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Exact Travelling Wave Solutions of Some Non-linear Evolution Equations Using Rational Sine-Cosine Function Method

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Abstract: In this paper, we establish a variety of exact travelling wave solutions by using rational sine cosine function method for Boussinesq Equation, Korteweg-de Vries Equation, Gardner Equation and Generalized Boussinesq Burgers Equations which are nonlinear evolution equations. It is shown that the rational sine cosine function method provides a powerful mathematical tool for solving many nonlinear evolution equations in applied mathematics, mathematical physics and engineering.

Keywords: NLEEs, Exact Travelling Wave Solutions, Rational Sine-Cosine method.
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1. Introduction

The studies of Nonlinear Evolution Equations (NLEEs) have attracted the attention of many scientists especially in Fluid Mechanics. It is important to find the exact solutions of the relevant NLEEs. In the recent years, various methods for obtaining exact traveling solutions of NLEEs have been developed and extended, as for instance, the Cole-Hopf transformation method [1, 2], the Darboux transformation method [3], the tanh method [4], the Hirota's bilinear method [5], the sine-Gordon equation expansion method [6] etc.

In this paper, we apply the Rational Sine-Cosine Function Method [7] for obtaining exact travelling wave solutions to Boussinesq Equation, Korteweg-de Vries Equation, Gardner Equation and Generalized Boussinesq Burgers Equations. By this method, one can solve many other NLEEs arose in Nonlinear Science and Engineering. Samir Hamdi, Brian Morse, Bernard Halphen and William Schiesser, [8] discussed conservation laws and invariants of motion for nonlinear internal waves. Bin Zheng, [9] studied travelling Wave Solutions for some nonlinear evolution equations by the first integral method. J. L. Bona, M. Chen and J. C. Saut, [10] generalized Boussinesq equation and other systems for small amplitude long waves in nonlinear dispersive media.

This paper is organized as follows: In section 2, the Rational Sine-Cosine Function Method is discussed. In section 3, this method is applied to the nonlinear evolution equations. In section 4, conclusions and discussions are given.

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2. Mathematical Formulation

Nonlinear Evolution Equations can be written as

$$M\left(u, u_t, u_{tt}, u_x, u_{xx}, u_{xt}, u_{xxt}, u_{xtt}\ldots\right) = 0 \tag{1}$$

Consider the following transformation,

$$u(x,t) = U(\eta) \tag{2}$$

where $\eta = x - ct$, (c is a parameter). Now,

$$\begin{aligned}
\frac{\partial\left(..\right)}{\partial t} &= \frac{\partial\eta}{\partial t}\frac{d\left(..\right)}{d\eta} = -c\frac{d(..)}{d\eta},\\
\frac{\partial^2\left(..\right)}{\partial t^2} &= c^2\frac{d^2(..)}{d\eta^2}\ etc.\\
\frac{\partial\left(..\right)}{\partial x} &= \frac{\partial\eta}{\partial x}\frac{d\left(..\right)}{d\eta} = \frac{d(..)}{d\eta},\\
\frac{\partial^2\left(..\right)}{\partial x^2} &= \frac{d^2(..)}{d\eta^2}\ etc.\\
\frac{\partial^2\left(..\right)}{\partial x\partial t} &= -c\frac{d^2(..)}{d\eta^2}\ etc.
\end{aligned} \tag{3}$$

Using the changes shown in Equation (3), the NLEE Equation (1) reduces to a Nonlinear Ordinary Differential Equation (NLODE) of the form

$$N(U, U', U'', U''', \ldots) = 0, \tag{4}$$

Where,

$$U' = \frac{dU}{d\eta}, U'' = \frac{d^2U}{d\eta^2}, \ldots$$

If all the terms in the Reduced NLODE Equation (4) contain derivatives, the equation is to be integrated out as the terms contain derivatives and the integration constants are taken as zero for the convenience of our calculation. Then, we assume the solution of Equation (4) as

$$U(\eta) = \frac{a_0}{1 + a_1\cos\mu\eta} \tag{5}$$

and

$$U(\eta) = \frac{a_0}{1 + a_1\sin\mu\eta} \tag{6}$$

Where a_0, a_1 and $\mu$are parameters. If we take Equation (5) as the solution of Equation (4), we have,

$$\begin{aligned}
U' &= \frac{a_0a_1\mu\sin\mu\eta}{\left(1 + a_1\cos\mu\eta\right)^2},\\
U'' &= \frac{2a_0a_1^2\mu^2 + a_0a_1\mu^2\cos\mu\eta - a_0a_1^2\mu^2\cos^2\mu\eta}{\left(1 + a_1\cos\mu\eta\right)^3},\ etc.
\end{aligned} \tag{7}$$

