X. The complement of $sg\alpha$-closed set is called $sg\alpha$-open. The family of all $sg\alpha$-open subsets of (X,τ) is denoted by $sg\alpha O(X)$.*

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The family of all $sg\alpha$-open (respectively $sg\alpha$-closed) sets of X is denoted by $sg\alpha(\tau)$ (respectively $sg\alpha C(X)$). We set $sg\alpha O(X,x) = \{U | U \in sg\alpha(\tau) \text{ and } x \in U\}$. In [6] shown that the set $sg\alpha(\tau)$ forms a topology, which is finer than τ.

Definition 2.3. *The intersection of all $sg\alpha$-closed sets containing A is called the $sg\alpha$-closure [6] of A and is denoted by $sg\alpha$-$\mathrm{Cl}(A)$. A set A is $sg\alpha$-closed if and only if $sg\alpha$-$\mathrm{Cl}(A) = A$ [6].*

3. Properties of $sg\alpha$-continuous Functions

Definition 3.1. *A function $f : (X,\tau) \to (Y,\sigma)$ is called :*

(1). $sg\alpha$-continuous [7] at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists a $U \in sg\alpha(X,x)$ such that $f(U) \subset V$;

(2). $sg\alpha$-continuous [7] if it has this property at each point of X.

Theorem 3.2 ([7]). *The following statements are equivalent for a function $f : (X,\tau) \to (Y,\sigma)$:*

(1). f is $sg\alpha$-continuous;

(2). $f : (X, sg\alpha(\tau)) \to (Y,\sigma)$ is continuous;

(3). for every open set V of Y, $f^{-1}(V)$ is $sg\alpha$-open in X;

(4). for every closed set V of Y, $f^{-1}(V)$ is $sg\alpha$-closed in X.

Lemma 3.3 ([6]). *Let $A \subset B \subset X$, A be a $sg\alpha$-open set in B and B an open subset of (X,τ), then $A \in sg\alpha(\tau)$.*

Theorem 3.4. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and $\Lambda = \{U_i : i \in I\}$ be a cover of X such that $U_i \in sg\alpha(\tau)$ for each $i \in I$. If $f|_{U_i}$ is continuous for each $i \in I$, then f is $sg\alpha$-continuous.*

Proof. Suppose that V is any open subset of (Y,σ). Since $f|_{U_i}$ is $sg\alpha$-continuous for each $i \in I$, it follows that $(f|_{U_i})^{-1}(V)$ is open in U_i. We have $f^{-1}(V) = \bigcup_{i\in I} (f^{-1}(V) \cap U_i) = \bigcup_{i\in I} (f|_{U_i})^{-1}(V)$. Then by Lemma 3.3, we obtain $f^{-1}(V) \in sg\alpha(\tau)$, which means that f is $sg\alpha$-continuous. □

Theorem 3.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and $x \in X$. If there exists an open set U of X such that $x \in U$, and the restriction of f to U is $sg\alpha$-continuous at x, then f is $sg\alpha$-continuous at x.*

Proof. Suppose that F is an open subset of (Y,σ) containing $f(x)$. Since $f|_U$ is $sg\alpha$-continuous at x, there exists a $sg\alpha$-open set V of U containing x such that $f(V) = (f|_U)\ (V) \subset F$. Since U is open in X containing x, it follows from Lemma 3.3 that $V \in sg\alpha(\tau)$ containing x. Thus, f is $sg\alpha$-continuous at x. □

Definition 3.6. *Let and let be a net in . We say that x is a -limit of and we write if for every-neighbourhood A of x in X there exists a such that for all .*

Theorem 3.7. *${}^\star sg\alpha$-continuous is identical with the union of the $sg\alpha$-frontiers of the inverse images of $sg\alpha$-open sets containing $f(x)$.*

