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Generalization of Homeomorphisms

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Abstract: In this paper, we first introduce a new class of closed functions called $sg\alpha$-closed functions also introduce a new class of homeomorphisms called $sg\alpha^*$-homeomorphisms, which are weaker than homeomorphisms. We also prove that the set of all $sg\alpha^*$-homeomorphisms forms a group under the operation composition of functions.

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1. Introduction

The notion homeomorphisms plays a very important role in General topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. Malghan [5] introduced the concept of generalized closed maps in topological spaces. In this paper, we first introduce a new class of closed maps called $sg\alpha$-closed maps in topological space and then we introduce and study $sg\alpha^*$-homeomorphisms and prove that the set of all $sg\alpha^*$-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. We recall the following definitions and some results, which are used in the sequel.

Definition 2.1. *A subset A of a space (X, τ) is called :*

(1). semiopen [3] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$,

(2). α-open [4] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

The complement of an α-open set is called an α-closed set. The α-closure of a subset A of X, denoted by $\alpha\,\mathrm{Cl}_X(A)$ briefly $\alpha\,\mathrm{Cl}(A)$ is defined to be the intersection of all α-closed sets of X containing A.

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Definition 2.2. *A subset A of a space (X,τ) is called a semi-generalized α-closed (briefly sgα-closed) [6] if $\alpha\,\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in (X,τ). The complement of a sgα-closed set is called a sgα-open set.*

Definition 2.3. *A function $f : (X,\tau) \to (Y,\sigma)$ is called :*

(1). sgα-continuous [7] if $f^{-1}(V)$ is sgα-closed in (X,τ) for every closed set V in (Y,σ),

(2). sgα-irresolute if $f^{-1}(V)$ is sgα-closed in (X,τ) for every sgα-closed set V in (Y,σ),

(3). irresolute [2] if $f^{-1}(V)$ is semiclosed (semiopen) in (X,τ) for each semiclosed (semiopen) set V of (Y,σ).

Definition 2.4 ([6]). *Let (X,τ) be a topological space and $E \subseteq X$. We define the sgα-closure of E to be the intersection of all sgα-closed sets of X containing E and is denoted by sgα-$\mathrm{Cl}(E)$.*

Theorem 2.5 ([6]). *Let (X,τ) be a topological space and $E \subseteq X$. The following properties are hold:*

(1). sgα-$\mathrm{Cl}(E)$ is the smallest sgα-closed set containing E and,

(2). E is sgα-closed if and only if sgα-$\mathrm{Cl}(E) = E$.

Theorem 2.6 ([6]). *For any two subsets A and B of (X,τ),*

(1). If $A \subseteq B$, then sgα-$\mathrm{Cl}(A) \subseteq$ sgα-$\mathrm{Cl}(B)$,

(2). sgα-$\mathrm{Cl}(A \cap B) \subseteq$ sgα-$\mathrm{Cl}(A) \cap$ sgα-$\mathrm{Cl}(B)$.

Theorem 2.7 ([6]). *Suppose that $B \subseteq A \subseteq X$, B is a sgα-closed set relative to A and that A is open and sgα-closed in (X,τ). Then B is sgα-closed in (X,τ).*

Corollary 2.8 ([6]). *If A is a sgα-closed set and F a closed set, then $A \cap F$ is a sgα-closed set.*

Theorem 2.9 ([6]). *A set A is sgα-open in (X,τ) if and only if $F \subseteq \mathrm{Int}(A)$ whenever F is semiclosed in (X,τ) and $F \subseteq A$.*

Definition 2.10 ([6]). *Let (X,τ) be a topological space and $E \subseteq X$. We define the sgα-interior of E to be the union of all sgα-open sets of X contained in E and is denoted by sgα-$\mathrm{Int}(E)$.*

Lemma 2.11 ([6]). *For any $E \subseteq X$, $\mathrm{Int}(E) \subseteq$ sgα-$\mathrm{Int}(E) \subseteq E$.*

