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Characterization of Signed Double-Star Admitting Minus Dominating Function

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Abstract: Given a signed graph $G = (V, E, \sigma)$, a function $f : V \to \{-1, 0, 1\}$ is a minus dominating function of G if $f(u) + \sum_{v \in N(u)} \sigma(uv)f(v) \geq 1$ for all $u \in V$. In this paper we characterize double star to admit an MDF and give some sufficient conditions for a general graph G to admit an MDF.

MSC: 05C22.

Keywords: Signed graphs, Minus domination, Minus Dominating Function.

1. Introduction

The main aim of this paper is to study minus dominating functions of a locally finite signed graph. In this paper we are going to see the characterization Theorem for few graphs to admit *Minus Dominating Function* (henceforth abbreviated as 'MDF'). The main motivation of the paper is that not all *pure* signed graphs, viz., signed graph possessing at least one negative edge will admit an MDF, thereby raising the basic problem that under what condition do they admit MDF.

The study of minus domination in graphs began in the year 1999 [4] by Jean Dunbar, Stephen Hedetniemi et al. The concept of minus domination in signed graph was introduced by B.D. Acharya [1] in the article *Minus Domination in a Signed graph*, in which he generalized the definition of minus domination of a graph to *signed* graph and he obtained necessary and sufficient condition for Stars to admit a minus dominating function. In this paper we extend his work.

Throughout this paper, unless mentioned otherwise, we will consider locally finite simple and *pure signed* graphs without self-loops as treated in most of the standard text-books on finite graph theory such as [5].

Signed graph is an ordered triple $G = (V, E, \sigma)$ where $G' = (V, E)$ is a graph called the *underlying* graph of G and $\sigma : E \to \{-1, +1\}$ is a function called a *signing* of G.

Definition 1.1. *Given a signed graph $G = (V, E, \sigma)$, a function $f : V \to \{-1, 0, 1\}$ is a minus dominating function of G if*

$$f(u) + \sum_{v \in N(u)} \sigma(uv)f(v) := f(N[u]) \geq 1 \quad \forall \ u \in V. \tag{1}$$

Definition 1.2. *Given a signed graph $G = (V, E, \sigma)$ and a function $f : V \to \{-1, 0, 1\}$, a vertex $v \in V(G)$ is said to be saturated under f, if $f(N[v]) \geq 1$, otherwise v is said to be unsaturated.*

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Remark 1.3. *By the above Definition 1.2, it is easy to see that given a signed graph $G = (V, E, \sigma)$, a function $f : V \to \{-1, 0, 1\}$ is a minus dominating function of G, If every vertex $v \in V(G)$ is saturated under f.*

Note 1. The number of positive(negative) edges incident to a given v in a signed graph $G = (V, E, \sigma)$ is denoted $d^+(v)(d^-(v), respectively)$ and is called the $positive(negative)$ degree of v. The $net - degree$ of v is defined by $d^{\pm}(v) = d^+(v) - d^-(v)$.

Note 2. For any signed $S = (V, E, \sigma)$ and for $u \in V(S)$, $N^+(u)$ denotes the set of all neighbors v of u such that $e(u, v)$ is a positive edge and $N^-(u)$ denotes the set of all neighbors v of u such that $e(u, v)$ is a negative edge. Also $|N^+(u)|$ and $|N^-(u)|$ denotes the $cardinality$ of the set.

2. Necessary Conditions for Double-Star to Admit an MDF

A tree containing exactly two non-pendent vertices is called a $double - star$. A double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, 1, ..., 1)$ is denoted by S_{k_1,k_2}. Suppose that $u, v \in V(S_{k_1,k_2})$ and $d(u) = k_1 + 1, d(v) = k_2 + 1$ then we say u, v as $central\ vertex$. Let X and Y be the set of all pendent edges incident to u and v, respectively. Then we say that S_{k_1,k_2} is a double-star with pendent sets X and Y. Also, $e(u, v)$ is called the $central\ edge$ of the double-star. Simple observations which will be used in many of the upcoming proofs are discussed in this section.

