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On Some Properties of the Real Matrix-Variate Gamma Function

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Abstract: In this paper, we prove that the real matrix-variate gamma function is logarithmically convex and as a consequence, we derive some inequalities involving the function. Also, we introduce the real matrix-variate digamma function and as applications, we derive some inequalities for certain ratios of the real matrix-variate gamma function. The methods of proofs are analytical in nature.

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1. Introduction

Arguably, the gamma function is one of the most important functions in mathematical analysis. It was descovered by Leonhard Euler in his pursuit to extend the factorial notation to non-integer values. It is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0. \tag{1}$$

Its logarithmic derivative, which is termed the digamma function is given as

$$\begin{aligned}\psi(x) &= \frac{d}{dx}\ln\Gamma(x)\\ &= -\gamma + \int_0^\infty \left(\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\right)dt,\end{aligned} \tag{2}$$

where γ is the Euler-Mascheroni constant. Due to its importance, the gamma function has been generalized and extended in different ways. See for instance [1–6, 8, 9, 11, 16]. In the midst of these generalizations, our focus will be on the one due Mathai and Haubold [11], which is the real matrix-variate gamma function. The real matrix-variate gamma function introduced by Mathai and Haubold [11] is defined as

$$\Gamma_n(x) = \int_{A=A'>0} |A|^{x-\frac{n+1}{2}}e^{-tr(A)}dA, \quad x > \frac{n-1}{2}, \tag{3}$$

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where the integration is carried out over $n \times n$ symmetric positive definite matrices, and A is $n \times n$ real symmetric positive definite matrix. It is clear that $\Gamma_n(x)$ reduces to $\Gamma(x)$ when $n = 1$. For further information on this function, one can refer to [7, 12] and the references therein.

When the integral (3) is evaluated, the function is seen in the form

$$\Gamma_n(x) = \pi^{\frac{n(n-1)}{4}} \Gamma(x) \Gamma\left(x - \frac{1}{2}\right) \ldots \Gamma\left(x - \frac{n-1}{2}\right), \quad x > \frac{n-1}{2}. \tag{4}$$

In [10], explicit evaluation of matrix-variate gamma and beta integrals in the complex domain for the order of the matrix $n = 1, 2$ were given. A formal definition of fractional integrals in the complex matrix-variate cases was also given.

In [15], the real matrix-variate gamma function was extended as

$$\Gamma_{b,n}(x) = \int_{A=A'>0} |A|^{x-\frac{n+1}{2}} e^{-tr(A+bA^{-1})} dA. \tag{5}$$

The particular case where $b = 0$, is reduced to equation (3). When $(n, b) = (1, 0)$, equation (5) is reduced to equation (1) and when $n = 1$, equation (5) is reduced to the form

$$\Gamma_{b,n}(x) = \int_0^\infty e^{-t-\frac{b}{t}} t^{x-1} \, dt, \qquad x > 0. \tag{6}$$

Nagar and other researchers further studied a number of properties of this function and some of its applications to statistical distribution theory [14]. In [17], the real matrix-variate extended gamma and beta functions and their density functions were defined. The extended real matrix-variate beta function was also applied to extend the real matrix-variate hypergeometric and confluent hypergeometric functions via zonal polynomials.

In the paper [13], a generalized extended matrix-variate gamma function denoted by $\Gamma_n^{(\alpha,\beta)}(x, b)$ was defined as

$$\Gamma_n^{(\alpha,\beta)}(x, b) = \int_{A=A'>0} |A|^{x-\frac{n+1}{2}} \phi\left(\alpha; \beta; -A - A^{\frac{-1}{2}} b A^{\frac{-1}{2}} dA\right). \tag{7}$$

From this definition, it is clear that for α=β, the generalized extended matrix-variate gamma function reduces to an extended matrix-variate gamma function as in equation (5). That is $\Gamma_n^{(\alpha,\alpha)}(x, b) = \Gamma_n(x, b)$. Further, if $\alpha = \beta$ and $b = 0$, then for $\Re(x) > \frac{n-1}{2}$, the generalized extended matrix-variate gamma function reduces to the real matrix-variate gamma function as in (3).