Proof. Since every open set is $sg\alpha$-open, the proof follows immediately. □

3. $sg\alpha$-Closed Functions

Definition 3.1. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be sgα-closed if the image of every closed set in (X,τ) is sgα-closed in (Y,σ).*

Example 3.2. *Let $X = \{a,b,c\}$, $\tau = \{\varnothing,\{a\},\{b\},\{a,b\},X\}$ and $\sigma = \{\varnothing,\{a\},\{b,c\},X\}$. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is a sgα-closed function.*

Theorem 3.3. *A function $f : (X,\tau) \to (Y,\sigma)$ is sgα-closed if and only if sgα-$\mathrm{Cl}(f(A)) \subseteq f(\mathrm{Cl}(A))$ for every subset A of (X,τ).*

Proof. Suppose that f is $sg\alpha$-closed and $A \subseteq X$. Then $f(\mathrm{Cl}(A))$ is $sg\alpha$-closed in (Y, σ). We have $f(A) \subseteq f(\mathrm{Cl}(A))$ and by Theorems 2.5 and 2.6, $sg\alpha\text{-}\mathrm{Cl}(f(A)) \subseteq sg\alpha\text{-}\mathrm{Cl}(f(\mathrm{Cl}(A))) = f(\mathrm{Cl}(A))$. Conversely, let A be any closed set in (X, τ). Then $A = \mathrm{Cl}(A)$ and so $f(A) = f(\mathrm{Cl}(A)) \supseteq sg\alpha\text{-}\mathrm{Cl}(f(A))$, by hypothesis. We have $f(A) \subseteq sg\alpha\text{-}\mathrm{Cl}(f(A))$ by Theorem 2.5. Therefore, $f(A) = sg\alpha\text{-}\mathrm{Cl}(f(A))$; hence $f(A)$ is $sg\alpha$-closed by Theorem 2.5. Therefore f is a $sg\alpha$-closed function. □

Theorem 3.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha$-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there exists a $sg\alpha$-open set V of (Y, σ) such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose that the function f is $sg\alpha$-closed. Let $S \subseteq Y$ and U be an open subset of (X, τ) such that $f^{-1}(S) \subseteq U$. Then $V = (f(U^c))^c$ is a $sg\alpha$-open set containing S such that $f^{-1}(V) \subseteq U$. For the converse, let S be a closed set of (X, τ). Then $f^{-1}((f(S))^c) \subseteq S^c$ and S^c is open in X. By assumption, there exists a $sg\alpha$-open set V of (Y, σ) such that $(f(S))^c \subseteq V$, follows that $f^{-1}(V) \subseteq S^c$ and so $S \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(S) \subseteq f((f^{-1}(V))^c) \subseteq V^c$ which implies $f(S) = V^c$. Since V^c is $sg\alpha$-closed, $f(S)$ is $sg\alpha$-closed and therefore f is $sg\alpha$-closed. □

Theorem 3.5. *If $f : (X, \tau) \to (Y, \sigma)$ is irresolute $sg\alpha$-closed and A is a $sg\alpha$-closed subset of (X, τ), then $f(A)$ is $sg\alpha$-closed.*

Proof. Let U be a semiopen set in (Y, σ) such that $f(A) \subseteq U$. Since f is irresolute, $f^{-1}(U)$ is a semiopen set containing A. Hence $\alpha\,\mathrm{Cl}(A) \subseteq f^{-1}(U)$ as A is $sg\alpha$-closed (X, τ). Since f is $sg\alpha$-closed, $f(\mathrm{Cl}(A))$ is a $sg\alpha$-closed set contained in the semiopen set U, which implies that $\alpha\,\mathrm{Cl}(f(\mathrm{Cl}(A))) \subseteq U$ and hence $\alpha\,\mathrm{Cl}(f(A)) \subseteq U$. Therefore, $f(A)$ is a $sg\alpha$-closed set. □

Remark 3.6. *The converse of the Theorem 3.5 is not true in general. The function f defined in Example 3.2 is $f(A)$ is $sg\alpha$-closed but not irresolute.*

The following example shows that the composition of two $sg\alpha$-closed functions is not a $sg\alpha$-closed function.

