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Laplace Approximation to the Posterior of Bayesian Weibull Model with Time-Varying Effect

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Abstract: In this paper, the estimation of the parameter of the Bayesian Weibull model with time-varying effect is considered. The Laplace approximation was developed and compared to the maximum likelihood approach achieved via the Newton-Raphson procedure. The efficiency of the two methods was analyzed using Monte-Carlo samples drawn from the time-varying Weibull model.

Keywords: Bayesian Weibull model, Newton-Raphson procedure, time-varying Weibull model.

1. Introduction

Rodrıguez [5] described four modelling approaches for parametric survival models namely; parametric families, proportional hazard, accelerated failure time and proportional odds. However, we decided to use the parametric families approach because of its simplicity and flexibility. Now, suppose T is an event time that follows Weibull distribution with shape and scale parameters define as(a, b), its density function, hazard function and survival function are (Alharpy and Ibrahim [2]; Collet [3]; Tseng [7]; Aalen [1]; Sparling [6]);

$$f\left(t \mid a, b\right) = \left(\frac{a}{b}\right)\left(\frac{t}{b}\right)^{a-1} exp\left[-\left(\frac{t}{b}\right)^{a}\right], t > 0,\ a,\ b > 0 \tag{1}$$

$$h\left(t \mid a, b\right) = \left(\frac{a}{b}\right)\left(\frac{t}{b}\right)^{a-1}, t > 0,\ a,\ b > 0 \tag{2}$$

$$S\left(t \mid a, b\right) = \exp\left[-\left(\frac{t}{b}\right)^{a}\right], t > 0,\ a,\ b > 0 \tag{3}$$

Using the parametric family approach, the covariate associated with the event time T can be modelled via the scale parameter b as,

$$b = \exp\left[x'\beta\right] \Rightarrow \log b = x'\beta \tag{4}$$

The form of parameterization in (4) is when the fixed time covariate or proportional assumption is satisfied. But when time-varying covariate $z(t)$ exist, (4) can be modified as;

$$b = \exp\left[x'\beta + \gamma z(t)\right] \tag{5}$$

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Where β and γ represent the parameters of fixed and time varying covariate. For the mortality model, $z(t)$ is a binary vector representing infection at the varying interval of time before the end of the udy. The matrix x represents the fixed covariate such as gender, age, marital status etc. substituting (5) in (1), we can obtain the density function for Weibull with fixed and time varying covariate effects as;

$$f\left(t \mid a, x,\ z(t)\right) = \left(\frac{a}{\exp\left[x'\beta + \gamma z(t)\right]}\right)\left(\frac{t}{\exp\left[x'\beta + \gamma z(t)\right]}\right)^{a-1} \exp\left[-\left(\frac{t}{\exp\left[x'\beta + \gamma z(t)\right]}\right)^{a}\right] \tag{6}$$

Now assuming $z(t)$ to be a piece-wise function as in the case of TB infection occurring with HIV infection, $z(t)$ can be define as;

$$z\left(t\right) = \begin{cases} 0, & t < t_c \\ 1, & t \geq t_c \end{cases}$$

Where t_c is the time at which the covariate $z(t)$ changes. This implies that $f\left(t \mid a, x,\ z(t)\right)$ will also be a piece-wise function. Therefore $f\left(t \mid a, x,\ z(t)\right)$ can be define as;

$$f\left(t \mid a, x, z(t)\right) = \begin{cases} f\left(t \mid a, x\right), & t < t_c \\ f\left(t \mid a, x, \gamma z(t)\right), & t \geq t_c \end{cases}$$

$$f\left(t \mid a, x,\ z\left(t\right)\right) = \begin{cases} \left(\frac{a}{\exp[x'\beta]}\right)\left(\frac{t}{\exp[x'\beta]}\right)^{a-1} \exp\left[-\left(\frac{t}{\exp[x'\beta]}\right)^{a}\right], & t < t_c \\ \left(\frac{a}{\exp[x'\beta+\gamma z(t)]}\right)\left(\frac{t}{\exp[x'\beta+\gamma z(t)]}\right)^{a-1} \exp\left[-\left(\frac{t}{\exp[x'\beta+\gamma z(t)]}\right)^{a}\right], & t \geq t_c \end{cases}$$