Proof. Suppose that f is not $sg\alpha$-continuous at a point x of X. Then there exists an open set V of Y containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \subset sg\alpha(\tau)$ containing x. Hence, we have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$ for every $sg\alpha$-open set U containing x. It follows that $x \in sg\alpha\,\mathrm{Cl}(X \backslash f^{-1}(V))$. We also have $x \in f^{-1}(V) \subset sg\alpha\,\mathrm{Cl}(f^{-1}(V))$. This means that $x \in sg\alpha Fr(f^{-1}(V))$. Now, let f be $sg\alpha$-continuous at $x \in X$ and V an open subset of Y containing $f(x)$. Then $x \in f^{-1}(V)$ is a $sg\alpha$-open set of X. Thus, $x \in sg\alpha\,\mathrm{Int}(f^{-1}(V))$ and therefore $x \notin sg\alpha Fr(f^{-1}(V))$ for every open set V containing $f(x)$. □

Definition 3.8.

(1). A filter base Λ is said to be $sg\alpha$-convergent to a point x in X if for any $U \in sg\alpha(\tau)$ containing x, there exists $B \in \Lambda$ such that $B \subset U$.

(2). A filter base Λ is said to be convergent to a point x in X if for any open set U of X containing x, there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.9. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-continuous, then for each point $x \in X$ and each filter base Λ in X $sg\alpha$-converging to x, the filter base $f(\Lambda)$ is convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X $sg\alpha$-converging to x. Since f is $sg\alpha$-continuous, then for any open set V of (Y, σ) containing $f(x)$, there exists $U \in sg\alpha O(X, x)$ such that $f(U) \subset V$. Since Λ is $sg\alpha$-converging to x, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\Lambda)$ is convergent to $f(x)$. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 3.10. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra $sg\alpha$-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exists $U \in sg\alpha O(X, x)$ and a closed set V of Y containing y such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 3.11. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is contra $sg\alpha$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in sg\alpha(\tau)$ containing x and a closed set V of Y containing y such that $f(U) \cap V = \varnothing$.*

Theorem 3.12. *If $f : (X, \tau) \to (Y, \sigma)$ is a $sg\alpha$-continuous function and (Y, σ) is a T_1-space, then $G(f)$ is contra $sg\alpha$-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is T_1, there exists an open set V in Y such that $f(x) \in V$ and $y \notin V$. Since f is $sg\alpha$-continuous, there exists $U \in sg\alpha O(X, x)$ such that $f(U) \subset V$. Therefore, $f(U) \cap (Y\backslash V) = \varnothing$ and $Y\backslash V$ is a closed subset of Y containing y. This shows that $G(f)$ is contra $sg\alpha$-closed. □

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two families of topological spaces with the same index set Λ. The product space of $\{X_\alpha : \alpha \in \Lambda\}$ is denoted by $\Pi\ \{X_\alpha{:}\alpha \in \Lambda\}$ (or simply ΠX_α). Let f_α: $X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product function f: $\Pi X_\alpha \to \Pi Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

Theorem 3.13. *If a function $f : X \to \Pi Y_\alpha$ is $sg\alpha$-continuous, then $P_\alpha \circ f$: $X \to Y_\alpha$ is $sg\alpha$-continuous for each $\alpha \in \Lambda$, where P_α is the projection of ΠY_α onto Y_α.*

Proof. Let V_α be any open set of Y_α. Then, $P_\alpha^{-1}(V_\alpha)$ is open in ΠY_α and hence $(P_\alpha \circ f)^{-1}\ (V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is $sg\alpha$-open in X. Therefore, $P_\alpha \circ f$ is $sg\alpha$-continuous. □

Theorem 3.14. *If a function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is $sg\alpha$-continuous, then f_α: $X_\alpha \to Y_\alpha$ is $sg\alpha$-continuous for each $\alpha \in \Lambda$.*

Proof. Let V_α be any open set of Y_α. Then $P_\alpha^{-1}(V_\alpha)$ is open in ΠY_α and $f^{-1}(P_\alpha^{-1}(V_\alpha)) = f_\alpha^{-1}(V_\alpha) \times \Pi\{X_\alpha\!: \alpha \in \Lambda \setminus \{\alpha\}\}$. Since f is $sg\alpha$-continuous, $f^{-1}(P_\alpha^{-1}(V_\alpha))$ is $sg\alpha$-open in ΠX_α. Since the projection P_α of ΠX_α onto X_α is open continuous, $f_\alpha^{-1}(V_\alpha)$ is $sg\alpha$-open in X_α and hence f_α is $sg\alpha$-continuous. □

Now, we recall the following definitions.