Example 3.7. *Let (X, τ), (X, σ) and f be as in Example 3.2. Let $Z = \{a, b, c\}$ and $\eta = \{\varnothing, \{a, c\}, Z\}$. Define a function $g : (X, \sigma) \to (Z, \eta)$ by $g(a) = g(b) = b$ and $g(c) = a$. Then both f and g are $sg\alpha$-closed functions but their composition $g \circ f$: $(X, \tau) \to (Z, \sigma)$ is not a $sg\alpha$-closed function, since for the closed set $\{c\}$ in (X, τ), $(g \circ f)(\{c\}) = \{a\}$, which is not $sg\alpha$-closed in (Z, η).*

Corollary 3.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $sg\alpha$-closed function and $g : (Y, \sigma) \to (Z, \eta)$ be $sg\alpha$-closed irresolute function, then their composition $g \circ f : (X, \tau) \to (Z, \sigma)$ is $sg\alpha$-closed.*

Proof. Let A be a closed subset of (X, τ). Then by hypothesis $f(A)$ is a $sg\alpha$-closed set in (Y, σ). Since g is $sg\alpha$-closed and irresolute by Theorem 3.5, $g(f(A)) = (g \circ f)(A)$ is $sg\alpha$-closed in (Z, η) and hence $g \circ f$ is $sg\alpha$-closed . □

Theorem 3.9. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two functions such that their composition $g \circ f : (X, \tau) \to (Z, \eta)$ be a $sg\alpha$-closed function. Then the following statements are true.*

(1). If f is continuous and surjective, then g is $sg\alpha$-closed.

(2). If g is $sg\alpha$-irresolute and injective, then f is $sg\alpha$-closed.

Proof.

(1). Let A be a closed set of (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is $sg\alpha$-closed, $(g \circ f)(f^{-1}(A))$ is $sg\alpha$-closed in (Z, η). Then $g(A)$ is $sg\alpha$-closed in (Z, η), since f is surjective. Therefore, g is $sg\alpha$-closed.

(2). Let B be a closed set of (X,τ). Since $g \circ f$ is $sg\alpha$-closed, $(g \circ f)(B)$ is $sg\alpha$-closed in (Z,η). Since g is $sg\alpha$-irresolute, $g^{-1}((g \circ f)(B))$ is $sg\alpha$-closed in (Y,σ). Then $f(B)$ is $sg\alpha$-closed in (Y,σ), since g is injective. Thus, f is $sg\alpha$-closed. □

As for the restriction f_A of a function $f : (X,\tau) \to (Y,\sigma)$ to a subset A of (X,τ), we have the following:

Theorem 3.10. *Let (X,τ) and (Y,σ) be topological spaces. Then*

(1). If $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha$-closed and A is a closed subset of (X,τ), then f_A: $(A,\tau_A) \to (Y,\sigma)$ is $sg\alpha$-closed.

(2). If $f : (X,\tau) \to (Y,\sigma)$ is irresolute $sg\alpha$-closed and A is an open subset of (X,τ), then f_A: $(A,\tau_A) \to (Y,\sigma)$ is $sg\alpha$-closed.

(3). If $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha$-closed (resp. closed) and $A = f^{-1}(B)$ for some closed (resp. $sg\alpha$-closed) set B of (Y,σ), then f_A: $(A,\tau_A) \to (Y,\sigma)$ is $sg\alpha$-closed.

Proof.

(1). Let B be a closed set of A. Then $B = A \cap F$ for some closed set F of (X,τ) and so B is closed in (X,τ). By hypothesis, $f(B)$ is $sg\alpha$-closed in (Y,σ). But $f(B) = f_A(B)$ and therefore f_A is $sg\alpha$-closed.