Consequently, the hazard function $h\left(t \mid a, x, z(t)\right)$ associated with T can be obtained as;

$$h\left(t \mid a, x, z(t)\right) = \begin{cases} \left(\frac{a}{\exp[x'\beta]}\right)\left(\frac{t}{\exp[x'\beta]}\right)^{a-1}, & t < t_c \\ \left(\frac{a}{\exp[x'\beta+\gamma z(t)]}\right)\left(\frac{t}{\exp[x'\beta+\gamma z(t)]}\right)^{a-1}, & t \geq t_c \end{cases}$$

The cumulative hazard function $H\left(t \mid a, x,\ z(t)\right)$ associated with T can be obtained as;

$$H\left(t \mid a, x, z(t)\right) = \begin{cases} \int_0^t \left(\frac{a}{\exp[x'\beta]}\right)\left(\frac{u}{\exp[x'\beta]}\right)^{a-1} du, & t < t_c \\ \int_0^t \left(\frac{a}{\exp[x'\beta+\gamma z(u)]}\right)\left(\frac{u}{\exp[x'\beta+\gamma z(u)]}\right)^{a-1} du, & t \geq t_c \end{cases}$$

Thus;

For $t < t_c$,

$$\begin{aligned} H\left(t \mid a, x,\ z(t)\right) &= \int_0^t \left(\frac{a}{\exp\left[x'\beta\right]}\right)\left(\frac{u}{\exp\left[x'\beta\right]}\right)^{a-1} du \\ &= \left(\frac{1}{\exp\left[x'\beta\right]}\right)^{a} \int_0^t a u^{a-1} du \\ H\left(t \mid a, x, z(t)\right) &= \left(\frac{t}{\exp\left[x'\beta\right]}\right)^{a} \end{aligned}$$

For $t \geq t_c$,

$$\begin{aligned} H\left(t \mid a, x, z(t)\right) &= \int_0^t \left(\frac{a}{\exp\left[x'\beta + \gamma\right]}\right)\left(\frac{u}{\exp\left[x'\beta + \gamma\right]}\right)^{a-1} du \\ &= \int_0^{t_c} \left(\frac{a}{\exp\left[x'\beta + \gamma z(t)\right]}\right)\left(\frac{u}{\exp\left[x'\beta + \gamma z(t)\right]}\right)^{a-1} du + \int_{t_c}^t \left(\frac{a}{\exp\left[x'\beta + \gamma z(t)\right]}\right)\left(\frac{u}{\exp\left[x'\beta + \gamma z(t)\right]}\right)^{a-1} du \\ &= \int_0^{t_c} \left(\frac{a}{\exp\left[x'\beta\right]}\right)\left(\frac{u}{\exp\left[x'\beta\right]}\right)^{a-1} du + \int_{t_c}^t \left(\frac{a}{\exp\left[x'\beta + \gamma\right]}\right)\left(\frac{u}{\exp\left[x'\beta + \gamma\right]}\right)^{a-1} du \end{aligned}$$

$$= \left(\frac{t_c}{\exp\left[x'\beta\right]}\right)^a + \int_{t_c}^{t}\left(\frac{a}{\exp\left[x'\beta+\gamma\right]}\right)\left(\frac{u}{\exp\left[x'\beta+\gamma\right]}\right)^{a-1} du$$

$$H\left(t \mid a, x,\ z(t)\right) = \left(\frac{t_c}{\exp\left[x'\beta\right]}\right)^a + \left(\frac{t}{\exp\left[x'\beta+\gamma\right]}\right)^a - \left(\frac{t_c}{\exp\left[x'\beta+\gamma\right]}\right)^a$$

Therefore,

$$H\left(t \mid a, x, z(t)\right) = \begin{cases} \left(\frac{t}{\exp[x'\beta]}\right)^a, & t < t_c \\ \left(\frac{t_c}{\exp[x'\beta]}\right)^a + \left(\frac{t}{\exp[x'\beta+\gamma]}\right)^a - \left(\frac{t_c}{\exp[x'\beta+\gamma]}\right)^a, & t \geq t_c \end{cases}$$

