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Operator Method to Solve Fractional Order Linear Differential Equations via Proportional α Derivative

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Abstract: In this study, we proposed an operator method for fractional order linear differential equations with constant coefficients. The form of fractional derivative, used in this article is proportional α derivative, introduced recently. Furthermore, to demonstrate the efficiency of the proposed operator method for fractional differential equations some numerical examples have discussed. One may verify the result for $\alpha = 1$.

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Keywords: Conformable fractional derivative, proportional α derivative, linear fractional differential equations.

1. Introduction

We are familiar with ordinary derivative or classical differential equations under the ordinary calculus. But now scenario has changed as the ordinary space has generalized into fractional space which has given a new idea to the researchers to think beyond the classical sense.

Till now there are many definitions of the fractional derivative and Integral, one may visit [1–3, 5] for the brief history of fractional calculus. Initial definitions are based on different ideas with some limitations and restrictions in which most popular definitions are Grunwald Letnicov, Riemann Liouville, Caputio. But these definition are too complicated to compute and do not posses even fundamental properties of a derivative.

In 2014 R. Khalil [7], introduced new definition of fractional derivative as Conformable derivative in classical sense. It seems to be convenient to evaluate and hold many fundamental properties like product rule, quotient rule, chain rule Rolle's theorem, Taylor series expansion etc which were not covered by previous definitions. This idea has generalized by U. N. Katugampola [8] who use exponential function in the place of polynomial. Many applications and results [4, 6, 9, 11–15] have been produced by this definition. But due to non linearity in the form of product of the independent variable and derivative of the function some complexity arises. Later the concept of Proportional derivative [15] has been searched on the basis of control theory and also explains various properties [16]. Using the definition of α proportional derivative here we presents the solution of some linear fractional order differential equations.

The rest of the paper is organized as follows. In Section 2, the basic definition of α proportional derivative and its properties are described in detail. In Section 3, some results related to operator method are proved. In Section 4, some numerical

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examples are presented to validate theoretical findings and demonstrate the performance of the proposed method. At last, in section 5. conclusive remarks has been concatenated to the discussed the work.

2. Basic Definitions

The main purpose of this section is to recall that some preliminaries definitions and properties for ensuing section.

Definition 2.1 (Fractional derivative by R. Khalil)**.** *For given function* $g : [0, \infty) \longrightarrow R$*, conformable fractional derivative of order* α *is defined as*

$$D^{\alpha}(g)(x) = \lim_{\epsilon \longrightarrow 0} \frac{g(x + \epsilon x^{1-\alpha}) - g(x)}{\epsilon}; \quad \forall\ x > 0, \quad 0 < \alpha \leqslant 1. \tag{1}$$

Definition 2.2 (Fractional derivative by Udita Katugampola)**.** *It is the generalization of conformable fractional derivative. For given function* $g : [0, \infty) \longrightarrow R$*, fractional derivative of order* α *is defined as:*

$$D^{\alpha}(g)(x) = \lim_{\epsilon \longrightarrow 0} \frac{g(xe^{\epsilon x^{-\alpha}}) - g(x)}{\epsilon}; \quad \forall\ x > 0, \quad 0 < \alpha \leqslant 1, \tag{2}$$

both the definitions (1) and (2) preserves the following property. If $g(t)$ is differentiable function, then $D^{\alpha}g(x) = x^{1-\alpha}g'(x)$.

Definition 2.3 (Fractional integral)**.** *Let* $x \geq a \geq 0$ *and* $g(x)$ *is defined on* $(a, x]$ *for* $\alpha \in (0, 1]$*. Then* α *fractional integral of* g *is defined by*

$$I_a^{\alpha}(g) = \int_{\alpha} g(x)dx^{\alpha} = \int_{a} \frac{g(x)}{x^{1-\alpha}}dx.$$

Provided above Riemann integral exists.