If we take Equation (6) as the solution of Equation (4), we must have,

$$\begin{aligned}
U' &= -\frac{a_0a_1\mu\cos\mu\eta}{\left(1 + a_1\sin\mu\eta\right)^2},\\
U'' &= \frac{2a_0a_1^2\mu^2 + a_0a_1\mu^2\sin\mu\eta - a_0a_1^2\mu^2\sin^2\mu\eta}{\left(1 + a_1\sin\mu\eta\right)^3},\ etc.
\end{aligned} \tag{8}$$

In one method known as the Rational Cosine Function Method, we substitute Equations (5) and (7) into the Reduced Nonlinear Ordinary Differential Equation (RNLODE) eqn. and we balance suitable exponents of $\cos\mu\eta$. Then, equating the co-efficient of like power of $\cos\mu\eta$ is equal to zero. In this way, we obtain a system of algebraic equations among the unknown parameters. Solving such a system of algebraic equations for the unknown parameters and substituting their values into Equation (5), we obtain the solutions of the RNLOD Equation. (4) and NLEE Equation (1). In another method known as the Rational Sine Function Method, we substitute Equations (6) and (8) into Equation (4) and we balance suitable exponents of $\sin\mu\eta$. Then, equating the co-efficient of like power of $\sin\mu\eta$ is equal to zero. Thus, we obtain a system of algebraic equations among the unknown parameters.

Solving the system of equations for the unknown parameters and substituting their values into Equation (6), we will obtain the solution of the RNLODE Equation (4) and NLEE Equation (1).

3. Applications

3.1. Boussinesq Equation

The Boussinesq Equation arises in several physical areas such as in the studies of

(i). propagation of long waves in shallow water,

(ii). one dimensional nonlinear lattice waves,

(iii). vibrations in a nonlinear string,

(iv). propagation of ion-acoustic waves in plasmas etc.

Mathematically,

$$u_{tt} = \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxx}, \tag{9}$$

where α, β and γ are real constants. Then, putting $\eta = x - ct$, where c is parameter, $u(x,t) = U(\eta)$ and using the changes shown in equations (3), we write,

$$c^2U'' - \alpha U'' - \beta(U^2)'' - \gamma U'''' = 0$$

where the primes denote differentiations with respect to η. Integrating the above equation twice with respect to η and taking the integration constants as zero, we get,

$$\gamma U'' + (\alpha - c^2)U + \beta U^2 = 0 \tag{10}$$

Substituting Equation (5) and (7) into Equation (10), we obtain,

$$\begin{aligned} &2\gamma a_0 a_1^2 \mu^2 + \gamma a_0 a_1 \mu^2 \cos\mu\eta - \gamma a_0 a_1^2 \mu^2 \cos^2\mu\eta + (\alpha - c^2)a_0 + 2(\alpha - c^2)a_0 a_1 \cos\mu\eta \\ &+(\alpha - c^2)a_0 a_1^2 \cos^2\mu\eta + \beta a_0^2 + \beta a_0^2 a_1 \cos\mu\eta = 0 \end{aligned} \tag{11}$$

For Equation (11) to be valid, we must have,

$$\begin{aligned} 2\gamma a_1^2\mu^2 + (\alpha - c^2) + \beta a_0 &= 0 \\ \gamma\mu^2 + 2(\alpha - c^2) + \beta a_0 &= 0 \\ (\alpha - c^2) - \gamma\mu^2 &= 0 \end{aligned} \tag{12}$$

Solving the above system of equations

$$c^2 = \alpha - \gamma\mu^2$$
$$\Rightarrow c = \pm\sqrt{\alpha - \gamma\mu^2}$$
$$\gamma\mu^2 + 2(\alpha - \alpha + \gamma\mu^2) + \beta a_0 = 0$$
$$\Rightarrow 3\gamma\mu^2 + \beta a_0 = 0$$
$$\Rightarrow a_0 = -\frac{3\gamma\mu^2}{\beta}$$
$$2\gamma a_1^2\mu^2 + (\alpha - \alpha + \gamma\mu^2) - 3\gamma\mu^2 = 0$$
$$\Rightarrow a_1 = \pm 1$$

So,

$$c = \pm\sqrt{\alpha - \gamma\mu^2}$$
$$a_0 = -\frac{3\gamma\mu^2}{\beta}$$
$$a_1 = \pm 1 \tag{13}$$