The survival function $S\left(t \mid a, x,\ z(t)\right)$ follows as;

$$S\left(t \mid a, x, z(t)\right) = \exp\left[-H\left(t \mid a, x, z(t)\right)\right]$$

$$S\left(t \mid a, x, z(t)\right) = \begin{cases} \exp\left[-\left(\frac{t}{\exp[x'\beta]}\right)^a\right], & t < t_c \\ \exp\left\{-\left[\left(\frac{t_c}{\exp[x'\beta]}\right)^a + \left(\frac{t}{\exp[x'\beta+\gamma]}\right)^a - \left(\frac{t_c}{\exp[x'\beta+\gamma]}\right)^a\right]\right\}, & t \geq t_c \end{cases}$$

2. Parameter Estimation for Weibull Time-Varying Covariate Model

Following Lessafre and Lawson (2013) the likelihood of a parametric survival model with right censored times can be defined as;

$$L(\theta) = \prod_{uce} f(t_{uce}|\theta) \prod_{ce} S(t_{ce}|\theta) \tag{7}$$

Where uce denote uncensored and ce denotes right censored. The likelihood in (7) can be simplified if a censoring indicator s_i that takes 0 for censored and 1 for uncensored is assumed. Thus,

$$L(\theta) = \prod_{i=1}^{n} h\left(t_i \mid \theta\right)^{s_i} S(t_i|\theta) \tag{8}$$

For the case of time-dependent covariate, we define c_i to be the time dependent indicator as a piece-wise function;

$$c_i = \begin{cases} 0, & \text{if the value of covariates is not updated} \\ 1, & \text{if the value of covariates is updated} \end{cases}$$

The likelihood for Weibull time-dependent covariate model is;

$$L\left(a, \beta, \gamma\right) = \prod_{i=1}^{n}\left(1-c_i\right) h\left(t_i \mid a, \beta, x\right)^{s_i} S\left(t_i \mid a, \beta, x\right) + \prod_{i=1}^{n} c_i h\left(t_i \mid a, \beta, \gamma, x\right)^{s_i} S\left(t_i \mid a, \beta, \gamma,\ x\right)$$

$$\begin{aligned} L\left(a, \beta, \gamma\right) = &\prod_{i=1}^{n}\left(1-c_i\right)\left[\left(\frac{a}{\exp\left[x_i{'}\beta\right]}\right)\left(\frac{t_i}{exp\left[x_i{'}\beta\right]}\right)^{a-1}\right]^{s_i} \exp\left[-\left(\frac{t_i}{\exp\left[x_i{'}\beta\right]}\right)^a\right] \\ &+ \prod_{i=1}^{n} c_i \left[\left(\frac{a}{\exp\left[x_i{'}\beta+\gamma\right]}\right)\left(\frac{t_i}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^{a-1}\right]^{s_i} \\ &\times \left(\exp\left\{-\left[\left(\frac{t_{ci}}{\exp\left[x_i{'}\beta\right]}\right)^a + \left(\frac{t_i}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^a - \left(\frac{t_{ci}}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^a\right]\right\}\right) \end{aligned}$$

The corresponding log-likelihood is;

$$\begin{aligned} l\left(a, \beta, \gamma\right) = &\sum_{i=1}^{n}\left(1-c_i\right)\left[s_i log\left[\left(\frac{a}{\exp\left[x_i{'}\beta\right]}\right)\left(\frac{t_i}{\exp\left[x_i{'}\beta\right]}\right)^{a-1}\right] - \left[\left(\frac{t_i}{\exp\left[x_i{'}\beta\right]}\right)^a\right]\right] \\ &+ \sum_{i=1}^{n} c_i \left\{s_i log\left[\left(\frac{a}{\exp\left[x_i{'}\beta+\gamma\right]}\right)\left(\frac{t_i}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^{a-1}\right]\right. \\ &\left. + \left[\left(\frac{t_{ci}}{\exp\left[x_i{'}\beta\right]}\right)^a + \left(\frac{t_i}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^a - \left(\frac{t_{ci}}{\exp\left[x_i{'}\beta+\gamma\right]}\right)^a\right]\right\} \end{aligned}$$

The model parameters a, β, γ can be obtained using Newthe ton-Raphson algorithm to find the solution of the likelihood equations given above. However, we have used the SPLUS function *nlminb* to obtain the results.