Lemma 2.1 (see [1])**.** *Let $G = (V, E, \sigma)$ be any signed graph having negative pendant vertex uv with u as its pendent vertex. Then, for any minus dominating function f of S one must have $f(u) \in \{0, 1\}$.*

We have for any Signed graph G, $E^+(G)$ and $E^-(G)$ denoting the set of positive and negative edges of G respectively. Now, we will generalize this as follows,

Definition 2.2. *Given a signed graph $G(V, E, \sigma)$. If $A \subseteq E$ then, $N^+(A)$ and $N^-(A)$ denoted the set of all positive and negative edges in A respectively.*

Remark 2.3. *If $A = E$. Then, $N^+(A) = E^+(G)$ and $N^-(A) = E^-(G)$. Henceforth, X and Y denotes the set of pendent edges associated with the central vertices u and v of signed double star S_{k_1,k_2} respectively.*

Note 3. By the definition of Double-Star we can easily see that both $|N^+(X)|$ and $|N^-(X)|$ cannot be empty at the same time.

Lemma 2.4. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. Then, for any MDF f of S_{k_1,k_2},*

(1). if $f(u) = 1$ then pendent edges associated with u cannot be negative.

(2). if $f(u) = -1$ then pendent edges associated with u cannot be positive.

(3). if $f(u) = 0$ then pendent vertices associated with u must be assigned 1.

(4). if $|N^+(X)| \neq \emptyset$ and $|N^-(X)| \neq \emptyset$. Then,

(a). $f(u) = 0$

(b). $f(u_i) = 1$, for $1 \leq i \leq k_1 - 1$, where $u_i^\prime s$ are the pendent vertices associated with u.

Proof. To prove (1), let $f(u) = 1$ and, if possible, let u_1 be the pendent vertex in S_{k_1,k_2} such that $e(u_1, u)$ is a negative pendent edge associated with u. Clearly, $\sigma(u_1, u) = -1$, whence we get $f(N[u_1]) = f(u_1) + \sigma(uu_1)f(u) = f(u_1) - 1 \leq 0$, the last inequality being due to the fact that $f(u_1) \in \{-1, 0, 1\}$. But, this contradicts the hypothesis that f is an MDF. Thus validity of (1) follows.

Towards proving (2), let $f(u) = -1$ and, if possible, let u_1 be the pendent vertex in S_{k_1,k_2} such that $e(u, u_1)$ is a positive pendent edge associated with u. Clearly, $\sigma(u, u_1) = 1$, whence we get $f(N[u_1]) = f(u_1) + \sigma(uu_1)f(u) = f(u_1) - 1 \leq 0$, the last inequality being due to the fact that $f(u_1) \in \{-1, 0, 1\}$. But, this contradicts the hypothesis that f is an MDF. Thus validity of (2) follows.

Towards proving (3), let $f(u) = 0$ and, if possible, let u_1 be any pendent vertex in S_{k_1,k_2} such that $e(u, u_1)$ is a pendent edge associated with u. Clearly, $f(N[u_1]) = f(u_1) + \sigma(uu_1)f(u) = f(u_1)$. Since $f(u_1) \in \{-1, 0, 1\}$ and $f(N[u_1]) \leq 1$ the only possibility is $f(u_1) = 1$. But u_1 was an arbitrary pendent vertex hence validity of (3) follows.

For proving (4)-(a), let $|N^+(X)| \neq \emptyset$ and $|N^-(X)| \neq \emptyset$. If possible, suppose $f(u) \in \{1, -1\}$. if $f(u) = 1$, then pendent edges associated with u cannot be negative (by (1)), this implies $|N^-(X)| = \emptyset$, contradicting the hypothesis. Else $f(u) = -1$, then pendent edges associated with u cannot be positive (by (2)), this implies $|N^+(X)| = \emptyset$, again contradiction to the hypothesis. Therefore, $f(u) = 0$ is the only possibility.

Now, from (3), (4)-(b) follows. □

Remark 2.5. *In the above Lemma it is evident that there is nothing special about the central vertex u. We can prove the same Lemma considering the other central vertex v also.*

Given any signed graph $G = (V, E, \sigma)$, let $\mathcal{M}(G)$ denote the set of all MDF's of G.

