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Some Properties of Spectral Theorem Over Hilbert Space

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Abstract: The aim of this paper is to study some properties over spectral theorem of finite dimensional Hilbert space H.

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1. Introduction

In the study of spectrum of an operator $T \in \beta(H)$, we have seen that if T is an operator on H, then the set of all eigen values of T is a non-empty finite set of the complex plane and the finite number of eigen spaces corresponding to different eigen value or closed subspace for normal operator. The spectral theorem gives the relation among the eigen values, closed eigen spaces and the projection on these spaces [2,3]. Some authors studied spectral theorem of Some operators in Hilbert spaces see [4,6]. In the sequel, the following definitions are needed [1].

2. Main Results

Definition 2.1. *Let T be an operator on H. Then the scalar is called an eigen value of T if there exists a non-zero vector x in H such that $Tx = \Omega x$.*

Definition 2.2. *If Ω is an eigen value of T, then any non-zero vector x in H such that $Tx = \Omega x$ is called an Eigenvector.*

Definition 2.3. *The closed subspace M_Ω is called the Eigen space of T corresponding to the eigen value Ω.*

Definition 2.4. *Let H be a Hilbert space and let $N \in \beta(H)$ and N^* be the adjoint of N. Then N is said to be normal if $NN^* = N^*N$ (i.e.,) N is said to be normal if it commutes with its adjoint. Every self-adjoint is normal.*

Theorem 2.5. *Let T be an operator on a finite dimensional Hilbert space H with $\Omega_1, \Omega_2, \ldots, \Omega_m$ as the distinct eigen values of T and with $M_1, M_2, \ldots, M_m$ be their corresponding eigen spaces. If $P_1, P_2, \ldots, P_m$ are the projections on the spaces, then the following conditions are equivalent.*

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(1). The M_i's are pair-wise orthogonal and span H.

(2). The P_i's are pair-wise orthogonal and $P_1 + P_2 + \cdots + P_m = I$ and $T = \Omega_1 P + \Omega_2 P + \cdots + \Omega_m P$.

(3). T is a normal operator on H.

Theorem 2.6. *Let T be any arbitrary operator on H and N a normal operator on H. Then if T is commute with N, then T also commutes with N^*.*

Proof. From hypothesis, we get $TN = NT$. We shall show that $TN^n = N^n T$, for any positive integer n by induction. The result is true for $n = 1$. Let us assume that it is true for $(n-1)$. That is $TN^{n-1} = N^{n-1}T$. Now $TN^n = (TN^{n-1})N = (N^{n-1}T)N = N^{n-1}(TN)$. But $N^{n-1}(TN) = N^{n-1}(NT) = (N^{n-1}N)T = N^n T$, which proves that the conclusion is true for n. Therefore it is true for all values of n.

Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be a polynomial with n degree whose coefficients are complex. Then we have

$$\begin{aligned} p(N) &= a_0 I + a_1 N + \cdots + a_n N^n. \\ Tp(N) &= T(a_0 I + a_1 N + \cdots + a_n N^n) \\ &= a_0 TI + a_1 TN + \cdots + a_n TN^n \\ &= (a_0 IT + a_1 NT + \cdots + a_n N^n T) \\ &= (a_0 I + a_1 N + \cdots + a_n N^n)T \\ &= p(N)T. \end{aligned}$$

Hence we have proved that $Tp(N) = p(N)T$. Thus we have also shown that if T commutes with N, then T also commutes with any polynomial in N. Let $N = \Omega_1 P_1 + \Omega_2 P_2 + \cdots + \Omega_n P_n$ be the spectral resolution of the normal operator N. Then $N^* = (\Omega_1 P_1 + \Omega_2 P_2 + \cdots + \Omega_m P_m)^* = \Omega_1 P_1^* + \Omega_2 P_2^* + \cdots + \Omega_m P_m^*$. Since $P_i^* = P_i$ for all n, we get, $N^* = \Omega_1 P_1^* + \Omega_2 P_2^* + \cdots + \Omega_n P_n^*$. Each operator P_i is a polynomial in N. Therefore N^* is also a polynomial in N. So T commutes with N^*. □

References

[1] Paul R. Halmos, *Introduction to Hilbert Space and the theory of Spectral Multiplicity*, Courier Dover Publications, (1957).

[2] Paul R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, (1982).

[3] Edwin Hewitt and Karl Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, (1965).

[4] B. M. Nzimbi, G. P. Pokhariyal and S. K. Moindi, *A note on metric equivalence of some Operators*, Far East J. Math. Sci., 75(2)(2008), 301-18.

[5] B. M. Nzimbi, G. P. Pokhariyal and J. M. Khalaghai, *A Note on Self-Adjoint and Skew Adjoint Operators and their Extensions*, Pioneer Journal of Mathematics and Mathematical Sciences, 4(14)(2014), 127-29.

[6] P. G. Rugiri, S. W. Luketero and J. M. Khalaghai, *On equality of spectra and essential spectra of operators in Hilbert spaces*, Mathematical Theory and Modelling, 5(10)(2015), 110- 114.

[7] Titchmarsh and Edward Charles, *Eigen function expansions: Part 1*, Oxford University, Clarendon Press, (1946).

[8] Weidmann and Joachim, *Linear operators in Hilbert spaces, Graduate Texts in Mathematics*, Vol. 68, Springer-Verlag, Berlin, New York, (1980).

[9] Young and Nicholas, *An introduction to Hilbert space*, Cambridge University Press, (1988).