Definition 3.15. *A space (X, τ) is said to be*

(1). $sg\alpha$-compact [7] if every $sg\alpha$-open cover of X has a finite subcover;

(2). $sg\alpha$-compact relative to X if every cover of A by $sg\alpha$-open sets of X has a finite subcover.

Theorem 3.16. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-continuous and A is $sg\alpha$-compact relative to X, then $f(A)$ is compact in Y.*

Proof. Let $\{H_\alpha : \alpha \in I\}$ be any cover of $f(A)$ by open sets of the subspace $f(A)$. For each $\alpha \in I$, there exists a open set A_α of Y such that $H_\alpha = K_\alpha \cap f(A)$. For each $x \in A$, there exists $\alpha_x \in I$ such that $f(x) \in A_{\alpha_x}$ and there exists $U_x \in sg\alpha(\tau)$ containing x such that $f(U_x) \subset A_{\alpha_x}$. Since the family $\{U_x : x \in K\}$ is a cover of A by $sg\alpha$-open sets of K, there exists a finite subset A_0 of A such that $A \subset \{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subset \bigcup\{f(U_x) : x \in A_0\}$ which is a subset of $\bigcup\{A_{\alpha_x} : x \in A_0\}$. Thus, $f(A) = \bigcup\{A_{\alpha_x} : x \in A_0\}$ and hence $f(A)$ is compact. □

Definition 3.17. *A space (X, τ) is said to be:*

(1). coutably $sg\alpha$-compact if every $sg\alpha$-open countably cover of X has a finite subcover;

(2). $sg\alpha$-Lindelof if every $sg\alpha$-open cover of X has a countable subcover;

(3). $sg\alpha$-closed compact if every $sg\alpha$-closed cover of X has a finite subcover;

(4). countably $sg\alpha$-closed compact if every countably cover of X by $sg\alpha$-closed sets has a finite subcover.

Theorem 3.18. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $sg\alpha$-continuous surjective function. Then the following statements hold:*

(1). If X is $sg\alpha$-Lindelof, then Y is Lindelof;

(2). If X is countably $sg\alpha$-compact, then Y is countably compact.

Proof.

(1). Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y. Since f is $sg\alpha$-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a $sg\alpha$-open cover of X. Since X is $sg\alpha$-Lindelof, there exists a countable subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and hence Y is Lindelof.

(2). Similar to (1). □

Theorem 3.19. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $sg\alpha$-continuous surjective function. Then the following statements hold:*

(1). If X is $sg\alpha$-closed compact, then Y is compact;

(2). If X is $sg\alpha$-closed Lindelof, then Y is Lindelof;

(3). If X is countably $sg\alpha$-closed compact, then Y is countably compact.

Proof. The proof is similar to Theorem 3.18. □

4. Separation Axioms

Definition 4.1. *A space (X, τ) is said to be:*

(1). $sg\alpha$-T_1 [11] if for each pair of distinct points x and y of X, there exist $sg\alpha$-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

(2). $sg\alpha$-T_2 [11] if for each pair of distinct points x and y in X, there exist disjoint $sg\alpha$-open sets U and V in X such that $x \in U$ and $y \in V$.

Recall, that a subset B_x of a topological space (X, τ) is said to be a $sg\alpha$-neighbourhood of a point $x \in X$ [11] if there exists a $sg\alpha$-open set U such that $x \in U \subset B_x$.