Definition 2.4. *Let* $g(x)$ *be real valued function defined on interval* $[0, \infty)$*. For a given number* $0 \leqslant \alpha \leqslant 1$*, we define new conformable fractional derivative called proportional* α *derivative by the following formula:*

$$D^{\alpha}g(x) = \alpha g'(x) + \beta g(x), \quad where \quad \beta = 1 - \alpha$$

provided $g(x)$ *is differentiable.*

2.1. Some basic properties of proportional α derivative:

(a). $D^{\alpha}(ag + bh) = aD^{\alpha}(g) + bD^{\alpha}(h)$ (Linearity)

(b). $D^{\alpha}(D^{\gamma})g = D^{\gamma}(D^{\alpha})g; \quad 0 \leq \alpha, \gamma \leq 1, \quad D^{\alpha}(.) \equiv \frac{d^{\alpha}}{dx^{\alpha}}(.)$ (Commutativity)

(c). $D^{\alpha}(D^{\gamma})g \neq D^{\alpha+\gamma}g; \quad 0 \leq \alpha, \gamma \leq 1, \quad D^{\alpha}(.) \equiv \frac{d^{\alpha}}{dx^{\alpha}}(.)$ (Law of Indices)

(d). If g and h are two α differentiable functions then
$D^{\alpha}(g.h) \neq (D^{\alpha}g).h + g(D^{\alpha}h)$ (Leibnitz's rule)

Definition 2.5. α *-Integral of a continuous function* $g(t)$ *defined for positive real numbers is defined and denoted as:*

$$I^{\alpha}(g(x)) = \frac{1}{\alpha}e^{-\frac{\beta}{\alpha}x} \int g(x).e^{\frac{\beta}{\alpha}x}dx + ce^{-\frac{\beta}{\alpha}x}; \ \alpha \in (0, 1], \beta = 1 - \alpha \tag{3}$$

where c is constant.

2.2. Proportional α derivative of some elementary functions.

(a). $D^\alpha((bx)^n) = \beta(bx)^n + bn\alpha(bx)^{n-1}$,

(b). $D^\alpha(e^{bx}) = (\beta + b\alpha)e^{bx}$,

(c). $D^\alpha(\sin(bx)) = \beta\sin(bx) + b\alpha\cos(bx)$,

(d). $D^\alpha(\cos(bx)) = \beta\cos(bx) - b\alpha\sin(bx)$,

(e). $D^\alpha(\log(bx)) = \beta\log(bx) + b\alpha(bx)^{-1}$,

(f). $D^\alpha(\lambda) = \beta\lambda$, where, $0 \le \alpha \le 1, \beta = 1-\alpha$ and λ is constants.

3. Proposed Operator Method for Fractional Order Linear Differential Equations with Constant Coefficients

The general linear fractional differential equation with constant coefficients is

$$[(D^\alpha)^n + b_n(D^\alpha)^{n-1} + ... + b_2D^\alpha + b_1]y(x) = g(x); \quad \alpha \in [0,1] \tag{4}$$

In short $L(D^\alpha)y(x) = g(x)$, where $(D^\alpha)^n \equiv D^\alpha D^\alpha D^\alpha ...ntimes$ and $b_1, b_2, ...b_n$ are constants.

Complete solution of differential equation (1) is $y(x) = y_c + y_p$ in which y_c denotes the solution of the homogeneous part and y_p is the particular integral. Now first we find the solution of $(D^\alpha - a)y(x) = f(x)$.

$$\begin{aligned} y &= \frac{1}{(D^\alpha - a)} f(x) \\ &= \frac{1}{(\beta + \alpha D - a)} f(x) \\ &= \frac{1}{\alpha} e^{-\frac{(\beta-a)x}{\alpha}} \int e^{\frac{(\beta-a)x}{\alpha}} f(x)dx + ce^{-\frac{(\beta-a)x}{\alpha}} \end{aligned}$$

Here solution of basis is $e^{-\frac{(\beta-a)x}{\alpha}}$ and $y_p = \frac{1}{\alpha} e^{-\frac{(\beta-a)x}{\alpha}} \int e^{\frac{(\beta-a)x}{\alpha}} f(x)dx$.