Now, substituting Equations (13) into Equations (5) and remembering that $\eta = x - ct$, we obtain the solution of the NLEE Equation (9) as

$$u(x,t) = -\frac{3\gamma\mu^2}{\beta[1 \pm \cos\mu\{x \mp (\sqrt{\alpha - \gamma\mu^2})t\}]} \tag{14}$$

For $\mu = 1$, we have the solution,

$$u_1(x,t) = -\frac{3\gamma}{\beta[1 \pm \cos\{x \mp (\sqrt{\alpha - \gamma})t\}]} \tag{15}$$

3.2. Korteweg-de Vries Equation

The Korteweg-de Vries Equation, also known as the kdV equation, arises in various areas of Nonlinear Physics that includes Fluid Dynamics. Mathematically can be written as

$$u_t + u_x + \varepsilon(uu_x + u_{xxx}) = 0 \tag{16}$$

where ε is a small perturbation parameter. Then, putting $\eta = x - ct$, where c is parameter, $u(x,t) = U(\eta)$ and using the changes shown in equations (3), we write,

$$-cU' + U' + \varepsilon(UU' + U''') = 0$$
$$\Rightarrow (1-c)U' + \varepsilon(UU' + U''') = 0 \tag{17}$$

where the primes denote differentiations with respect to η. Integrating the above equation twice with respect to η and taking the integration constants as zero, we get,

$$(1-c)U + \varepsilon(\frac{1}{2}U^2 + U'') = 0$$
$$\Rightarrow \varepsilon U'' + (1-c)U + \frac{\varepsilon}{2}U^2 = 0 \tag{18}$$

Substituting Equations (5) and (7) into Equation (18), we obtain,

$$\begin{aligned}&4\varepsilon a_0 a_1^2\mu^2 + 2a_0a_1\varepsilon\mu^2\cos\mu\eta - 2\varepsilon a_0a_1^2\mu^2\cos^2\mu\eta + 2(1-c)a_0 + 4(1-c)a_0a_1\cos\mu\eta\\&+2(1-c)a_0a_1^2\cos^2\mu\eta + \varepsilon a_0^2 + \varepsilon a_0^2 a_1\cos\mu\eta = 0\end{aligned} \tag{19}$$

For Equation (19) to be valid, we must have,

$$\begin{aligned}4\varepsilon a_1^2\mu^2 + 2(1-c) + \varepsilon a_0 &= 0\\ 2\varepsilon\mu^2 + 4(1-c) + \varepsilon a_0 &= 0\\ -\varepsilon\mu^2 + (1-c) &= 0\end{aligned} \tag{20}$$

The solutions of the above system of equations are

$$\begin{aligned}c = 1-\varepsilon\mu^2,\\ 2\varepsilon\mu^2 + 4(1-1+\varepsilon\mu^2) + \varepsilon a_0 = 0\\ \Rightarrow a_0 = -6\mu^2,\end{aligned}$$

and,

$$\begin{aligned}4\varepsilon a_1^2\mu^2 + 2(1-1+\varepsilon\mu^2) - 6\varepsilon\mu^2 = 0\\ \Rightarrow 4\varepsilon a_1^2\mu^2 = 4\varepsilon\mu^2\\ \Rightarrow a_1 = \pm 1\end{aligned}$$

So,

$$\begin{aligned}c &= 1-\varepsilon\mu^2\\ a_0 &= -6\mu^2\\ a_1 &= \pm 1\end{aligned} \tag{21}$$

Now, substituting equations (21) into equation (5) and remembering that $\eta = x - ct$, we obtain the solution of the NLEEs Equations (16) as

$$u(x,t) = -\frac{6\mu^2}{1 \pm \cos\mu\{x-(1-\varepsilon\mu^2)t\}} \tag{22}$$

For $\mu = 1$, we have the solution,

$$u_1(x,t) = -\frac{6}{1 \pm \cos\{x-(1-\varepsilon)t\}} \tag{23}$$

3.3. Gardner Equation

The Gardner Equation, also known as the mixed Korteweg-de Vries (kdV)- Modified Korteweg-de Vries (mkdV) equation, arises in various areas of Nonlinear Physics that includes Fluid Dynamics. Mathematically,

$$u_t = \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} \tag{24}$$

where α, β and γ are real constants. Then, putting $\eta = x - ct$, where c is parameter to be determined latter, $u(x,t) = U(\eta)$ and using the changes shown in equations (3), we write,

$$-cU' = \alpha UU' + \beta U^2U' + \gamma \dot{U'''} = 0$$

Integrating the above equation twice with respect to η and taking the integration constants as zero, we get,