In this paper, we prove that the real matrix-variate gamma function is logarithmically convex. Subsequently, we derive some new inequalities involving the function. Also, we introduce the real matrix-variate digamma function and as applications, we derive some inequalities for certain ratios of the real matrix-variate gamma function. We present our findings in the following section.

2. Main Results

Theorem 2.1. *The real matrix-variate gamma function, $\Gamma_n(x)$ is logarithmically convex. In other words, for $x, y > \frac{n-1}{2}$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality*

$$\Gamma_n\left(\frac{x}{a} + \frac{y}{b}\right) \leq [\Gamma_n(x)]^{\frac{1}{a}} [\Gamma_n(y)]^{\frac{1}{b}}, \tag{8}$$

is satisfied.

Proof. Let $x, y > \frac{n-1}{2}$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then by (3), we have

$$\Gamma_n\left(\frac{x}{a} + \frac{y}{b}\right) = \int_{A=A'>0} |A|^{\left(\frac{x}{a}+\frac{y}{b}\right)-\frac{n+1}{2}} e^{-tr(A)} dA.$$

Since $\frac{1}{a} + \frac{1}{b} = 1$, then we have

$$\begin{aligned}\Gamma_n\left(\frac{x}{a} + \frac{y}{b}\right) &= \int_{A=A'>0} |A|^{\frac{x}{a}+\frac{y}{b}-\frac{n+1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)} e^{-tr(A)\left(\frac{1}{a}+\frac{1}{b}\right)} dA \\ &= \int_{A=A'>0} |A|^{\frac{x}{a}-\frac{n+1}{2a}} e^{-\frac{tr(A)}{a}} |A|^{\frac{y}{b}-\frac{n+1}{2b}} e^{-\frac{tr(A)}{b}} dA \\ &= \int_{A=A'>0} |A|^{\left(x-\frac{n+1}{2}\right)\frac{1}{a}} e^{-\frac{tr(A)}{a}} |A|^{\left(y-\frac{n+1}{2}\right)\frac{1}{b}} e^{-\frac{tr(A)}{b}} dA.\end{aligned}$$

Now by applying Holder's inequality, we obtain

$$\begin{aligned}&\int_{A=A'>0} |A|^{\left(x-\frac{n+1}{2}\right)\frac{1}{a}} e^{-\frac{tr(A)}{a}} |A|^{\left(y-\frac{n+1}{2}\right)\frac{1}{b}} e^{-\frac{tr(A)}{b}} dA \\ &\qquad\le \left(\int_{A=A'>0} \left[|A|^{\left(x-\frac{n+1}{2}\right)\frac{1}{a}} e^{-\frac{tr(A)}{a}}\right]^a dA\right)^{\frac{1}{a}} \left(\int_{A=A'>0} \left[|A|^{\left(y-\frac{n+1}{2}\right)\frac{1}{b}} e^{-\frac{tr(A)}{b}}\right]^b dA\right)^{\frac{1}{b}} \\ &\qquad= \left(\int_{A=A'>0} |A|^{x-\frac{n+1}{2}} e^{-tr(A)} dA\right)^{\frac{1}{a}} \left(\int_{A=A'>0} |A|^{y-\frac{n+1}{2}} e^{-tr(A)} dA\right)^{\frac{1}{b}} \\ &\qquad= \left[\Gamma_n(x)\right]^{\frac{1}{a}} \left[\Gamma_n(y)\right]^{\frac{1}{b}},\end{aligned}$$

which gives (8). This concludes the proof. □

Corollary 2.2. *For $x > \frac{n-1}{2}$, the inequality*

$$\Gamma_n(x)\,\Gamma_n''(x) \ge \left[\Gamma_n'(x)\right]^2, \tag{9}$$

is satisfied.

