Spectral Properties of M-class $A_k^*$ Operator

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Abstract: The Banach algebra on a non-zero complex Hilbert space $H$ of all bounded linear operators are denoted by $B(H)$. An operator $T$ is defined as an element in $B(H)$. If $T$ belongs to $B(H)$, then $T^*$ means the adjoint of $T$ in $B(H)$. An operator $T$ is called class $A(k)$ if $|T|^2 \leq (T^*|T|^{2k}T)^{\frac{k}{k+1}}$ for $k > 0$. An operator $T$ is called $A_k$ if $|T|^2 \leq (|T|^{k+1})^{\frac{k}{k+1}}$ for some positive integer $k$. S. Panayappan [11] introduced class $A_k^*$ operator as “an operator $T$ is called class $A_k^*$ if $|T^k|^2 \geq |T^s|^2$ where $k$ is a positive integer” and studied Weyl and Weyl type theorems for the operator [9]. In this paper we introduced extended class $A_k^*$ operator and studied some of its spectral properties. We also show that extended class $A_k^*$ operators are closed under tensor product.

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1. Introduction

An operator $T$ is defined in $B(H)$ is an element in $B(H)$. Weyl and Weyl type theorems where studied for the following class of operators. Furuta et al introduced class $A(k), k > 0$ as a class of operators including p-hyponormal and log-hyponormal operators and studied Weyl type theorems. L.A.Coburn studied Weyl’s theorem for non normal operators [3] then M. Berkani studied generalized Weyl’s theorem for hyponormal operators [1, 2]. Panayappan extended this concept and introduced class $A_k$ operators and verified Weyl’s theorem [11]. In 2016, D. Senthil Kumar studied aluthge transformation for N-class $A_k$ operators [10]. In 2013, Panayappan et al introduced a new class of operators in a different manner called class $A_k^*$ operator, quasi class $A_k^*$ operators and studied Weyl and Weyl type theorems and also proved tensor product of two quasi class $A_k^*$ operators is closed [9]. An operator $T$ is called class $A_k^*$ if $|T^k|^2 \geq |T^s|^2$ where $k$ is a positive integer.

If $k = 1$ then class $A_k^*$ operator coincides with hyponormal operator [9]. In this paper, we extended class $A_k^*$ operator as a new class of operator named M-class $A_k^*$ operators and studied some of its spectral properties.

Definition 1.1. An operator $T \in B(H)$ is said to be M-Class $A_k^*$ operator if there exists positive real numbers $M, k$ such that $|T^s|^2 \leq M \left(|T^k|^\frac{k}{k+1}\right)$.

Proposition 1.2. If $M = 1$, then M-Class $A_k^*$ operator coincides with class $A_k^*$ operator. If $M = 1$ and $k = 1$, then M-Class $A_k^*$ operator coincides with hyponormal operator. Hence, Hyponormal operator $\Rightarrow$ class $A_k^*$ operator $\Rightarrow$ M-Class $A_k^*$ operator.

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2. Spectral Properties of M-Class $A^*_k$ Operators

In this section, first we prove using matrix representation that the restriction of M-Class $A^*_k$ operators to an invariant subspace is also M-Class $A^*_k$, and if $T$ is M-Class $A^*_k$ operator, then Weyl’s theorem hold for $T$, $T^*$ and $f(T)$ for $f \in H(\sigma(T))$ and if $T^*$ has SVEP, then a-Weyl’s theorem hold for $T$, $T^*$ and $f(T)$ for $f \in H(\sigma(T))$.

**Theorem 2.1.** If $T$ is M-Class $A^*_k$ operator for positive real numbers $M$ and $k$, then $T|_N$ is also M-Class $A^*_k$ operator where $N$ is an invariant subspace of $T$.