(2). Let C be a closed set of A. Then C is $sg\alpha$-closed relative to A. Since A is both open and $sg\alpha$-closed, C is $sg\alpha$-closed, by Theorem 2.7. Since f is both irresolute and $sg\alpha$-closed, $f(C)$ is $sg\alpha$-closed in (Y,σ), by Theorem 3.5. Since $f(C) = f_A(C)$, f_A is $sg\alpha$-closed.

(3). Let D be a closed set of A. Then $D = A \cap H$ for some closed set H in (X,τ). Now $f_A(D) = f(D) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since f is $sg\alpha$-closed, $f(H)$ is $sg\alpha$-closed and so $B \cap f(H)$ is $sg\alpha$-closed in (Y,σ) by Corollary 2.8. Therefore, f_A is a $sg\alpha$-closed function. □

The next theorem shows that normality is preserved under continuous $sg\alpha$-closed functions.

Theorem 3.11. *If $f : (X,\tau) \to (Y,\sigma)$ is a continuous, $sg\alpha$-closed function from a normal space (X,τ) onto a space (Y,σ), then (Y,σ) is normal.*

Proof. Let A and B be two disjoint closed subsets of (Y,σ). Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of (X,τ). Since (X,τ) is normal, there exist disjoint open sets U and V of (X,τ) such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is $sg\alpha$-closed, by Theorem 3.4, there exist disjoint $sg\alpha$-open sets G and H in (Y,σ) such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U and V are disjoint, $\alpha\,\mathrm{Int}(G)$ and $\alpha\,\mathrm{Int}(H)$ are also disjoint α-open and hence $sg\alpha$-open sets in (Y,σ). Since A is closed, A is semiclosed and $A \subseteq G$, $B \subseteq H$, we have by Theorem 2.9, $A \subseteq \alpha\,\mathrm{Int}(G)$. Similarly $B \subseteq \alpha\,\mathrm{Int}(H)$ and hence (Y,σ) is α-normal. □

Analogous to a $sg\alpha$-closed function, we define a $sg\alpha$-open function as follows:

Definition 3.12. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to a $sg\alpha$-open function if the image $f(A)$ is $sg\alpha$-open in (Y,σ) for each open set A in (X,τ).*

Theorem 3.13. *For any bijective function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(1). $f^{-1} : (Y,\sigma) \to (X,\tau)$ is $sg\alpha$-continuous,

(2). f is $sg\alpha$-open,

(3). f is $sg\alpha$-closed.

Proof. (1) $\Rightarrow$ (2): Let U be an open subset of (X,τ). By assumption $(f^{-1})^{-1}(U) = f(U)$ is $sg\alpha$-open in (Y,σ) and so f is $sg\alpha$-open.

(2) $\Rightarrow$ (3): Let F be a closed subset of (X,τ). Then F^c is open in (X,τ). By assumption, $f(F^c)$ is $sg\alpha$-open in (Y,σ). Then $f(F^c) = (f(F))^c$ is $sg\alpha$-open in (Y,σ) and therefore $f(F)$ is $sg\alpha$-closed in (Y,σ). Hence f is $sg\alpha$-closed.

(3) $\Rightarrow$ (1): Let F be a closed set in (X,τ). By assumption $f(F)$ is $sg\alpha$-closed in (Y,σ). But $f(F) = (f^{-1})^{-1}(F)$ and therefore f^{-1} is $sg\alpha$-continuous on Y. □

Definition 3.14. *Let x be a point of (X,τ) and V be a subset of X. Then V is called a $sg\alpha$-neighbourhood [7] of x in (X,τ) if there exists a $sg\alpha$-open set U of (X,τ) such that $x \in U \subseteq V$.*

In the next two theorems, we obtain various characterizations of $sg\alpha$-open functions.

Theorem 3.15. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. Then the following statements are equivalent:*

(1). f is a $sg\alpha$-open function.

(2). For a subset A of (X,τ), $f(\mathrm{Int}(A)) \subseteq sg\alpha\text{-}\mathrm{Int}(f(A))$.