3. Bayesian Weibull Time-Varying Covariate Model

The Bayesian approach to the estimation of the parameter $\mathbf{\Omega} = \{a, \beta, \gamma\}$, can be defined as:

$$p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right) = \frac{p\left(\mathbf{\Omega} = \{a, \beta, \gamma\}\right) \times L\left(t, x|a, \beta, \gamma\right)}{\int_a \int_\beta \int_\gamma p\left(\mathbf{\Omega} = \{a, \beta, \gamma\}\right) \times L\left(t, x|a, \beta, \gamma\right) da d\beta d\gamma} \tag{9}$$

$$p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right) \propto p\left(\mathbf{\Omega} = \{a, \beta, \gamma\}\right) \times L\left(t, x|a, \beta, \gamma\right) \tag{10}$$

Where $p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right)$ and $p\left(\mathbf{\Omega} = \{a, \beta, \gamma\}\right)$ are the posterior and prior distributions of the parameter space $\mathbf{\Omega} = \{a, \beta, \gamma\}$. Equation (9) is the standard posterior of a procedure estimation procedure for Weibull time-varying covariate model. However, because of the intractability of the denominator of (9), it is often drop in practice as it only ensure the summing of $p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right)$ to one (Lesafree & Lawson, 2013). Thus, the estimation of the Bayesian model in this paper is developed using (10). The Laplace approximation involves using the maximum likelihood procedure to estimate the parameter of the posterior in (10). The log of (10) can be defined as:

$$log\left[p\left(\mathbf{\Omega}= \{a, \beta, \gamma\} \,|t, x\right)\right] = \log\left[p\left(\mathbf{\Omega} = \{a, \beta, \gamma\}\right)\right] + log\left[L\left(t, x|a, \beta,\ \gamma\right)\right] \tag{11}$$

One of the popular prior distribution usually assumed for this kind of model is the product of exponential distribution with parameter τ for parameter a and Gaussian distribution with parameter μ and σ^2 for the joint vector $\omega = \{\beta, \gamma\}$. Thus, we can rewrite (11) as:

$$\begin{aligned}
log\left[p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right)\right] = \log &\left\{\tau \exp(-a\tau) \times \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\omega-\mu)^2}{2\sigma^2}\right]\right\} \\
&+ \sum_{i=1}^{n} (1-c_i) \left[s_i log\left[\left(\frac{a}{\exp\left[x_i{}'\beta\right]}\right)\left(\frac{t_i}{\exp\left[x_i{}'\beta\right]}\right)^{a-1}\right] - \left[\left(\frac{t_i}{\exp\left[x_i{}'\beta\right]}\right)^{a}\right]\right] \\
&+ \sum_{i=1}^{n} c_i \left\{s_i log\left[\left(\frac{a}{\exp\left[x_i{}'\beta+\gamma\right]}\right)\left(\frac{t_i}{exp\left[x_i{}'\beta+\gamma\right]}\right)^{a-1}\right]\right. \\
&+ \left.\left[\left(\frac{t_{ci}}{\exp\left[x_i{}'\beta\right]}\right)^{a} + \left(\frac{t_i}{\exp\left[x_i{}'\beta+\gamma\right]}\right)^{a} - \left(\frac{t_{ci}}{\exp\left[x_i{}'\beta+\gamma\right]}\right)^{a}\right]\right\}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\log\left[p\left(\mathbf{\Omega} = \{a, \beta, \gamma\} \,|t, x\right)\right] = log&\left[\tau \exp(-a\tau)\right] + log\left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\omega-\mu)^2}{2\sigma^2}\right]\right] \\
&+ \sum_{i=1}^{n} (1-c_i) \left[s_i log\left[\left(\frac{a}{\exp\left[x_i{}'\beta\right]}\right)\left(\frac{t_i}{\exp\left[x_i{}'\beta\right]}\right)^{a-1}\right] - \left[\left(\frac{t_i}{\exp\left[x_i{}'\beta\right]}\right)^{a}\right]\right] \\
&+ \sum_{i=1}^{n} c_i \left\{s_i log\left[\left(\frac{a}{exp\left[x_i{}'\beta+\gamma\right]}\right)\left(\frac{t_i}{exp\left[x_i{}'\beta+\gamma\right]}\right)^{a-1}\right]\right. \\
&+ \left.\left[\left(\frac{t_{ci}}{exp\left[x_i{}'\beta\right]}\right)^{a} + \left(\frac{t_i}{exp\left[x_i{}'\beta+\gamma\right]}\right)^{a} - \left(\frac{t_{ci}}{exp\left[x_i{}'\beta+\gamma\right]}\right)^{a}\right]\right\}
\end{aligned} \tag{13}$$