Theorem 2.6. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. If $|N^+(X)| < |N^-(X)|$ or $|N^+(Y)| < |N^-(Y)|$. Then, $\mathcal{M}(S_{k_1,k_2}) = \emptyset$.*

Proof. Let $|N^+(X)| < |N^-(X)|$. This implies that there are more negative pendent edges corresponding to vertex u than positive pendent edges.

Claim: If $|N^+(X)| = \emptyset$. Then, u is unsaturated.

Since u is the central vertex with $deg(u) = k_1$. Let $v, u_1, u_2, ..., u_{k_1-1}$ be the vertices adjacent to u. So we have,

$$f(N[u]) = f(u) + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i)f(u_i) \tag{2}$$

But $|N^+(X)| = \emptyset$ implies that $f(u) \in \{-1, 0\}$ (by Lemma 2.4 and Note 3).

Case (i): $f(u) = -1$.

Then, by Lemma 2.1. $f(u_i) \in \{0, 1\}$, $\forall\ i$ where $1 \leq i \leq k_1 - 1$. Now, equation (2) implies, $f(N[u]) = -1 + \sigma(uv)f(v) - \sum_{i=1}^{k_1-1} f(u_i)$. As $\sigma(uv)f(v) \leq 1$ and $f(u_i) \in \{0, 1\}$, the above equation has an upper bound 0, in other words $f(N[u]) \leq 0$. Therefore, in this case u is unsaturated.

Case (ii): $f(u) = 0$.

Then by Lemma 2.4 $f(u_i) = 1$, $\forall\ i$ where $1 \leq i \leq k_1 - 1$. Now equation (2) implies, $f(N[u]) = 0 + \sigma(uv)f(v) - \sum_{i=1}^{k_1-1} f(u_i) = \sigma(uv)f(v) - (k_1 - 1)$. We have $k_1 - 1 \geq 1$ and also $\sigma(uv)f(v) \leq 1$. Therefore, $f(N[u]) \leq 0$. Hence, u is unsaturated.

By case (i) and case (ii) claim holds.

Therefore, $|N^+(X)| \neq \emptyset$ but in hypothesis we have $|N^+(X)| < |N^-(X)|$ this implies $|N^-(X)| \neq \emptyset$. So, we have $|N^+(X)| \neq \emptyset$ and $|N^-(X)| \neq \emptyset$. By Lemma 2.3 we can write $f(u) = 0$ and $f(u_i) = 1$, for $1 \leq i \leq k_1 - 1$, where $u_i's$ are the pendent vertices associated with u.

Now by the above discussion, equation (2) implies,

$$f(N[u]) = 0 + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i) = \sigma(uv)f(v) + |N^+(X)| - |N^-(X)|.$$

By hypothesis we can write, $|N^+(X)| < |N^-(X)|$. In other words $|N^+(X)| - |N^-(X)| < 0$ which in turn implies and implied by $|N^+(X)| - |N^-(X)| \leq -1$.

These observations with the fact that $f(u)\sigma(uv) \leq 1$ for any $u, v \in V(S_{k_1,k_2})$ leads to the conclusion that $f(N[u]) \leq 0$, making u as unsaturated. Hence the Theorem. □

Theorem 2.7. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. If $|N^+(X)| = |N^-(X)|$ and $|N^-(Y)| \neq \emptyset$. Then, $\mathcal{M}(S_{k_1,k_2}) = \emptyset$.*

Proof. Let $|N^+(X)| = |N^-(X)|$ and $|N^-(Y)| \neq \emptyset$. If possible, let f be an MDF of S_{k_1,k_2}. Since u and v are the central vertices with degree k_1 and k_2 respectively. Let $u_i's$ be the pendent vertices associated with u, for $1 \leq i \leq k_1 - 1$ and similarly let $v_j's$ be the pendent vertices associated with v, for $1 \leq j \leq k_2 - 1$. we have,

$$f(N[u]) = f(u) + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i)f(u_i) \tag{3}$$