Proof. Since $\Gamma_n(x)$ is log-convex, then $[\ln\Gamma_n(x)]'' \ge 0$ for all $x > \frac{n-1}{2}$. This implies that,

$$\begin{aligned}\left[\ln\Gamma_n(x)\right]'' = \left[\frac{\Gamma_n'(x)}{\Gamma_n(x)}\right]' &= \frac{\Gamma_n''(x)\,\Gamma_n(x) - \Gamma_n'(x)\,\Gamma_n'(x)}{\left[\Gamma_n(x)\right]^2} \\ &= \frac{\Gamma_n''(x)\,\Gamma_n(x) - \left[\Gamma_n'(x)\right]^2}{\left[\Gamma_n(x)\right]^2} \ge 0.\end{aligned}$$

Hence, $\Gamma_n''(x)\,\Gamma_n(x) - \left[\Gamma_n'(x)\right]^2 \ge 0$ which completes the proof. □

Theorem 2.3. *For $x, y > \frac{n-1}{2}$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality*

$$\Gamma_n(x + y) \le \left[\Gamma_n(ax)\right]^{\frac{1}{a}} \left[\Gamma_n(by)\right]^{\frac{1}{b}}, \tag{10}$$

is satisfied.

Proof. Let $x, y > \frac{n-1}{2}$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then by (3), we have

$$\Gamma_n(x + y) = \int_{A=A'>0} |A|^{(x+y)-\frac{n+1}{2}} e^{-tr(A)} dA$$

$$= \int_{A=A'>0} |A|^{x-\frac{n+1}{2a}} e^{-\frac{tr(A)}{a}} |A|^{y-\frac{n+1}{2b}} e^{-\frac{tr(A)}{b}} dA,$$

and by applying Holder's inequality, we obtain

$$\begin{aligned}
&\int_{A=A'>0} |A|^{x-\frac{n+1}{2a}} e^{-\frac{tr(A)}{a}} |A|^{y-\frac{n+1}{2b}} e^{-\frac{tr(A)}{b}} dA \\
&\leq \left(\int_{A=A'>0} \left[|A|^{x-\frac{n+1}{2a}} e^{-\frac{tr(A)}{a}} \right]^a dA \right)^{\frac{1}{a}} \left(\int_{A=A'>0} \left[|A|^{y-\frac{n+1}{2b}} e^{-\frac{tr(A)}{b}} \right]^b dA \right)^{\frac{1}{b}} \\
&= \left(\int_{A=A'>0} |A|^{ax-\frac{n+1}{2}} e^{-tr(A)} dA \right)^{\frac{1}{a}} \left(\int_{A=A'>0} |A|^{by-\frac{n+1}{2}} e^{-tr(A)} dA \right)^{\frac{1}{b}} \\
&= \left[\Gamma_n(ax) \right]^{\frac{1}{a}} \left[\Gamma_n(by) \right]^{\frac{1}{b}}
\end{aligned}$$

which gives (10) and this completes the proof. □

The following lemma which is well known in the literature is called Young's inequality for scalars.

Lemma 2.4. *Let $x, y \geq 0$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then,*

$$xy \leq \frac{x^a}{a} + \frac{y^b}{b}, \tag{11}$$

or equivalently,

$$x^{\frac{1}{a}} y^{\frac{1}{b}} \leq \frac{x}{a} + \frac{y}{b}. \tag{12}$$

Corollary 2.5. *For $x, y \geq \frac{n-1}{2}$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality*

$$\Gamma_n(x+y) \leq \frac{\Gamma_n(ax)}{a} + \frac{\Gamma_n(by)}{b} \tag{13}$$

is satisfied.