Since solution of homogeneous part of the first order fractional differential equation forms family of exponential curves so we can proceed with the same argument for higher order fractional linear differential equations.

Considering that $y = e^{mx}$ satisfy the homogeneous part of (4), we make Auxiliary equation $L(\beta + \alpha m) = 0$ and find its roots and observe the following:

(a). If $m = \lambda_1, \lambda_2, ...\lambda_n$ be the distinct roots of $L(\beta + \alpha m) = 0$ then $y_c = \sum_{i=1}^{n} c_i e^{\frac{(\lambda_i - \beta)x}{\alpha}}$

(b). If $m = \lambda, \lambda, ...n\ times$, then $y_c = \sum_{i=1}^{n} c_i x^{i-1} e^{\frac{(\lambda-\beta)x}{\alpha}}$

(c). For each pair of real root $m = \gamma \pm \delta$, $\; y_c = e^{\frac{\gamma-\beta}{\alpha}x}(c_1 \cosh \frac{\delta-\beta}{\alpha}x + c_2 \sinh \frac{\delta-\beta}{\alpha}x)$

(d). For each pair of complex root $m = \gamma \pm i\delta$, $\; y_c = e^{\frac{\gamma-\beta}{\alpha}x}(c_1 \cos \frac{\delta-\beta}{\alpha}x + c_2 \sin \frac{\delta-\beta}{\alpha}x)$.

Now we will discussed the operator method to solve sequential linear fractional differential equations. For this purpose we are interested to find the particular solution for such fractional differential equations as

$$y_p = \frac{1}{L(D^\alpha)} g(x); \quad L(D^\alpha) = (D^\alpha)^n + b_n(D^\alpha)^{n-1} + ... + b_2D^\alpha + b_1 \tag{5}$$

it is obvious that $[L^{-1}(D^\alpha)]\,(L(D^\alpha))\,y = y$.

Theorem 3.1. *If $a_1, a_2, ...a_n$ be the distinct roots of $L(D^\alpha)$ then*

$$y_p = \frac{1}{\alpha^n} e^{-\frac{(\beta - a_1)x}{\alpha}} \int e^{\frac{(a_2-a_1)x}{\alpha}} \int e^{\frac{(a_3-a_2)x}{\alpha}} ..ntimes... \int e^{\frac{(\beta-a_n)x}{\alpha}} g(x)dx^n \tag{6}$$

Proof. We have

$$\begin{aligned} &[(D^\alpha - a_1)(D^\alpha - a_2)...(D^\alpha - a_n)]y(x) = g(x) \qquad (7)\\ \Rightarrow &[(D^\alpha - a_1)(D^\alpha - a_2)...(D^\alpha - a_{n-1})]y(x) = \frac{1}{(D^\alpha - a_n)} g(x)\\ &= \frac{1}{\alpha} e^{-\frac{(\beta-a_n)x}{\alpha}} \int e^{\frac{(\beta-a_n)x}{\alpha}} g(x)dx \end{aligned}$$

and

$$\begin{aligned} [(D^\alpha - a_1)(D^\alpha - a_2)...(D^\alpha - a_{n-2})]y(x) &= \frac{1}{(D^\alpha - a_{n-1})(D^\alpha - a_n)} g(x)\\ &= \frac{1}{(D^\alpha - a_{n-1})} \left(\frac{1}{\alpha} e^{-\frac{(\beta-a_n)x}{\alpha}} \int e^{\frac{(\beta-a_n)x}{\alpha}} g(x)dx \right)\\ &= \frac{1}{\alpha^2} e^{-\frac{(\beta-a_{n-1})x}{\alpha}} \int e^{\frac{(a_n-a_{n-1})x}{\alpha}} \int e^{\frac{(\beta-a_n)x}{\alpha}} g(t)dx^2 \end{aligned}$$