(3). For each $x \in X$ and for each neighbourhood U of x in (X,τ), there exists a $sg\alpha$-neighbourhood W of $f(x)$ in (Y,σ) such that $W \subseteq f(U)$.

Proof. (1) $\Rightarrow$ (2): Suppose f is $sg\alpha$-open. Let $A \subseteq X$. Then $\mathrm{Int}(A)$ is open in (X,τ) and so $f(\mathrm{Int}(A))$ is $sg\alpha$-open in (Y,σ). But $f(\mathrm{Int}(A)) \subseteq f(A)$. Therefore, by Lemma 2.11, $f(\mathrm{Int}(A)) \subseteq sg\alpha\text{-}\mathrm{Int}(f(A))$.

(2) $\Rightarrow$ (3): Suppose (2) holds. Let $x \in X$ and U be an arbitrary neighbourhood of x in (X,τ). Then there exists an open set G such that $x \in G \subseteq U$. By assumption, $f(G) = f(\mathrm{Int}(G)) \subseteq sg\alpha\text{-}\mathrm{Int}(f(G))$. This implies $f(G) = sg\alpha\text{-}\mathrm{Int}(f(G))$. By Lemma 2.11, we have $f(G)$ is $sg\alpha$-open in (Y,σ). Further, $f(x) \in f(G) \subseteq f(U)$ and so (3) holds, by taking $W = f(G)$.

(3) $\Rightarrow$ (1): Suppose (3) holds. Let U be any open set in (X,τ), $x \in U$ and $f(x) = y$. Then $y \in f(U)$ and for each $y \in f(U)$, by assumption there exists a $sg\alpha$-neighbourhood W_y of y in (Y,σ) such that $W_y \subseteq f(U)$. Since W_y is a $sg\alpha$-neighbourhood of y, there exists a $sg\alpha$-open set V_y in (Y,σ) such that $y \in V_y \subseteq W_y$. Therefore, $f(U) = \bigcup \{V_y : y \in f(U)\}$ is a $sg\alpha$-open set in (Y,σ). Thus, f is a $sg\alpha$-open function. □

Theorem 3.16. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is $sg\alpha$-open if and only if for any subset B of (Y,σ) and for any closed set S containing $f^{-1}(B)$, there exists a $sg\alpha$-closed set A of (Y,σ) containing B such that $f^{-1}(A) \subseteq S$.*

Proof. Similar to Theorem 3.4. □

Corollary 3.17. *A function $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha$-open if and only if $f^{-1}(sg\alpha\text{-}\mathrm{Cl}(B)) \subseteq \mathrm{Cl}(f^{-1}(B))$ for every subset B of (Y,σ).*

Proof. Suppose that f is $sg\alpha$-open. Let B any subset of Y, then $f^{-1}(B) \subseteq \mathrm{Cl}(f^{-1}(B))$. By Theorem 3.16, there exists a $sg\alpha$-closed set A of (Y,σ) such that $B \subseteq A$ and $f^{-1}(A) \subseteq \mathrm{Cl}(f^{-1}(B))$. Since A is $sg\alpha$-closed set (Y,σ), follows $f^{-1}(sg\alpha\text{-}\mathrm{Cl}(B)) \subseteq f^{-1}(A) \subseteq \mathrm{Cl}(f^{-1}(B))$. Conversely, let S be any subset of (Y,σ) and F be any closed set containing $f^{-1}(S)$. Put $A = sg\alpha\text{-}\mathrm{Cl}(S)$. Then A is $sg\alpha$-closed and $S \subseteq A$. By assumption, $f^{-1}(A) = f^{-1}(sg\alpha\text{-}\mathrm{Cl}(S)) \subseteq \mathrm{Cl}(f^{-1}(S)) \subseteq A$ and therefore by Theorem 3.16, f is $sg\alpha$-open. □

Finally in this section, we define another new class of functions called $sg\alpha^*$-closed functions which are stronger than $sg\alpha$-closed functions.