The model parameters a, β, γ can again be obtained using the Newton-Raphson algorithm to find the solution of equation (13). However, we have used the SPLUS function *optim* to obtain the results.

4. Simulation Studies

The estimation methods were evaluated using the following simulation procedure. The following parameters; $a = 0.5$, $\beta = 2$ and $\gamma = 1$ was set as true values. The covariate $x_i \sim Binomial(1, 0.5)$, $t_{ci} \sim Weibull(shape = 2, scale = 0.04)$. Censoring rates 20% was used to check the effect of censoring rate on parameter estimates. Also, sample sizes $n = 50$, and $n = 500$ were used to study the effect of sample size on parameter estimates. Performance metrics used to assess the estimating methods are standard error (SE), bias and Mean Square error (MSE). The formulas based on 1000 repetition of the study are;

$$SE(\widehat{\theta}) = \sqrt{\frac{\sum_{j=1}^{1000}\left(\widehat{\theta}_j - \overline{\widehat{\theta}}\right)^2}{1000}}$$
$$bias(\widehat{\theta}) = \frac{\sum_{j=1}^{1000}\left(\widehat{\theta}_j - \theta\right)}{1000}$$
$$MSE(\widehat{\theta}) = \frac{\sum_{j=1}^{1000}\left(\widehat{\theta}_j - \theta\right)^2}{1000}$$

5. Results and Discussion

Table 1 shows the results for small and large sample size 50 and 500 and censoring rate 20%. For all the performance metrics used, the results of the proposed Laplace Bayesian method are better than the competing MLE method. Specifically, the proposed method is more stable in terms of low average standard error, consistent in terms of low average bias as well as efficient in terms of low mean square error. However, the exciting results are more attributable to the fixed and time-varying effect parameters γ & β which is the main focus.

Statistic	Parameter	$n = 50$		$n = 500$	
		MLE	LAPLACE BAYES	MLE	LAPLACE BAYES
$\widehat{\theta}$	$a = 0.5$	0.2739	0.3098	0.2758	0.3165
	$\beta = 2$	1.1014	1.9995	1.1139	1.9947
	$\gamma = 1$	2.9292	1.0002	3.0226	1.0019
$SE(\widehat{\theta})$	a	0.0259	0.0275	0.0094	0.0089
	β	0.3482	0.0003	0.1076	0.0008
	γ	0.7468	0.0001	0.2694	0.0003
$bias(\widehat{\theta})$	a	0.2261	0.1902	0.2242	0.1835
	β	0.8986	0.0005	0.8861	0.0053
	γ	1.9292	0.0002	2.0226	0.0019
$MSE(\widehat{\theta})$	a	0.0511	0.0362	0.0503	0.0337
	β	0.8074	0.0000	0.7852	0.0000
	γ	3.7220	0.0000	4.0910	0.0000

Table 1: Simulation results for average bias ($bias(\widehat{\theta})$), average standard error ($SE(\widehat{\theta})$) and average mean square error ($MSE(\widehat{\theta})$) based on 1000 replications for sample size $n = 50,\ 500$ and censoring rate 20%.

6. Conclusion

In this paper, we have presented a new estimation strategy for Bayesian time-varying model using Laplace approximation. The results from the performance analysis showed that the Laplace approximation procedure is more consistent, efficient and precise than the maximum likelihood estimator (MLE).

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