Proof. Let $x, y \geq 0$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then from Theorem 2.3, it is obtained that,

$$\Gamma_n(x+y) \leq \left[\Gamma_n(ax) \right]^{\frac{1}{a}} \left[\Gamma_n(by) \right]^{\frac{1}{b}}. \tag{14}$$

Also, by (12), we have,

$$\left[\Gamma_n(ax) \right]^{\frac{1}{a}} \left[\Gamma_n(by) \right]^{\frac{1}{b}} \leq \frac{\Gamma_n(ax)}{a} + \frac{\Gamma_n(by)}{b}. \tag{15}$$

Now combining (14) and (15) gives the resule (13). □

In what follows, we introduce the real matrix-variate digamma function and then apply it to establish some inequaties for certain ratios of the real matrix-variate gamma function.

Proposition 2.6. *Let the real matrix-variate digamma function be defined as the logarithmic derivative of the real matrix-variate gamma function. That is,*

$$\Psi_n(x) = \frac{d}{dx} \ln \Gamma_n(x) = \frac{\Gamma_n'(x)}{\Gamma_n(x)}, \quad x > \frac{n-1}{2}.$$

Then the series representation

$$\Psi_n(x) = \sum_{i=1}^{n} \psi\left(x - \frac{i-1}{2} \right), \quad x > \frac{n-1}{2}, \tag{16}$$

is valid, where $\psi(x)$ is the ordinary digamma function.

Proof. Recall from (4) that

$$\Gamma_n(x) = \pi^{\frac{n(n-1)}{4}} \Gamma(x) \Gamma\left(x - \frac{1}{2}\right) \Gamma(x-1) \Gamma\left(x - \frac{3}{2}\right) \dots \Gamma\left(x - \frac{n-1}{2}\right),$$

for $x > \frac{n-1}{2}$. Then,

$$\ln \Gamma_n(x) = \frac{n(n-1)}{4} \ln \pi + \ln \Gamma(x) + \ln \Gamma\left(x - \frac{1}{2}\right) + \cdots + \ln \Gamma\left(x - \frac{n-1}{2}\right),$$

which implies that

$$\begin{aligned}
\Psi_n(x) = \frac{d}{dx} \ln \Gamma_n(x) &= \frac{\Gamma'(x)}{\Gamma(x)} + \frac{\Gamma'\left(x - \frac{1}{2}\right)}{\Gamma\left(x - \frac{1}{2}\right)} + \frac{\Gamma'(x-1)}{\Gamma(x-1)} + \cdots + \frac{\Gamma'\left(x - \frac{n-1}{2}\right)}{\Gamma\left(x - \frac{n-1}{2}\right)} \\
&= \psi(x) + \psi\left(x - \frac{1}{2}\right) + \psi(x-1) + \psi\left(x - \frac{3}{2}\right) + \cdots + \psi\left(x - \frac{n-1}{2}\right) \\
&= \sum_{i=1}^{n} \psi\left(x - \frac{i-1}{2}\right).
\end{aligned}$$

This concludes the proof. □

Proposition 2.7. *The real matrix-variate digamma function, $\Psi_n(x)$ has the representation*

$$\Psi_n(x) = -n\gamma + \sum_{i=1}^{n} \int_0^\infty \frac{e^{-t} - e^{-\left(x - \frac{i-1}{2}\right)t}}{1 - e^{-t}} \, dt. \tag{17}$$

Proof. This is obtained from (16) by using (2). □

Proposition 2.8. *The function $\Psi_n(x)$ is increasing for all $x > \frac{n-1}{2}$.*

Proof. Method 1: Since $\psi(x)$ is increasing for all $x > 0$, it follows easily from (16) that

$$\Psi_n'(x) = \sum_{i=1}^{n} \psi'\left(x - \frac{i-1}{2}\right) > 0,$$

which concludes the proof.