Definition 3.18. *A function f: $(X,\tau) \to (Y,\sigma)$ is said to be a $sg\alpha^*$-closed function if the image $f(A)$ is $sg\alpha$-closed in (Y,σ) for every $sg\alpha$-closed set A in (X,τ).*

For example, the function f in Example 3.2 is a $sg\alpha^*$-closed function.

Remark 3.19. *Since every closed set is a $sg\alpha$-closed set we have every $sg\alpha^*$-closed function is a $sg\alpha$-closed function. The converse is not true in general as seen from the following example.*

Example 3.20. *Let $X = Y = \{a,b,c\}$, $\tau = \{\varnothing, \{a,b\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{a,b\}, Y\}$ and $f : (X,\tau) \to (Y,\sigma)$ be the identity function. Then f is $sg\alpha$-closed but not $sg\alpha^*$-closed, since $\{a,c\}$ is a $sg\alpha$-closed set in (X,τ), but its image under f is $\{a,c\}$, which is not $sg\alpha$-closed in (Y,σ).*

Theorem 3.21. *A function $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha^*$-closed if and only if $sg\alpha\text{-Cl}(f(A)) \subseteq f(sg\alpha\text{-Cl}(A))$ for every subset A of (X,τ).*

Proof. Similar to Theorem 3.3. □

Analogous to $sg\alpha^*$-closed function we can also define $sg\alpha^*$-open function.

Theorem 3.22. *For any bijective function $f : (X,\tau) \to (Y,\sigma)$, the following are equivalent:*

(1). f^{-1} :$(Y,\sigma) \to (X,\tau)$ is $sg\alpha$-irresolute,

(2). f is a $sg\alpha^*$-open,

(3). f is a $sg\alpha^*$-closed function.

Proof. Similar to Theorem 3.13. □

Theorem 3.23. *If f: $(X,\tau) \to (Y,\sigma)$ is irresolute $sg\alpha$-closed functions, then it is $sg\alpha^*$-closed.*

Proof. Follows from Theorem 3.5. □

The following example shows that the converse of Theorem 3.23 is not true in general.

Example 3.24. *Let $X = \{a,b,c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{b,c\}, X\}$. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is $sg\alpha^*$-closed but none of irresolute $sg\alpha$-closed.*

Lemma 3.25 ([6]). *Let A be a subset of X. Then $p \in sg\alpha\text{-Cl}(A)$ if and only if for any $sg\alpha$-neighborhood N of p in X, $A \cap N \neq \varnothing$.*

Definition 3.26. *Let A be a subset of X. A function $r : X \to A$ is called a $sg\alpha$-continuous retraction if r is $sg\alpha$-continuous and the restriction r_A is the identity mapping on A.*

Definition 3.27. *A topological space (X,τ) is called a $sg\alpha$-Hausdorff if for each pair x, y of distinct points of X, there exists $sg\alpha$-neighborhoods U_1 and U_2 of x and y, respectively, that are disjoint.*

Example 3.28. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$. Clearly, the topological space (X,τ) is a $sg\alpha$-Hausdorff space.*

Theorem 3.29. *Let A be a subset of X and $r : X \to A$ be a $sg\alpha$-continuous retraction. If X is $sg\alpha$-Hausdorff, then A is a $sg\alpha$-closed set of X.*

Proof. Suppose that A is not $sg\alpha$-closed. Then there exists a point x in X such that $x \in sg\alpha\text{-Cl}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is $sg\alpha$-continuous retraction. Since X is $sg\alpha$-Hausdorff, there exists disjoint $sg\alpha$-open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary $sg\alpha$-neighborhood of x. Then $W \cap U$ is a $sg\alpha$-neighborhood of x. Since $x \in sg\alpha\text{-Cl}(A)$, by Lemma 3.25, we have $(W \cap U) \cap A \neq \varnothing$. Therefore there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \nsubseteq V$ because $y \in W$. This is contrary to the $sg\alpha$-continuity of r. Consequently, A is a $sg\alpha$-closed set of X. □

