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Some Global Properties of Almost Pseudo Ricci Symmetric Manifolds Satisfying Codazzi Type of Ricci Tensor

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Abstract: The object of the present paper is to study some global properties of almost pseudo Ricci symmetric manifold satisfying Codazzi type of Ricci tensor. Among others, it is proved that if a compact, orientable $A(PRS)_n$ satisfying Codazzi type of Ricci tensor without boundary admits a non-isometric conformal transformation, then the $A(PRS)_n$ is isometric to a sphere. Also we obtain a sufficient condition for a compact, orientable $A(PRS)_n$ satisfying Codazzi type of Ricci tensor without boundary to be conformal to a sphere in E_{n+1}.

MSC: 53B35, 53B05.

Keywords: Almost pseudo Ricci symmetric manifold, compact, orientable manifold of this without boundary.

1. Introduction

In a recent paper De and Gazi [4] introduced a type of Riemannian manifold $(M^n, g)(n \geq 2)$, whose curvature tensor $\tilde{R}$ of type $(0, 4)$ satisfies the condition

$$(\nabla_X \tilde{R})(U,Y,Z,W) = [A(X)+B(X)]\tilde{R}(U,Y,Z,W) + A(U)\tilde{R}(X,Y,Z,W) + A(Y)\tilde{R}(U,X,Z,W) + A(Z)\tilde{R}(U,Y,X,W) + A(W)\tilde{R}(U,Y,Z,X), \tag{1}$$

where A, B are two non-zero 1-forms defined by

$$g(X,\mu) = A(X), \qquad g(X,\nu) = B(X), \tag{2}$$

for all vector fields X, ∇ denotes the operator of covariant differentiation with respect to the metric g. $\tilde{R}$ is defined by $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, where R is the curvature tensor of type $(1,3)$. Such a manifold was called an almost pseudo symmetric manifold and the 1-forms A and B were called the associated 1-forms of the manifold. The name almost pseudo symmetric was chosen because if $A = B$ in (1) then the manifold reduces to a pseudo symmetric manifold which is denoted by $(PS)_n$, introduced by M. C. Chaki [1]. It is to be noted that the notion of pseudo symmetry in the sense of Chaki [1] is different from that of R. Deszcz [7]. It may be mentioned that almost pseudo symmetric manifold is not

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a particular case of a weakly symmetric manifold introduced by Tamassy and Binh [10]. An n-dimensional almost pseudo symmetric manifold has been denoted by $A(PS)n$.

In the paper [4] De and Gazi have constructed concrete examples of $A(PS)n$ by the following theorems:

Theorem 1.1. *Let* $(\mathbb{R}^4, g)$ *be a Riemannian space endowed with the metric* g *given by*

$$ds^2 = g_{ij}dx^i dx^j = (X^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$*. Then* $(\mathbb{R}^4, g)$ *is an* $A(PS)n$ *with non-zero and non-constant scalar curvature.*

Theorem 1.2. *Let* $(\mathbb{R}^4, g)$ *be a Riemannian space endowed with the metric* g *given by*

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

$(i, j = 1, 2, 3, 4)$*, where* $q = \frac{e^{X^1}}{k^2}$ *and* k *is non-zero constant. Then* $(\mathbb{R}^4, g)$ *is an* $A(PS)n$ *whose scalar curvature is non-zero and non-constant.*

De and Gazi [4] also shown the physical significance of such a manifold in the field of relativity by proving the following theorem:

Theorem 1.3. *If an* $A(PS)n$ *spacetime satisfies cyclic Ricci tensor, the matter content is perfect fluid whose velocity vector field is the associated vector field of the space-time, then the acceleration vector of the fluid must be zero and the expansion scalar also be so.*