$$\gamma U'' + cU + \frac{\alpha}{2}U^2 + \frac{\beta}{3}U^3 = 0 \tag{25}$$

Substituting Equations (5) and (7) into Equation (25), we obtain,

$$2\gamma a_1^2\mu^2 + \gamma a_1\mu^2 \cos\mu\eta - \gamma a_1^2\mu^2\cos^2\mu\eta + c + 2ca_1\cos\mu\eta + ca_1^2\cos^2\mu\eta + \frac{\alpha}{2}a_0 + \frac{\alpha}{2}a_0a_1\cos\mu\eta + \frac{\beta}{3}a_0^2 = 0 \tag{26}$$

For Equation (26) to be valid, we must have,

$$\begin{aligned} 2\gamma a_1^2\mu^2 + c + \frac{\alpha}{2}a_0 + \frac{\beta}{3}a_0^2 &= 0 \\ \gamma\mu^2 + 2c + \frac{\alpha}{2}a_0 &= 0 \\ c - \gamma\mu^2 &= 0 \end{aligned} \tag{27}$$

Solving the above system of equations

$$\begin{aligned} c &= \gamma\mu^2 \\ \gamma\mu^2 + 2\gamma\mu^2 + \frac{\alpha}{2}a_0 &= 0 \\ \Rightarrow 3\gamma\mu^2 + \frac{\alpha}{2}a_0 &= 0 \\ \Rightarrow a_0 &= -\frac{6\gamma\mu^2}{\alpha} \\ 2\gamma a_1^2\mu^2 + \gamma\mu^2 - 3\gamma\mu^2 + \frac{\beta}{3}\frac{36\gamma^2\mu^4}{\alpha^2} &= 0 \\ \Rightarrow a_1^2 - 1 + \frac{6\beta\gamma\mu^2}{\alpha^2} &= 0 \\ \Rightarrow a_1 &= \pm\sqrt{1 - \frac{6\beta\gamma\mu^2}{\alpha^2}} \end{aligned}$$

So,

$$\begin{aligned} c &= \gamma\mu^2 \\ a_0 &= -\frac{6\gamma\mu^2}{\alpha} \\ a_1 &= \pm\sqrt{1 - \frac{6\beta\gamma\mu^2}{\alpha^2}} \end{aligned} \tag{28}$$

Now, substituting Equations (28) into Equation (5) and remembering that $\eta = x - ct$, we obtain the solution of the NLEE Equation (24) as

$$u(x,t) = -\frac{6\gamma\mu^2}{\alpha\left[1 \pm \left(\sqrt{1 - \frac{6\beta\gamma\mu^2}{\alpha^2}}\right)\cos(x - \gamma\mu^2 t)\right]} \tag{29}$$

For $\mu = 1$, we have the solution,

$$u_1(x,t) = -\frac{6\gamma}{\alpha\left[1 \pm \left(\sqrt{1 - \frac{6\beta\gamma}{\alpha^2}}\right)\cos(x - \gamma t)\right]} \tag{30}$$

3.4. Generalized Boussinesq Burgers Equations

The Generalized Boussinesq-Burgers equations (GBBE) arise in the study of fluid flow and describe the propagation of shallow water waves. Mathematically can be written as

$$u_t + \alpha u u_x + \beta v_x = 0, \tag{31a}$$

$$v_t + \gamma(uv)_x + \lambda u_{xxx} = 0. \tag{31b}$$

Putting $u(x,t) = U(\eta)$, $v(x,t) = V(\eta)$ with $\eta = x - ct$, where c is parameter to be determined latter, and using the changes shown in equations (3), we write,

$$-cU' + \left(\frac{\alpha}{2}U^2\right)' + (\beta V)' = 0, \tag{32a}$$

$$-cV' + (\gamma UV)' + (\lambda U)''' = 0. \tag{32b}$$

where the primes denote differentiations with respect to η. Integrating the above equation twice with respect to η and taking the integration constants as zero, we get,

$$-cU + \frac{\alpha}{2}U^2 + \beta V = 0,$$
$$\Rightarrow V = \frac{1}{\beta}\left(cU - \frac{\alpha}{2}U^2\right). \tag{33}$$

Integrating equation (32b) once with respect to η and taking the integration constant to be zero, we get,

$$-cV + \gamma UV + \lambda U'' = 0 \tag{34}$$

Substituting this value of V into Equation (34), we obtain,

$$2\beta\lambda U'' - 2c^2U + (\alpha + 2\gamma)cU^2 - \alpha\gamma U^3 = 0. \tag{35}$$