Theorem 3.30. *Let $\{X_i | i \in I\}$ be any family of topological spaces. If $f : X \to \Pi X_i$ is a $sg\alpha$-continuous mapping, then $P_{r_1} \circ f : X \to X_i$ is $sg\alpha$-continuous for each $i \in I$, where P_{r_1} is the projection of ΠX_j on X_i.*

Proof. We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary open set in X_i. Then $P_{r_1}^{-1}(U_i)$ is open in ΠX_i. Since f is $sg\alpha$-continuous, we have $f^{-1}(P_{r_1}^{-1}(U_i)) = (P_{r_1} \circ f)^{-1}(U_i)$ is a $sg\alpha$-open set in X. Therefore, $P_{r_1} \circ f$ is $sg\alpha$-continuous. □

4. $sg\alpha^*$-Homeomorphisms

In this section, we introduced the following definition:

Definition 4.1. *A bijective function $f : (X, \tau) \to (Y, \sigma)$ is said to be $sg\alpha^*$-homeomorphism if both f and f^{-1} are $sg\alpha$-irresolute.*

We denote the family of all $sg\alpha^*$-homeomorphisms of a topological space (X, τ) onto itself by $sg\alpha^*\text{-}h(X, \tau)$.

Theorem 4.2. *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $sg\alpha^*$-homeomorphisms, then their composition $g \circ f$: $(X, \tau) \to (Z, \eta)$ is also $sg\alpha^*$-homeomorphism.*

Proof. Let U be a $sg\alpha$-open set in (Z, σ). Now, $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, V is $sg\alpha$-open in (Y, σ) and so again by hypothesis, $f^{-1}(V)$ is $sg\alpha$-open in (X, τ). Therefore, $g \circ f$ is $sg\alpha$-irresolute. Also for a $sg\alpha$-open set G in (X, τ), we have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis, $f(G)$ is $sg\alpha$-open in (Y, σ) and so again by hypothesis, $g(f(G))$ is $sg\alpha$-open in (Z, η). i.e., $(g \circ f)(G)$ is $sg\alpha$-open in (Z, η) and therefore $(g \circ f)^{-1}$ is $sg\alpha$-irresolute. Hence $g \circ f$ is a $sg\alpha^*$-homeomorphism. □

On $sg\alpha^*\text{-}h(X, \tau)$, we define a binary operation $*$: $sg\alpha^*\text{-}h(X, \tau) \times sg\alpha^*\text{-}h(X, \tau) \to sg\alpha^*\text{-}h(X, \tau)$ by $f * g = g \circ f$. It is easy to see that the operation $*$ is well define, see Theorem 4.2 and associative, also the identity function I: $(X, \tau) \to (X, \tau)$ belongs to $sg\alpha^*\text{-}h(X, \tau)$ serves as the identity element. From all of this, we obtain the following theorem

Theorem 4.3. *The set $sg\alpha^*\text{-}h(X, \tau)$ is a group under the composition of functions.*

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $sg\alpha^*$-homeomorphism. Then f induces an isomorphism from the group $sg\alpha^*$-$h(X, \tau)$ onto the group $sg\alpha^*\text{-}h(Y, \sigma)$.*

Proof. Using the function f, we define a function θ_f: $sg\alpha^*\text{-}h(X, \tau) \to sg\alpha{*}\text{-}(Y, \sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in sg\alpha^*\text{-}h(X, \tau)$. Then θ_f is a bijection. Further, for all $h_1, h_2 \in sg\alpha^*\text{-}h(X, \tau)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, ${}^{\theta}f$ is a homeomorphism and so it is an isomorphism induced by f. □

Theorem 4.5. *$sg\alpha^*$-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof. Reflexivity and symmetry are immediate and transitivity follows from Theorem 4.2. □