So $A(PS)n$ has some importance in the general theory of relativity. Considering this aspect we are motivated to study such a manifold. In 2007 M. C. Chaki and T. Kawaguchi [3] introduced the notion of almost pseudo Ricci symmetric manifold. A non-flat Riemannian manifold $(M^n, g)(n > 3)$, is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \tag{3}$$

where A and B are two non-zero 1-forms and ∇ denotes the operator of covariant differentiation with respect to the metric g. In such a case A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If $A= B$ in (3), then $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [2]. Then pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$. In a recent paper [5] De and Gazi studied conformally flat almost pseudo Ricci symmetric manifolds and existence of conformally flat almost pseudo Ricci symmetric manifolds is proved by concrete examples. Also De and De [6] prove that if an almost pseudo symmetric manifold admits a type of semi-symmetric non-metric connection, then the manifold $A(PS)_n$ reduces to an $A(PRS)_n$ with non-zero scalar curvature. A Riemannian manifold (M^n, g) is said to satisfy Codazzi type of Ricci tensor [8] if its Ricci tensor $S(X, Y)$ of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{4}$$

The object of the present paper is to study some global properties of almost pseudo Ricci symmetric manifold satisfying Codazzi type of Ricci tensor. It is prove that if a compact, orientable $A(PRS)_n$ satisfying Codazzi type of Ricci tensor without boundary admits a non-isometric conformal transformation, then the $A(PRS)_n$ is isometric to a sphere. Also we obtain a sufficient condition for a compact, orientable $A(PRS)_n$ without boundary to be conformal to a sphere in E_{n+1}.

2. The Form of the Ricci Curvature in an $A(PRS)_n$

In the study of an $A(PRS)_n$ an important role is played by the 1-form B defined by (2).

Theorem 2.1. *In an $A(PRS)_n$, satisfying Codazzi type of Ricci tensor the Ricci tensor is of the form*

$$S(Y,Z) = rT(Y)T(Z)$$

where T is a non-zero 1-form defined by $T(X) = g(X,\rho)$ and r is the scalar curvature.

Proof. Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(LX,Y) = S(X,Y) \tag{5}$$

for all X and Y. From (3) it follows that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = B(X)S(Y,Z) - B(Y)S(X,Z), \tag{6}$$

since S is of Codazzi type, then from (6) we get

$$B(X)S(Y,Z) = B(Y)S(X,Z). \tag{7}$$

Putting $Y = Z = e_i$ where $\{e_i\}$, $i = 1,2,3,\dots,n$ is an orthonormal basis of the tangent space at each point of the manifold and i summed for $1 \le i \le n$. Then using (6), (7) implies

$$rB(X) = B(LX). \tag{8}$$

From (7) we get

$$\begin{aligned} B(\lambda)S(Y,Z) &= B(Y)S(\lambda,Z) \\ &= B(Y)g(LZ,\lambda) \\ &= B(Y)B(LZ). \end{aligned}$$

Hence

$$\begin{aligned} S(Y,Z) &= \frac{1}{B(\lambda)}B(Y)B(LZ) \\ &= \frac{r}{B(\lambda)}B(Y)B(Z) \\ &= r\frac{B(Y)}{\sqrt{B}(\lambda)}\frac{B(Z)}{\sqrt{B}(\lambda)} \\ &= rT(Y)T(Z), \end{aligned} \tag{9}$$

where $T(X) = \frac{B(X)}{\sqrt{B}(\lambda)}$. This prove the theorem. □

We define a unit vector ρ as follows

$$g(X,\rho) = T(X). \tag{10}$$

If $r = 0$ we get from (9) that $S = 0$ which is not admissible. Hence we can state following:

Theorem 2.2. *In an $A(PRS)_n$ satisfying Codazzi type of Ricci tensor the scalar curvature is non-zero.*

Using (10), from (9) we get

$$S(X,X) = r[g(X,\rho)]^2 \tag{11}$$

for all X. Hence $S(\rho,\rho) = r$, since ρ is an unit vector. Let θ be the angle between ρ and an arbitrary vector X, then

$$\begin{aligned} cos\theta &= \frac{g(X,\rho)}{\sqrt{g(\rho,\rho)}\sqrt{g(X,X)}} \\ &= \frac{g(X,\rho)}{g(X,X)} \end{aligned}$$