Method 2: Let $\frac{n-1}{2} < x < y$. Then, since $\psi(x)$ is increasing for all $x > 0$, we have

$$\begin{aligned}
\Psi_n(x) - \Psi_n(y) &= \sum_{i=1}^{n} \psi\left(x - \frac{i-1}{2}\right) - \sum_{i=1}^{n} \psi\left(y - \frac{i-1}{2}\right) \\
&= \sum_{i=1}^{n} \left[\psi\left(x - \frac{i-1}{2}\right) - \psi\left(y - \frac{i-1}{2}\right)\right] < 0,
\end{aligned}$$

which gives the desired result. Method 3: By direct differentiation and by using (9), we obtain

$$\Psi_n'(x) = \frac{\Gamma_n''(x)\Gamma_n(x) - [\Gamma_n'(x)]^2}{[\Gamma_n(x)]^2} \geq 0,$$

which also gives the desired result. These conclude the proof. □

Theorem 2.9. *The function*

$$f(x) = \frac{\Gamma_n(kx)}{[\Gamma_n(x)]^k}, \quad k \geq 1, \tag{18}$$

is increing on $\left(\frac{n-1}{2}, \infty\right)$ and consequently, the inequality

$$\left[\frac{\Gamma_n(y)}{\Gamma_n(x)}\right]^k \leq \frac{\Gamma_n(ky)}{\Gamma_n(kx)}, \tag{19}$$

holds for $\frac{n-1}{2} < x < y$.

Proof. For $\frac{n-1}{2} < x < y$ and $k \geq 1$, let $g(x) = \ln f(x)$. Then $g(x)' = k[\Psi_n(kx) - \Psi_n(x)] \geq 0$, which follows from the increasing property of $\Psi_n(x)$. Thus $g(x)$ is increasing. In view of this, $f(x)$ is also increasing. Hence for $\frac{n-1}{2} < x < y$, we have

$$\frac{\Gamma_n(kx)}{[\Gamma_n(x)]^k} \leq \frac{\Gamma_n(ky)}{[\Gamma_n(y)]^k},$$

which when rearranged gives (19). This completes the proof. □

Theorem 2.10. *Let $\lambda \geq 1$. Then the function*

$$f(x) = \frac{\Gamma_n(1+x)^\lambda}{\Gamma_n(1+\lambda x)}, \tag{20}$$

is decreasing. Hence the inequalities

$$\frac{[\Gamma_n(2)]^\lambda}{\Gamma_n(1+\lambda)} \leq \frac{\Gamma_n(1+x)^\lambda}{\Gamma_n(1+\lambda x)} \leq \frac{\Gamma_n(1)^\lambda}{\Gamma_n(1)}, x \in [0,1], \tag{21}$$

and

$$\frac{\Gamma_n(1+x)^\lambda}{\Gamma_n(1+\lambda x)} \leq \frac{[\Gamma_n(2)]^\lambda}{\Gamma_n(1+\lambda)}, x \in (1,\infty), \tag{22}$$

are valid.

Proof. Let $h(x) = \ln f(x) = \lambda \ln \Gamma_n(1+x) - \ln \Gamma_n(1+\lambda x)$. Then

$$\begin{aligned} h'(x) &= \frac{\lambda \Gamma_n'(1+x)}{\Gamma_n(1+x)} - \frac{\lambda \Gamma_n'(1+\lambda x)}{\Gamma_n(1+\lambda x)} \\ &= \lambda [\Psi_n(1+x) - \Psi_n(1+\lambda x)] \leq 0, \end{aligned}$$

which implies that $h(x)$ is decreasing. In view of this, $f(x)$ is also decreasing. Then for $x \in [0,1]$, we have $f(1) \leq f(x) \leq f(0)$, which gives inequality (21). Also, for $x \in (1,\infty)$, we have $f(x) \leq f(1)$, which gives inequality (22). □

3. Conclusion

In this work, we have proved that the real matrix-variate gamma function is logarithmically convex and consequently, we derived some inequalities involving the function. We also introduced the real matrix-variate digamma function and as applications, we derived some inequalities for certain ratios of the real matrix-variate gamma function.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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