Theorem 4.2. *If an injective function $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-continuous and Y is a T_1-space, then X is a $sg\alpha$-T_1-space.*

Proof. Suppose that Y is T_1. For any distict points x and y in X, there exist open sets V and W such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is $sg\alpha$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $sg\alpha$-open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is $sg\alpha$-T_1. □

Theorem 4.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a $sg\alpha$-continuous injective function and (Y, σ) is a T_2-space, then (X, τ) is $sg\alpha$-T_2-space.*

Proof. For any pair of distinct points x and y in X, there exist disjoint open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is $sg\alpha$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $sg\alpha$-open in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \varnothing$ because $U \cap V = \varnothing$. This shows that X is $sg\alpha$-T_2. □

Lemma 4.4 ([6])**.** *The intersection of an open and $sg\alpha$-open subset of (X, τ) is $sg\alpha$-open in (X, τ).*

Theorem 4.5. *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous function and $g : (X, \tau) \to (Y, \sigma)$ is a $sg\alpha$-continuous function and Y is a T_2-space, then the set $E = \{x \in X : f(x) = g(x)\}$ is $sg\alpha$-closed set in X.*

Proof. If $x \in E^c$, then it follows that $f(x) \neq g(x)$. Since Y is T_2, there exist disjoint open sets V and W of Y such that $f(x) \in V$ and $g(x) \in W$. Since f is continuous and g is $sg\alpha$-continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is $sg\alpha$-open in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Put $A = f^{-1}(V) \cap g^{-1}(W)$. By Lemma 4.4, A is $sg\alpha$-open in X. Therefore, $f(A) \cap g(A) = \varnothing$ and it follows that $x \notin sg\alpha\text{-Cl}(E)$. This shows that E is $sg\alpha$-closed in X. □

Definition 4.6. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly $sg\alpha$-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in sg\alpha O(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 4.7. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is strongly $sg\alpha$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in sg\alpha O(X, x)$ and an open set V of Y containing y such that $f(U) \cap V = \varnothing$.*

Theorem 4.8. *If $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-continuous and (Y, σ) is Hausdorff, then $G(f)$ is strongly $sg\alpha$-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $f(x) \neq y$. Since Y is Hausdorff, there exist open sets V and W in Y containing $f(x)$ and y, respectively, such that $V \cap W = \varnothing$. Since f is $sg\alpha$-continuous, there exists $U \in sg\alpha O(X, x)$ such that $f(U) \subset V$. Therefore, $f(U) \cap W = \varnothing$ and then by Lemma 4.7, $G(f)$ is strongly $sg\alpha$-closed in $X \times Y$. □

Theorem 4.9. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective with the strongly $sg\alpha$-closed graph, then (X, τ) is $sg\alpha$-T_1.*

Proof. Suppose that x and y are two distinct points of X. Then $f(x) \neq f(y)$. Hence there exist a $sg\alpha$-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \varnothing$. Hence $y \notin U$. This implies that (X, τ) is $sg\alpha$-T_1. □

Theorem 4.10. *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective function with the strongly $sg\alpha$-closed graph, then (Y, σ) is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists a point x in X such that $f(x) = y_2$. Hence $(x, y_1) \notin G(f)$. Then by Lemma 4.7, there exist a $sg\alpha$-open set U and an open set V containing x and y_1, respectively, such that $f(U) \cap V = \varnothing$. Hence $y_2 \notin V$. This means that (Y, σ) is T_1. □

Definition 4.11. *A function $f : (X, \tau) \to (Y, \sigma)$ has a ultra $sg\alpha$-closed graph if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in sg\alpha O(X, x)$, $V \in sg\alpha O(Y, y)$ such that $(U \times sg\alpha\text{-Cl}(V)) \cap G(f) = \varnothing$.*

Lemma 4.12. *A function $f : (X, \tau) \to (Y, \sigma)$ has a ultra $sg\alpha$-closed graph if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in sg\alpha O(X, x)$, $V \in sg\alpha O(Y, y)$ such that $f(U) \cap sg\alpha\text{-Cl}(V) = \varnothing$.*

Proof. Follows from Definition 4.11. □

Definition 4.13. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $sg\alpha$-irresolute if $f^{-1}(V) \in sg\alpha(\tau)$ for each $V \in sg\alpha(\sigma)$.*