**Proof.** Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $H$ onto $N$ and $T|_N = T_1 = (PTP)|_N$ and $TP = PTP$. Since $T$ is M-class $A^*_k$ operator and $P$ is a projection on $N$, $P \left( M \left| T^k \right|^2 - |T^*|^2 \right) P \geq 0$. By Hansen’s Inequality [4, 10],

$$P \left( M \left| T^k \right|^2 \right) P \leq M \left( P \left| T^k \right|^2 \right) \frac{1}{k} = \begin{pmatrix} M \left| T^k_1 \right|^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M \left| T^k_1 \right|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$M \begin{pmatrix} \left| T^k_1 \right|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq P(M \left| T^k \right|^2)P \geq P|T^*|^2 P = \begin{pmatrix} |T^k_1|^2 + |T^k_2|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $M \left| T^k_1 \right|^2 - |T^k_1|^2 \geq |T^k_2|^2 \geq 0$. Hence $T|_N$ is M-Class $A^*_k$ operator on an invariant subspace $N$ of $T$. □

**Theorem 2.2.** If $T$ is M-Class $A^*_k$ operator for positive real numbers $M$ and $k$, $\lambda \in \sigma_T(T)$ where $\lambda \neq 0$ and $T$ is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $Ker(T - \lambda) \oplus \text{ran}(T - \lambda)^*$, then $T_3$ is M-Class $A^*_k$ operator and $T_2 = 0$.

**Proof.** Let $P$ be the orthogonal projection of $H$ onto $Ker(T - \lambda)$. Since $T$ is M-Class $A^*_k$ operator, $M \left| T^k \right|^2 - |T^*|^2 \geq 0$ this implies that $0 \leq P \left( M \left| T^k \right|^2 - |T^*|^2 \right) P$, where $P|T^*|^2 P = \begin{pmatrix} |\lambda|^2 & T_2^* T_2 \\ 0 & 0 \end{pmatrix}$ and $P|T^k|^2 P = \begin{pmatrix} |\lambda|^{2k} & 0 \\ 0 & 0 \end{pmatrix}$. Therefore,

$$P \left( M \left| T^k \right|^2 \right) P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq P|T^*|^2 P = \begin{pmatrix} |\lambda|^2 & T_2^* T_2 \\ 0 & 0 \end{pmatrix}.$$ 

Hence $T_2 T_2^* = 0$ implies that $T_2 = 0$. Therefore,

$$0 \leq M \left| T^k \right|^2 - TT^* = \begin{pmatrix} 0 & 0 \\ 0 & M \left| T^k_3 \right|^2 - |T^*_3|^2 \end{pmatrix}.$$ 

Hence $T_3$ is M-Class $A^*_k$ operator. □
Theorem 2.3. If $T$ is $M$-Class $A_k^*$ operator for positive real numbers $M$ and $k$ and $(T - \lambda)x = 0$ for all $\lambda \neq 0$ and $x \in H$ then $(T - \lambda)^*x = 0$.

Proof. Using Schwarz’s and Holder McCarthy inequalities,

$$\|\lambda\|^2 \|x\|^2 = \|Tx\|^2$$

$$= f(\sigma(T)) - \pi_{00}(T)$$

$$= \langle T^*Tx, x \rangle$$

$$= \langle (T^*T)x, x \rangle$$

$$= \langle |T|^2x, x \rangle$$

$$\leq \left\langle M \left( |T|^k \right)^{2/k} x, x \right\rangle$$

$$\leq \left\langle M \left( |T|^k x, T^k x \right) \right\rangle^{2/k} \|x\|^{2(1-2/k)}$$

$$\leq \left\langle M \left( |T|^k x, T^k x \right) \right\rangle^{2/k} \|x\|^{2((k-2)/k)}$$

$$= M \|\lambda\|^2 \|x\|^2.$$

Hence $|\lambda|^2 \langle x, x \rangle = \langle T^*Tx, x \rangle = \left\langle M \left( |T|^k \right)^{2/k} x, x \right\rangle$. Since, $\left\langle M \left( |T|^k \right)^{2/k} x \right\rangle$ and $x$ are linearly independent. Therefore,

$$M \left( |T|^k \right)^{2/k} x = |\lambda|^2 x$$

$$\left\| \left( M |T|^k \right)^{2/k} x, x \right\|^2 = \left\| \left( M |T|^k \right)^{2/k} - \langle T^*T \rangle x, x \right\| = 0.$$

Therefore, $(TT^*)x = M \left( |T|^k \right)^{2/k} x = |\lambda|^2 x = 0 \Rightarrow (T - \lambda)^*x = 0.$

Corollary 2.4. If $T$ is $M$-Class $A_k^*$ operator for positive real numbers $M$ and $k$, $0 \neq \lambda \in \sigma_T(T)$ then $T$ is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $\text{Ker}(T - \lambda) \oplus \text{ran}(T - \lambda)^*$, where $T_3$ is $M$-Class $A_k^*$ and $\text{Ker}(T_3 - \lambda) = \{0\}$.