Theorem 4.6. *If $f : (X, \tau) \to (Y, \sigma)$ is $sg\alpha^*$-homeomorphism, then $sg\alpha\text{-Cl}(f^{-1}(B)) = f^{-1}(sg\alpha - \text{Cl}(B))$ for all $B \subseteq Y$.*

Proof. Since f is $sg\alpha^*$-homeomorphism, f is $sg\alpha$-irresolute. Since $sg\alpha$-Cl$(f(B))$ is a $sg\alpha$-closed set in (Y,σ), $f^{-1}(sg\alpha$-Cl$(f(B)))$ is $sg\alpha$-closed in (X,τ). Now, $f^{-1}(B) \subseteq f^{-1}(sg\alpha$-Cl$(B))$ and so by Theorem 2.6, $sg\alpha$-Cl$(f^{-1}(B)) \subseteq f^{-1}(sg\alpha$-Cl$(B))$. Again since f is a $sg\alpha^*$-homeomorphism, f^{-1} is $sg\alpha$-irresolute. Since $sg\alpha$-Cl$(f^{-1}(B))$ is $sg\alpha$-closed in (X,τ), $(f^{-1})^{-1}(sg\alpha$-Cl$(f^{-1}(B))) = f(sg\alpha$-Cl$(f^{-1}(B)))$ is $sg\alpha$-closed in (Y,σ). Now, $B \subseteq (f^{-1})^{-1}(f^{-1}(B))) \subseteq (f^{-1})^{-1}(sg\alpha$-Cl$(f^{-1}(B))) = f(sg\alpha$-Cl$(f^{-1}(B)))$ and so $sg\alpha$-Cl$(B) \subseteq f(sg\alpha$-Cl$(f^{-1}(B)))$. Therefore, $f^{-1}(sg\alpha$-Cl$(B)) \subseteq f^{-1}(f(sg\alpha$-Cl$(f^{-1}(B)))) \subseteq sg\alpha$-Cl$(f^{-1}(B)$ and hence the equality holds. □

Corollary 4.7. *If $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha^*$-homeomorphism, then $sg\alpha$-Cl$(f(B)) = f(sg\alpha$-Cl$(B))$ for all $B \subseteq X$.*

Proof. Since $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha^*$-homeomorphism, f^{-1}: $(Y,\sigma) \to (X,\tau)$ is also $sg\alpha^*$-homeomorphism. Therefore, by Theorem 4.6, $sg\alpha$-Cl$((f^{-1})^{-1}(B)) = (f^{-1})^{-1}(sg\alpha$-Cl$(B))$ for all $B \subseteq X$. i.e., $sg\alpha$-Cl$(f(B)) = f(sg\alpha$-Cl$(B))$. □

Corollary 4.8. *If $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha^*$-homeomorphism, then $f(sg\alpha$-Int$(B)) = sg\alpha$-Int$(f(B))$ for all $B \subseteq X$.*

Proof. For any set $B \subseteq X$, $sg\alpha$-Int$(B) = (sg\alpha$-Cl$(B^c))^c$. Thus,

$$\begin{aligned} f(sg\alpha\text{-Int}(B)) &= f((sg\alpha\text{-Cl}(B^c))^c) \\ &= (f(sg\alpha\text{-Cl}(B^c)))^c \\ &= (sg\alpha\text{-Cl}(f(B^c)))^c, \textit{ by Corollary } 4.7 \\ &= (sg\alpha\text{-Cl}((f(B))^c))^c = sg\alpha\text{-Int}(f(B)). \end{aligned}$$

□

Corollary 4.9. *If $f : (X,\tau) \to (Y,\sigma)$ is $sg\alpha^*$-homeomorphism, then $f^{-1}(sg\alpha$-Int$(B)) = sg\alpha$-Int$(f^{-1}(B))$ for all $B \subseteq Y$.*

Proof. Since f^{-1}: $(Y,\sigma) \to (X,\tau)$ is also $sg\alpha^*$-homeomorphism, the proof follows from Corollary 4.8. □

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