Theorem 4.14. *If $f : (X, \tau) \to (Y, \sigma)$ is a $sg\alpha$-irresolute function and (Y, σ) is a $sg\alpha$-T_2 space, then $G(f)$ is ultra $sg\alpha$-closed.*

Proof. Similar proof of Theorem 3.12. □

Theorem 4.15. *If $f : (X, \tau) \to (Y, \sigma)$ is surjective and has a ultra $sg\alpha$-closed graph $G(f)$, then (Y, σ) is both $sg\alpha$-T_2 and $sg\alpha$-T_1 space.*

Proof. Let y_1, y_2 $(y_1 \neq y_2) \in Y$. The surjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, x_2) \in (X \times Y) \setminus G(f)$. The ultra $sg\alpha$-closedness of $G(f)$ provides $U \in sg\alpha O(X, x_1)$, $V \in sg\alpha O(Y, y_2)$ such that $f(U) \cap sg\alpha\text{-Cl}(V) = \varnothing$. Whence one infers that $y_1 \notin sg\alpha\text{-Cl}(V)$. This means that there exists $W \in sg\alpha O(Y, y_1)$ such that $W \cap V = \varnothing$. So, Y is $sg\alpha$-T_2 and hence $sg\alpha$-T_1. □

Theorem 4.16. *A space (X, τ) is $sg\alpha$-T_2 if and only if the identity function $i : X \to X$ has a ultra $sg\alpha$-closed graph.*

Proof. Necessity. Let (X, τ) be $sg\alpha$-T_2. Since the identity function $i : X \to X$ is $sg\alpha$-irresolute, it follows from Theorem 4.14 that $G(i)$ is ultra $sg\alpha$-closed. Sufficiency: Let $G(i)$ be ultra $sg\alpha$-closed. Then the surjectivity of i and ultra $sg\alpha$-closedness of $G(i)$ together imply, by Theorem 4.15, that X is $sg\alpha$-T_2. □

Theorem 4.17. *If $f : (X, \tau) \to (Y, \sigma)$ is an injection and $G(f)$ is ultra $sg\alpha$-closed, then (X, τ) is a $sg\alpha$-T_1 space.*

Proof. Since f is injective, for any pair of distinct point x_1, $x_2 \in X$, $f(x_1) \neq f(x_2)$. Then $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is ultra $sg\alpha$-closed there exist $U \in sg\alpha O(X, x_1)$, $V \in sg\alpha O(Y, f(x_2))$ such that $f(U) \cap sg\alpha\text{-Cl}(V) = \varnothing$. Therefore $x_2 \notin U$. Similarly we can obtain a set $W \in sg\alpha O(X, x_2)$ such that $x_1 \notin W$. Hence (X, τ) is $sg\alpha$-T_1. □

Theorem 4.18. *If $f : (X, \tau) \to (Y, \sigma)$ is a bijective function with ultra $sg\alpha$-closed graph, then both (X, τ) and (Y, σ) are $sg\alpha$-T_1 space.*

Proof. The proof is an immediate consequence of Theorem 4.15 and 4.17. □

Definition 4.19. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly $sg\alpha$-irresolute [8] if for each point $x \in X$ and each $V \in sg\alpha O(Y, f(x))$, there exists $U \in sg\alpha O(X, x)$ such that $f(U) \subset sg\alpha\text{-Cl}(V)$.*

Theorem 4.20. *If a weakly $sg\alpha$-irresolute function $f : (X, \tau) \to (Y, \sigma)$ is an injection with ultra $sg\alpha$-closed graph $G(f)$, then (X, τ) is $sg\alpha$-T_2.*