An operator $T$ is called normaloid if $r(T) = \|T\|$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. An operator $T$ is called hereditarily normaloid, if every part of it is normaloid. If $0 = \mu = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. An operator $T$ is called polaroid where $\pi(T)$ is the set of poles of the resolvent of $T$ and $\sigma(T)$ is the set of all isolated points of $\sigma(T)$. An operator $T$ is said to be isolid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. An operator $T$ is said to be reguloid if for every isolated point $\lambda$ of $\sigma(T)$, $\lambda - T$ is relatively regular. An operator $T$ is known as relatively regular if and only if $\text{ker} T$ and $(T(X))$ are complemented. Hence, we can say that Polaroid$\Rightarrow$Reguloid$\Rightarrow$Isolid.

Theorem 2.5. If $T$ is $M$-Class $A_k^*$ operator for positive real numbers $M$ and $k$, then for $\lambda \in C$, if $\sigma(T) = \lambda$ then $T = \lambda$.

Proof.

Case (A): Let $\lambda = 0$. It is obvious that, Hynonormal operator $\subset k$-paranormal $\subset$ normaloid [11]. Therefore $M$-Class $A_k^*$ operator is also normaloid. Therefore $T = 0$.

Case (B): Let $\lambda \neq 0$. Since $T$ is $M$-Class $A_k^*$ operator then $T$ is invertible, so is also $M$-Class $A_k^*$. Hence it is also normaloid. We know that, if $\lambda \in T$ then $\frac{1}{\lambda} \in T^{-1}$. Hence $\|T\| \ |T^{-1}| = |\lambda| \ |\frac{1}{\lambda}| = 1 \Rightarrow T$ is covexoid (i.e) $w(T) = \{\lambda\} \Rightarrow T = \lambda$. Since class $A_k^*$ operator are $k^*$ paranormal, by [8] class $A_k^*$ operators are normaloid by the inclusion property $M$-class $A_k^*$ operators are also normaloid and by [7] we have the following results.

Theorem 2.6. If $T$ is $M$-Class $A_k^*$ operator for positive real numbers $M$ and $k$, then
(1). T is Polaroid.

(2). T is isoloid.

(3). If \( \lambda \in \sigma(T) \) is a isolated point then \( E_\lambda H = \text{Ker}(T - \lambda) \) and hence \( \lambda \) is an eigen value of \( T \).

(4). If \( \lambda \neq 0 \) be an isolated point in \( \sigma(T) \), then \( E_\lambda \) is self adjoint and satisfies

\[ E_\lambda H = \text{Ker}(T - \lambda) = \text{Ker}(T - \lambda)^* \]

(5). \( T \) has SVEP, \( P(\lambda I - T) \leq 1 \) for every \( \lambda \in C \) and \( T^* \) is reguloid.

(6). Weyl’s theorem holds for \( T \) and \( T^* \). In addition, \( T^* \) has SVEP, then a-Weyl’s theorem holds for both \( T \) and \( T^* \) and for \( f(T) \) for every \( f \in H(\sigma(T)) \).

**Theorem 2.7.** If \( T \) is M-Class \( A_k \) operator for positive real numbers \( M \) and \( k \) then \( (T - \lambda) \) has finite ascent for \( \lambda \in C \).