Then, substituting Equations (5) and (7) into Equation (35), we obtain,

$$\begin{aligned}&2\beta\lambda(2a_0a_1^2\mu^2 + a_0a_1\mu^2\cos\mu\eta - a_0a_1^2\mu^2\cos^2\mu\eta) - 2c^2a_0(1 + 2a_1\cos\mu\eta + a_1^2\cos^2\mu\eta\\&+(\alpha + 2\gamma)ca_0^2(1 + a_1\cos\mu\eta) - \alpha\gamma a_0^3 = 0\end{aligned} \tag{36}$$

For Equation (36) to be valid, we must have,

$$4\beta\lambda a_1^2\mu^2 - 2c^2 + (\alpha + 2\gamma)ca_0 - \alpha\gamma a_0^2 = 0,$$
$$2\beta\lambda\mu^2 - 4c^2 + (\alpha + 2\gamma)ca_0 = 0$$
$$-\beta\lambda\mu^2 - c^2 = 0$$

The solutions of the above system of equations are

$$c = \pm i\mu\sqrt{\beta\lambda},$$
$$2\beta\lambda\mu^2 + 4\beta\lambda\mu^2 \pm i\mu\sqrt{\beta\lambda}(\alpha + 2\gamma)a_0 = 0$$

$$\Rightarrow a_0 = \pm\frac{6i\mu\sqrt{\beta\lambda}}{(\alpha+2\gamma)},$$

$$4\beta\lambda a_1^2\mu^2 + 2\beta\lambda\mu^2 \pm i\mu\sqrt{\beta\lambda}\frac{6i\mu\sqrt{\beta\lambda}}{(\alpha+2\gamma)}(\alpha+2\gamma) - \alpha\gamma\left(\pm\frac{6i\mu\sqrt{\beta\lambda}}{(\alpha+2\gamma)}\right)^2 = 0$$

$$\Rightarrow 4a_1^2 + 2 - 6 + \frac{36\alpha\gamma}{(\alpha+2\gamma)^2} = 0$$

$$\Rightarrow a_1 = \pm\sqrt{1-\frac{9\alpha\gamma}{(\alpha+2\gamma)^2}}$$

So,

$$\begin{aligned} c &= \pm i\mu\sqrt{\beta\lambda} \\ a_0 &= \pm\frac{6i\mu\sqrt{\beta\lambda}}{(\alpha+2\gamma)} \\ a_1 &= \pm\sqrt{1-\frac{9\alpha\gamma}{(\alpha+2\gamma)^2}} \end{aligned} \tag{37}$$

Now, substituting equations (37) into equation (5) and using Equation (26), we obtain the solution of the NLEEs Equations (31a) and (31b) as

$$u(x,t) = \pm\frac{6i\mu\sqrt{\beta\lambda}}{(\alpha+2\gamma)}\frac{1}{\left[1\pm\left(\sqrt{1-\frac{9\alpha\gamma}{(\alpha+2\gamma)^2}}\right)\cos\left\{\mu\left(x\mp i\mu\sqrt{\beta\lambda}t\right)\right\}\right]} \tag{38}$$

and

$$\begin{aligned} v(x,t) = &\pm\frac{6i\mu\sqrt{\beta\lambda}}{\beta(\alpha+2\gamma)}\frac{1}{\left[1\pm\left(\sqrt{1-\frac{9\alpha\gamma}{(\alpha+2\gamma)^2}}\right)\cos\left\{\mu\left(x\mp i\mu\sqrt{\beta\lambda}t\right)\right\}\right]} \\ &+ \frac{\alpha}{2\beta}\frac{36\beta\lambda\mu^2}{(\alpha+2\gamma)^2}\frac{1}{\left[1\pm\left(\sqrt{1-\frac{9\alpha\gamma}{(\alpha+2\gamma)^2}}\right)\cos\left\{\mu\left(x\mp i\mu\sqrt{\beta\lambda}t\right)\right\}\right]^2} \end{aligned} \tag{39}$$

4. Discussions and Conclusions

Rational Sine-Cosine function method is convenient and effective to solve Nonlinear Evolution Equations (NLEES) in water wave problem. To find solitary wave solutions of four NLEEs such as Boussinesq Equation (BE), Korteweg-de Vries (kdV), Gardner Equation (GE) and Generalized Boussinesq-Burgers equations (GBBE) rational sine-cosine function method has been used. These method can be used to solve many NLEEs arising in the study of shallow water wave problems.

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