[by hypothesis $g(\rho,\rho) = 1$]. Hence

$$[g(X,\rho)]^2 \leq g(X,X) = \mid X \mid^2$$

and if $r > 0$, then $r \mid X \mid^2 \geq r[g(X,\rho)]^2$. Thus we have from (11)

$$S(X,X) \leq r \mid X \mid^2 . \tag{12}$$

Let l^2 be the square of the length of the Ricci tensor S. Then

$$l^2 = S(Le_i, e_i) \tag{13}$$

where $\{e_i\}$, $i = 1,2,3,\ldots,n$ is an orthonormal basis of the tangent space at each point of the manifold. From (9) we have

$$\begin{aligned} S(Le_i, e_i) &= rT(Le_i)T(e_i) \\ &= rg(Le_i,\rho)g(e_i,\rho) \\ &= rg(L\rho,\rho) \\ &= rS(\rho,\rho) \\ &= r^2 \end{aligned}$$

i.e. $l^2 = r^2$. So we have the following:

Theorem 2.3. *In an $A(PRS)_n$, satisfying Codazzi type of Ricci tensor the length of the Ricci tensor is $l = r$.*

Since by assumption $A(PRS)_n$ satisfies Codazzi type of Ricci tensor, i.e. $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$, then by contraction it follows that the scalar curvature is constant. Hence from Theorem 2.3 we get the length of the Ricci tensor is constant.

3. Sufficient Condition for a Compact Orientable $A(PRS)_n$ to be Conformal to a Sphere in E_{n+1} Satisfying Codazzi Type of Ricci Tensor

We begin with the definition of conformality of one Riemannian manifold to another.

Let (M,g) and $(\bar{M},\bar{g})$ be two n-dimensional Riemannian manifolds. If there exit a one-one differentiable mapping $(M,g) \longrightarrow (\bar{M},\bar{g})$ such that the angle between any two vectors at point p of M is always equal to that of the corresponding two vectors at the corresponding point $\bar{p}$ of $\bar{M}$, then (M,g) is said to be conformal to $(\bar{M},\bar{g})$. Y. Watanabe [11] has given a sufficient condition for conformality of an n-dimensional sphere immersed in E_{n+1}. Its statement is as follows:

If in an n-dimensional Riemannian manifold M, there exist a non-parallel vector field X such that the condition

$$\int_M S(X,X)dv = \frac{1}{2}\int_M \mid dX \mid^2 dv + \frac{n-1}{n}\int_M (\delta X)^2 dv \tag{14}$$

holds, then M is conformal to a sphere in E_{n+1}, where dv is the volume element of M and dX and δX are the curl and divergence of X respectively.

In this section we consider a compact and orientable almost pseudo Ricci symmetric $A(PRS)_n = M$ without boundary having generator ρ (defined by (10)). It satisfies (9) and (10). Hence

$$S(\rho,\rho) = r.$$

In virtue of this and by taking ρ for X the condition (14) takes the following form

$$\int_M r dv = \frac{1}{2}\int_M \mid d\rho \mid^2 dv + \frac{n-1}{n}\int_M (\delta\rho)^2 dv. \tag{15}$$

From (9) we get $S(X,Y) = rT(X)T(Y)$ which implies that

$$S(X,\rho) = rT(X), \text{ since } T(\rho) = 1. \tag{16}$$

Suppose ρ be a parallel vector field ([9]). Then $\nabla_X \rho = 0$. Hence by Ricci identity we obtain

$$R(X,Y)\rho = 0. \tag{17}$$

Contracting X we get from (17)

$$S(Y,\rho) = 0. \tag{18}$$

From (16) we get since $r \neq 0$ and $T(X) \neq 0$ in an $A(PRS)_n$ which implies that

$$S(Y,\rho) \neq 0.$$

Hence ρ can not be a parallel vector field. Thus if in an n-dimensional compact, orientable almost pseudo Ricci symmetric manifold M without boundary the vector field ρ is a non-parallel vector field. If in such a case the condition (15) is satisfied, then by Watanabe's condition (14) M is conformal to a sphere in E_{n+1}. We can therefore state the following:

Theorem 3.1. *If a compact, orientable almost pseudo Ricci symmetric manifold $M = A(PRS)_n$ without boundary, satisfies the condition (15), then the manifold $A(PRS)_n$ is conformal to a sphere immersed in E_{n+1}.*

4. Killing Vector Field in a Compact, Orientable $A(PRS)_n (n \geq 3) = M$ Without Boundary

In this section we consider a compact, orientable $A(PRS)_n(n \geq 3) = M$ without boundary. It is known [11] that in such a manifold M the following relation holds

$$\int_M [S(X,X) - \mid \nabla X \mid^2 - (divX)^2] dv \leq 0, \tag{19}$$

for all X. If X is a Killing vector field, then $divX = 0$ ([13], p.43). Hence (19) takes the form

$$\int_M [S(X,X) - |\nabla X|^2] dv = 0. \tag{20}$$

We have $r \neq 0$ in an $A(PRS)_n$. Hence either $r > 0$ or $r < 0$. First we suppose that $r > 0$. Then by (13) $S(X,X) \leq r \mid X \mid^2$. Therefore

$$r \mid X \mid^2 - \mid \nabla X \mid^2 \geq S(X,X) - \mid \nabla X \mid^2 .$$

Consequently

$$\int_M [r \mid X \mid^2 - \mid \nabla X \mid^2] dv \geq \int_M [S(X,X) - \mid \nabla X \mid^2] dv$$

and by (20)

$$\int_M [r \mid X \mid^2 - \mid \nabla X \mid^2] dv \geq 0. \tag{21}$$

Nest we suppose that $r < 0$. Then

$$\int_M [r \mid X \mid^2 - \mid \nabla X \mid^2] dv \leq 0. \tag{22}$$

From (21) and (22) it follows that $X = 0$. This leads to the following result:

Theorem 4.1. *A compact, orientable $M = A(PRS)_n (n \geq 3)$ without boundary, satisfying Codazzi type of Ricci tensor there exists no non-zero Killing vector field in this manifold.*

We now state the two following corollaries of the above theorem which follow easily from it.

Corollary 4.2. *In a compact, orientable $A(PRS)_n (n \geq 3) = M$ without boundary, the vector field ρ is a Killing vector field, then the following relation holds.*

$$\int_M (r \mid \rho \mid^2 - \mid \nabla \rho \mid^2] dv \geq 0.$$

Corollary 4.3. *In a compact, orientable $A(PRS)_n (n \geq 3) = M$ without boundary the vector field ρ of the manifold cannot be a Killing vector field.*

5. Harmonic Vector Fields in a Compact, Orientable $A(PRS)_n(n \geq 3) = M$ without Boundary

A vector field X in a Riemannian manifold M is said to be harmonic if ([12], p.28)

$$curl\ X = 0 \ \ and \ \ div\ X = 0. \tag{23}$$

It is known ([12], p.26) that in a compact, orientable Riemannian manifold M the following relation holds for any vector field X.

$$\int_M [S(X,X) + \mid \nabla X \mid^2 - (divX)^2] dv = 0 \tag{24}$$

where dv denotes the volume element of M. Now considering a Compact, Orientable $A(PRS)_n (n \geq 3) = M$ without boundary, it follows from (24) that if in such a manifold X is a harmonic vector field, then

$$\int_M [S(X,X) + \mid \nabla X \mid^2] dv = 0 \tag{25}$$

(by (23)). Since in $A(PRS)_n$, $S(X,X) = r[g(X,\rho)]^2$ by (11) it follows from (25) that

$$\int_M [r(g(X,\rho))^2 + \mid \nabla X \mid^2] dv = 0. \tag{26}$$

Since $r > 0$, from (26) we get

$$g(X,\rho) = 0 \quad and \quad \nabla X = 0. \tag{27}$$

From the first of (27) it follows that X is orthogonal to ρ, and from the second it follows that the vector field X is parallel. Hence we have:

Theorem 5.1. *In a Compact, Orientable $A(PRS)_n(n \geq 3) = M$ without boundary, satisfying Codazzi type of Ricci tensor any harmonic vector field in the $A(PRS)_n$ is parallel and orthogonal to the vector field ρ of the manifold.*

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