Since both $|N^+(X)|$ and $|N^-(X)|$ cannot be empty simultaneously, we have for any MDF f of S_{k_1,k_2}, by Lemma 2.4 $f(u) = 0$ and $f(u_i) = 1$, for $1 \leq i \leq k_1 - 1$. Now, substituting the values of f in equation (3) we get,

$$f(N[u]) = \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i).1 = \sigma(uv)f(v) + |N^+(X)| - |N^-(X)|.$$

Since $|N^+(X)| = |N^-(X)|$ this yields $f(N[u]) = \sigma(uv)f(v)$. So, $f(N[u]) \geq 1$ if and only if $\sigma(uv)f(v) = 1$. But $|N^-(Y)| \neq \emptyset$ implies that $f(v) \in \{0, -1\}$ (by Lemma 2.4). Therefore, $f(N[u]) \geq 1$ if and only if $f(v) = -1$ and $\sigma(uv) = -1$.

Claim: v is unsaturated.

We have,

$$f(N[v]) = f(v) + \sigma(uv)f(u) + \sum_{j=1}^{k_2-1} \sigma(vv_j)f(v_j) \tag{4}$$

As discussed earlier for any MDF f of S_{k_1,k_2} we have, $f(v) = -1$, $f(u) = 0$ and $\sigma(uv) = -1$. Therefore, equation (4) implies, $f(N[v]) = -1 + \sum_{j=1}^{k_2-1} \sigma(vv_j)f(v_j)$.

Clearly, $\sigma(vv_j) \neq 1$ for any j. Otherwise, let us suppose that there exist some k, such that $\sigma(uv_k) = +1$, for $1 \leq k \leq k_2 - 1$. Then, it is evident that v_k will not be saturated as $f(N[v_k]) \leq 0$. Therefore, $\sigma(vv_j) \neq 1$ for any j, where $1 \leq j \leq k_2 - 1$. In other words, $\sigma(vv_j) = -1$ for all j. Since $f(v) = -1$ and $\sigma(vv_j) = -1$ for all j, by Lemma 2.1. we have $f(v_j) \in \{0, 1\}$. This discussion will yield that $f(N[v]) \leq -1$. Hence, claim holds. So the conclusion is that there exist no function f, which will be an MDF for S_{k_1,k_2}. □

Theorem 2.8. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. If $|N^+(X)| = |N^-(X)|$ and $\sigma(uv) = -1$. Then, $\mathcal{M}(S_{k_1,k_2}) = \emptyset$.*

Proof. Let $|N^+(X)| = |N^-(X)|$ and $\sigma(uv) = -1$. If possible, let f be an MDF of S_{k_1,k_2}. Since u and v are the central vertices with degree k_1 and k_2 respectively. Let $u_i's$ be the pendent vertices associated with u, for $1 \leq i \leq k_1 - 1$ and similarly let $v_j's$ be the pendent vertices associated with v, for $1 \leq j \leq k_2 - 1$. we have,

$$f(N[u]) = f(u) + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i)f(u_i) \tag{5}$$

Since both $|N^+(X)|$ and $|N^-(X)|$ cannot be empty simultaneously, we have for any MDF f of S_{k_1,k_2}, by Lemma 2.4 $f(u) = 0$ and $f(u_i) = 1$, for $1 \leq i \leq k_1 - 1$. Now, substituting the values of f in equation (5) we get,

$$f(N[u]) = \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i).1 = \sigma(uv)f(v) + |N^+(X)| - |N^-(X)|.$$

Since $|N^+(X)| = |N^-(X)|$ this yields $f(N[u]) = \sigma(uv)f(v)$. But $\sigma(uv) = -1$. So, $f(N[u]) \geq 1$ if and only if $f(v) = -1$.

Claim: v is unsaturated.

We have,

$$f(N[v]) = f(v) + \sigma(uv)f(u) + \sum_{j=1}^{k_2-1} \sigma(vv_j)f(v_j) \tag{6}$$

As discussed earlier for any MDF f of S_{k_1,k_2} we have, $f(v) = -1$, $f(u) = 0$ and $\sigma(uv) = -1$. Therefore, equation (6) implies, $f(N[v]) = -1 + \sum_{j=1}^{k_2-1} \sigma(vv_j)f(v_j)$.