With the same process we get

$$y_p = \frac{1}{\alpha^n} e^{-\frac{(\beta - a_1)x}{\alpha}} \int e^{\frac{(a_2-a_1)x}{\alpha}} \int e^{\frac{(a_3-a_2)x}{\alpha}} ..ntimes... \int e^{\frac{(\beta-a_n)x}{\alpha}} g(x)dx^n \tag{8}$$

□

Corollary 3.2. *If $\prod_{i=1}^{n} \frac{1}{(D^\alpha - a_i)} g(x) = \sum_{i=1}^{n} \frac{A_i}{(D^\alpha - a_i)} g(x)$. Then,*

$$y_p = \frac{1}{L(D^\alpha)} g(x) = \sum_{i=1}^{n} \frac{A_i}{\alpha} e^{-\frac{(\beta-a_i)x}{\alpha}} \int e^{\frac{(\beta-a_i)x}{\alpha}} g(x)dx \tag{9}$$

Theorem 3.3. *Followings are the formulas to find the particular solution of the corresponding function*

(a). $\frac{1}{L(D^\alpha)} e^{at} = \frac{1}{L(\beta + a\alpha)} e^{at};\ a \neq -\frac{\beta}{\alpha}$

(b). $\frac{1}{L(D^\alpha)^2} \sin ax \ or\ \cos ax = \frac{1}{L(\beta^2 - a^2\alpha^2 + 2\alpha\beta D)} \sin ax \ or\ \cos ax$

(c). $\frac{1}{L(D^\alpha)} p(x) = L(\beta) \left(1 + \frac{f(D)}{L(\beta)}\right)^{-1} p(x);\quad where,\ p(x) is\ polynomial\ and f(D) = L(D^\alpha) - L(\beta)$

(d). $\frac{1}{L(D^\alpha)} e^{ax} g(x) = e^{ax} \frac{1}{L(\beta + \alpha(D+a))} g(x)$

(e). If $(\beta + \alpha a) = 0, y_p = \frac{1}{L(D^\alpha)} e^{ax} = \frac{x^k e^{ax}}{k! \alpha^k}$*; where k is the multiplicity of the factor $(\beta + \alpha D)$ in $L(D^\alpha)$.*

4. Numerical Examples

In this section, to illustrate the proposed approach, some conformable fractional differential equations and conformable fractional differential equation will be solved. Here we consider $0 < \alpha \leq 1$ and $\beta = 1 - \alpha$.

Example 4.1. *Solve the following fractional differential equations*

(a). $(D^\alpha D^\alpha - 5D^\alpha + 6)y = e^{-3x}$,

(b). $(D^\alpha D^\alpha + 3D^\alpha + 2)y = \cos ax,$

(c). $(D^\alpha D^\alpha + 4D^\alpha + 3)y = x^2 + 3x - 3,$

(d). $(D^{1/2} + 2)^3 y = e^{-5x},$

(e). $(9D^{2/3}D^{2/3} - 6D^{2/3} + 10)y = 9\sin\frac{3}{2}x,$

(f). $(4D^{1/2}D^{1/2} - 4D^{1/2} + 1)y = e^x(1+x).$

Solution: Complete solution is given by

$$y(x) = y_c + y_p$$

(a) The A.E. is $(\beta + \alpha m - 3)(\beta + \alpha m - 2) = 0$ so that

$$y_c = c_1 e^{\frac{(3-\beta)x}{\alpha}} + c_2 e^{\frac{(2-\beta)x}{\alpha}}$$

and

$$y_p = \frac{1}{(D^\alpha D^\alpha - 5D^\alpha + 6)} e^{-3x} = \frac{1}{[(\beta - 3\alpha)^2 - 5(\beta - 3\alpha) + 6]} e^{-3x}$$

(b) Here $y_c = c_1 e^{-\frac{(1+\beta)x}{\alpha}} + c_2 e^{\frac{-(2+\beta)x}{\alpha}}$ and

$$\begin{aligned} y_p &= \frac{1}{[(3\alpha + 2\alpha\beta)D + \beta^2 + 3\beta + 2 - \alpha^2 a^2]} \cos ax \\ &= \frac{[a(3\alpha + 2\alpha\beta)\sin ax + (\beta^2 + 3\beta + 2 - \alpha^2 a^2)\cos ax}{a^2(3\alpha + 2\alpha\beta)^2 + (\beta^2 + 3\beta + 2 - \alpha^2 a^2)^2} \quad \text{(multiplying by conjugate and simplifying)} \end{aligned}$$