**Proof.** By Theorem 2.5, for \( \lambda \neq 0 \)

\[ \text{Ker}(T - \lambda) \subseteq \text{Ker}(T - \lambda)^* \]

Hence if \( x \in \text{ker}(T - \lambda)^2 \), then \( (T - \lambda)^2(T - \lambda)x = 0 \) for \( \lambda \neq 0 \). Hence \( \|T - \lambda\|x\|^2 = 0 \) implies \( x \in \text{ker}(T - \lambda) \). Hence \( \ker(T - \lambda)^2 = \ker(T - \lambda) \). If \( \lambda = 0 \), it is sufficient to prove \( \ker T^{2k} \subseteq \ker T^k \). Let \( x \in \ker T^{2k} \) and \( x \neq 0 \). By holder MC Carthy inequality,

\[
0 = \|T^{2k}\|^2 = \langle \|T^{2k}\|^2 x, x \rangle \\
\geq \langle \|T^{2k}\|^{2/k} x, x \rangle^k \\
\geq \langle \|T^{2k}\|^2 x, x \rangle^k \|x\|^{2k} \\
= \|Tx\|^{2/k} \|x\|^{2k}
\]

Hence \( x \in \ker T \subseteq \ker T^k \Rightarrow T \) has finite ascent.

**Theorem 2.8.** If \( T \) is M-Class \( A_k \) operator for positive real numbers \( M \) and \( k \) then \( f(w(T)) = w(f(T)) \forall f \in (\sigma(T)) \).

**Proof.** If \( T \) is M-Class \( A_k \) operator for positive real numbers \( M \) and \( k \) then \( T \) is of finite Ascent (by Theorem 2.9) by [5], Proposition 38.5 ind\((T - \lambda) \neq 0 \) for all complex numbers \( \lambda \). Therefore by Theorem 5 of [13] \( f(w(T)) = w(f(T)) \forall f \in (\sigma(T)) \).

**Theorem 2.9.** If \( T \) is M-Class \( A_k \) operator for positive real numbers \( M \) and \( k \), then Weyl’s theorem holds for \( f(T) \) for every \( f \in (\sigma(T)) \).

**Proof.** By Theorem 2.7, \( T \) is isoloid and Weyl’s theorem holds for \( T \). By lemma of [6],

\[ f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)), \text{ for every } f \in H(\sigma(T)). \]

By Theorem 2.8,

\[ f(w(T)) = w(f(T)) \forall f \in (\sigma(T)). \]

Hence, \( \sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T)) - \pi_{00}(T) = f(w(T)) = w(f(T)) \). Hence, Weyl’s theorem holds for \( f(T) \) \forall f \in H(\sigma(T)) \).
3. Tensor Product of M-Class $A_k^*$ Operators

In this section, we proved that M-Class $A_k^*$ operators are closed under tensor product.

**Theorem 3.1.** If $T \in B(H)$ and $S \in B(K)$ are non-zero operators, then $T \otimes S$ is M-Class $A_k^*$ operator if and only if $T$ and $S$ are M-Class $A_k^*$ operators. $|T^*|^2 \leq M |T^k|^2/k$.

**Proof.** Assume that $T$ and $S$ are M-Class $A_k^*$ operators. Then

$$M \left| (T \otimes S)^k \right|^{2/k} = M |T^k|^{2/k} \otimes M |S^k|^{2/k} \geq |T^*|^2 \otimes |S^*|^2 = |T^* \otimes S^*|^2$$

Hence, $T \otimes S$ is M-Class $A_k^*$ operator.

Conversely, assume that $T \otimes S$ is M-Class $A_k^*$ operator. Without loss of generality, it is enough to show that $T$ is M-Class $A_k^*$ operator. Since $|T^* \otimes S^*|^2 \leq M \left| (T \otimes S)^k \right|^{2/k}$. We have $|T^*|^2 \otimes |S^*|^2 \leq M |T^k|^{2/k} \otimes M |S^k|^{2/k}$. Therefore,

$$\|T^*\|^2 = \sup \left\{ \left| \langle T^* x, x \rangle \right| : x \in H \text{ and } \|x\| = 1 \right\} \leq \sup \left\{ \left| M |T^k|^{2/k} x, x \right| : x \in H \text{ and } \|x\| = 1 \right\} \leq M \sup \left\{ \left| (T^k)^{1/k} x \right|^2 : x \in H \text{ and } \|x\| = 1 \right\} \leq M \|T^k\|^2$$

Similarly, $\|S^*\|^2 \leq M \|S^*\|^2$. Hence both T and S are M-Class $A_k^*$ operators.

References