Proof. Since f is injective for any pair of distinct points x_1 and $x_2 \in X$, $f(x_1) \neq f(x_2)$. Therefore, $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. The $sg\alpha$-closedness of $G(f)$ gives $U \in sg\alpha O(X, x_1)$, $V \in sg\alpha O(Y, f(x_2))$ such that $f(U) \cap sg\alpha\text{-Cl}(V) = \varnothing$, where one obtains $U \cap f^{-1}(g\,\text{Cl}(V)) = \varnothing$. Consequently, $f^{-1}(sg\alpha\text{-Cl}(V)) \subset X \setminus U$. Since f is weakly $sg\alpha$-irresolute, it is so at x_2. Then there exists $W \in sg\alpha O(X, x_2)$ such that $f(W) \subset sg\alpha\text{-Cl}(V)$. It follows that $W \subset f^{-1}(sg\alpha\text{-Cl}(V)) \subset X \setminus U$. Whence one infer that $W \cap U = \varnothing$. Thus, for any pair of distinct points x_1, x_2 there exist $U \in sg\alpha O(X, x_1)$, $V \in sg\alpha O(X, x_2)$ such that $W \cap V = \varnothing$. This shows that (X, τ) is $sg\alpha$-T_2. □

5. $sg\alpha$-Quotient Functions

We introduce the following definition

Definition 5.1. *A surjective function $f : (X, \tau) \to (Y, \sigma)$ is said to be a $sg\alpha$-quotient function if f is $sg\alpha$-continuous and $f^{-1}(V)$ is open in (X, τ) implies V is a $sg\alpha$-open set in (Y, σ).*

Proposition 5.2. *Every quotient function is $sg\alpha$-quotient function.*

Proof. Follows from the definitions. □

The following example shows that $sg\alpha$-quotient function need not be a quotient function in general.

Example 5.3. *Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{a\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = f(d) = c$. Then f is a $sg\alpha$-quotient function but not a quotient function.*

Theorem 5.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-quotient function if and only if $(X, \tau) \to (Y, sg\alpha(\tau))$ is quotient function.*

Definition 5.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $sg\alpha$-open [9] if $f(U) \in sg\alpha(\sigma)$ for each $U \in \tau$.*

Proposition 5.6. *If a function $f : (X, \tau) \to (Y, \sigma)$ is surjective, $sg\alpha$-continuous and $sg\alpha$-open function, then f is a quotient function.*

Proof. We only need to prove that $f^{-1}(V)$ is open in (X, τ) implies V is a $sg\alpha$-open set in (Y, σ). Let $f^{-1}(V)$ is open in (X, τ). Then $f(f^{-1}(V))$ is $sg\alpha$-open, since f is $sg\alpha$-open. Hence V is a $sg\alpha$-open set of Y, as f is surjective, $f(f^{-1}(V)) = V$. Thus, f is a $sg\alpha$-quotient function. □

Proposition 5.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be an open surjective $sg\alpha$-irresolute function and $g : (Y, \sigma) \to (Z, \eta)$ be a $sg\alpha$-quotient function. Then the composition $g \circ f \colon (X, \tau) \to (Z, \eta)$ is a $sg\alpha$-quotient function.*

Proof. Let V be any open set in (Z, η). Then $g^{-1}(V)$ is a $sg\alpha$-open set, since g is a $sg\alpha$-quotient function. Since f is $sg\alpha$-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a $sg\alpha$-open in X. This shows that $g \circ f$ is $sg\alpha$-continuous. Also, assume that $(g \circ f)^{-1}(V)$ is open in (X, τ) for $V \subset Z$, that is $f^{-1}(g^{-1}(V))$ is open in (X, τ). Since f is open $f(f^{-1}(g^{-1}(V)))$ is open in (Y, σ). It follows that $g^{-1}(V)$ is open in (Y, σ), because f is surjective. Since g is a $sg\alpha$-quotient function, V is $sg\alpha$-open in Z. Thus, $g \circ f \colon (X, \tau) \to (Z, \eta)$ is a $sg\alpha$-quotient function. □

Proposition 5.8. *If $h : (X, \tau) \to (Y, \sigma)$ is a $sg\alpha$-quotient function and $g : (X, \tau) \to (Z, \eta)$ is a continuous function where (Z, η) is a space that is constant on each set $h^{-1}(\{y\})$, for $y \in Y$, then g induces a $sg\alpha$-continuous function $f : (Y, \sigma) \to (Z, \eta)$ such that $f \circ h = g$.*