(c) Here $y_c = c_1 e^{-\frac{(1+\beta)x}{\alpha}} + c_2 e^{\frac{-(3+\beta)x}{\alpha}}$ and

$$\begin{aligned} y_p &= \frac{1}{L(\beta)\left(1 + \frac{f(D)}{L(\beta)}\right)} x^2 + 3x - 3 \quad \textit{where} \quad f(D) = \alpha^2 D^2 + 2\alpha\beta D + 4\alpha D,\ L(\beta) = \beta^2 + 4\beta + 3 \\ &= \frac{1}{L(\beta)} \left(x^2 + 3x - 3 - \frac{2\alpha^2 + 2\alpha(\beta+2)(2x+3)}{L(\beta)} + \frac{8\alpha^2(2+\beta)^2}{[L(\beta)]^2} \right) \end{aligned}$$

(d) A.E is $\left(\frac{1}{2}(m+1) + 2\right)^3 = 0 \Rightarrow y_c = (c_1 + c_2 x + c_3 x^2)e^{-5x}$ and

$$\begin{aligned} y_p &= \frac{1}{(D^{1/2}+2)^3} e^{-5x} \\ &= \frac{2^3}{(D+5)^3} e^{-5x} = \frac{8x^3 e^{-5x}}{3!} \end{aligned}$$

(e) $m = -1 \pm 3i \Rightarrow y_c = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$ and

$$\begin{aligned} y_p &= \frac{1}{(1 + 4D^2 + 4D - 2 - 4D + 10)} 9\sin\frac{3}{2}x \\ &= \frac{x}{8D} 9\sin\frac{3}{2}x = -\frac{3x}{4}\cos\frac{3}{2}x \end{aligned}$$

(f) $m = 0, 0 \Rightarrow y_c = (c_1 + c_2 x)$ and

$$\begin{aligned} y_p &= \frac{4e^x}{(D+1)^2}(1+x) \\ &= \frac{e^x}{(D^{1/2})^2}(1+x) \\ &= e^x\left(I^{1/2}\left(I^{1/2}(1+x)\right)\right) \\ &= 4e^x(x-1) \end{aligned}$$

Here we also have from classical method

$$\frac{4e^x}{(D+1)^2}(1+x) = 4e^x(1+D)^{-2}(1+x) = 4e^x(x-1)$$

Example 4.2. *We find the particular solution of the equation*

$$(D^{1/3}-3)(D^{1/3}-2)y = xe^{2x}$$

by the methods given in Theorem 3.1, Corollary 3.2 and Theorem 3.3.

$$y_p = \frac{1}{(D^{1/3}-3)(D^{1/3}-2)}xe^{2x} = 9e^{7x}\int e^{-3x}\int xe^{-2x}dx^2 = \frac{9}{100}(7+10x)e^{2x}$$

$$y_p = \left(\frac{1}{(D^{1/3}-3)} - \frac{1}{(D^{1/3}-2)}\right)xe^{2x} = 3\left(e^{7x}\int xe^{-5x}dx - e^{4x}\int xe^{-2x}dx\right) = \frac{9}{100}(7+10x)e^{2x}$$

$$y_p = \frac{1}{(D-7)(D-4)}xe^{2x} = \frac{9}{100}(7+10x)e^{2x}$$

5. Conclusion

A number of analytical or numerical methods have been published for fractional differential equations (FDE). In fact, the solution of FDE basically depends upon respective fractional derivative. Here we have develop an operator method to find the particular solutions of the linear fractional differential equation with constant coefficients, based on the proportional α derivative. The solutions obtained by operator method works as similar to the solutions obtained as usual operator method in the case of ordinary differential equations useful in several physical problems.

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