Proof. Since g is constant on $h^{-1}(\{y\})$, for each $y \in Y$, the set $g(h^{-1}(\{y\}))$ is a point set in (Z, η). Let $f(y)$ denote this point, then it is clear that f is well defined and for each $x \in X$, $f(h(x)) = g(x)$. We claim that f is $sg\alpha$-continuous. Let V be any open set of (Z, η), then $g^{-1}(V)$ is open, as g is continuous. But $g^{-1}(V) = h^{-1}(f^{-1}(V))$ is open in (X, τ). Since h is a $sg\alpha$-quotient function, $f^{-1}(V)$ is $sg\alpha$-open in Y. □

Definition 5.9. *A surjective function $f : (X, \tau) \to (Y, \sigma)$ is said to be a strongly $sg\alpha$-quotient function if f is $sg\alpha$-continuous and $f^{-1}(V)$ is $sg\alpha$-open in (X, τ) implies V is open set in (Y, σ).*

Proposition 5.10. *Every strongly $sg\alpha$-quotient function is $sg\alpha$-quotient function.*

For example, the function in the Example 5.3 is a $sg\alpha$-quotient function but not strongly $sg\alpha$-quotient function.

Definition 5.11. *A function $f : (X, \tau) \to (Y, \sigma)$ is called a completely $sg\alpha$-quotient function if f is $sg\alpha$-irresolute and $f^{-1}(U)$ is $sg\alpha$-open in X implies U is open in Y.*

Definition 5.12. *A function $f : (X, \tau) \to (Y, \sigma)$ is called $sg\alpha^*$-open [9] if the image of every $sg\alpha$-open set in X is an $sg\alpha$-open in Y.*

Theorem 5.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective $sg\alpha^*$-open and $sg\alpha$-irresolute function and $g : (Y, \sigma) \to (Z, \eta)$ be a completely $sg\alpha$-quotient function. Then $g \circ f$ is a completely $sg\alpha$-quotient function.*

Proof. Let V be a $sg\alpha$-open set in Z. Then $g^{-1}(V)$ is a $sg\alpha$-open set in Y because g is a completely $sg\alpha$-quotient function. We claim that $g \circ f$ is $sg\alpha$-irresolute. Since f is $sg\alpha$-irreoslute, $f^{-1}(g^{-1}(V))$ is a $sg\alpha$-open set in X, that is $g \circ f$ is $sg\alpha$-irresolute. Suppose $(g \circ f)^{-1}(V)$ is an $sg\alpha$-open set in X for $V \subset Z$, that is, $f^{-1}(g^{-1}(V))$ is a $sg\alpha$-open set in X. Since f is $sg\alpha^*$-open, $f(f^{-1})$ is a $sg\alpha$-open set in Y, and $g^{-1}(V)$ is a $sg\alpha$-open set in Y because f is surjective. Since g is completely $sg\alpha$-quotient function, V is an open set in Z. Thus, $g \circ f$ is completely $sg\alpha$-quotient function. □

Proposition 5.14. *Every completely $sg\alpha$-quotient function is strongly $sg\alpha$-quotient function.*

Proof. Suppose V is an open set in Y then it is a $sg\alpha$-open set in Y. Since f is $sg\alpha$-irresolute, $f^{-1}(V)$ is a $sg\alpha$-open in X. Thus V is open in Y gives $f^{-1}(V)$ is a $sg\alpha$-open set in X. Suppose $f^{-1}(V)$ is a $sg\alpha$-open set in X. Since f is a completely $sg\alpha$-quotient function, V is an open set in Y. Hence f is strongly $sg\alpha$-quotient function. □

Theorem 5.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be a strongly $sg\alpha$-quotient function and $g : (Y, \sigma) \to (Z, \eta)$ be a $sg\alpha$-quotient function, then $g \circ f$ is a completely $sg\alpha$-quotient function.*

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