Clearly, $\sigma(vv_j) \neq 1$ for any j. Otherwise, let us suppose that there exist some k, such that $\sigma(uv_k) = +1$, for $1 \leq k \leq k_2 - 1$. Then, it is evident that v_k will not be saturated as $f(N[v_k]) \leq 0$. Therefore, $\sigma(vv_j) \neq 1$ for any j, where $1 \leq j \leq k_2 - 1$. In other words, $\sigma(vv_j) = -1$ for all j. Since $f(v) = -1$ and $\sigma(vv_j) = -1$ for all j, by Lemma 2.1. we have $f(v_j) \in \{0, 1\}$. This discussion will yield that $f(N[v]) \leq -1$. Hence, claim holds. So the conclusion is that there exist no f, which will be an MDF for S_{k_1,k_2}. □

Remark 2.9. *In the above Theorems, there is nothing sacred about $N^+(X)$ and $N^-(X)$, we can prove the theorem for $N^+(Y)$ and $N^-(Y)$ in a similar manner.*

Theorem 2.10. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. If $|N^+(Y)| = |N^-(Y)|$ and $|N^-(X)| \neq \emptyset$. Then, $\mathcal{M}(S_{k_1,k_2}) = \emptyset$.*

Proof. Proof is similar to that of Theorem 2.6. □

Theorem 2.11. *Let S_{k_1,k_2}, be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. If $|N^+(Y)| = |N^-(Y)|$ and $\sigma(uv) = -1$. Then, $\mathcal{M}(S_{k_1,k_2}) = \emptyset$.*

Proof. Proof is similar to that of Theorem 2.7. □

3. Characterization Theorem for Double-Star to Admit an MDF

Now we will see the necessary and sufficient condition for Double-Star to admit a minus dominating function.

Theorem 3.1. *For any positive integer $k_1, k_2 \geq 2$. Let S_{k_1,k_2} be any pure signed double-star. Let u, v be the central vertices of S_{k_1,k_2}. $\mathcal{M}(S_{k_1,k_2}) \neq \emptyset$ if and only if the following conditions holds for S_{k_1,k_2},*

(1). $|N^+(X)| \geq |N^-(X)|$ and $|N^+(Y)| \geq |N^-(Y)|$.

(2). If $|N^+(X)| = |N^-(X)|$. Then, $|N^-(Y)| = \emptyset$ and $\sigma(uv) = +1$.

(3). If $|N^+(Y)| = |N^-(Y)|$. Then, $|N^-(X)| = \emptyset$ and $\sigma(uv) = +1$.

Proof. Necessity follows from Theorem 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9. Hence, to prove sufficiency we define a function $f : V(S_{k_1,k_2}) \to \{-1, 0, 1\}$ in such a way that every vertex in S_{k_1,k_2} will be saturated under f. Here the definition of f

depends on the negative sign assigned to the edges. Let the conditions (i), (ii) and (iii) hold for S_{k_1,k_2}. Then from (1) we have,

$$|N^+(X)| \geq |N^-(X)| \ \ and \ \ |N^+(Y)| \geq |N^-(Y)|,$$

The above inequality will give rise to following cases for any S_{k_1,k_2},

Case (a): $|N^+(X)| > |N^-(X)|$ and $|N^+(Y)| > |N^-(Y)|$,

Case (b): $|N^+(X)| = |N^-(X)|$ and $|N^+(Y)| = |N^-(Y)|$,

Case (c): $|N^+(X)| = |N^-(X)|$ and $|N^+(Y)| > |N^-(Y)|$,

Case (d): $|N^+(X)| > |N^-(X)|$ and $|N^+(Y)| = |N^-(Y)|$,

Clearly, Case (b) will contradict the necessary conditions (2) and (3). Therefore, it suffices to discuss all other cases except Case (b),

Case (a): Suppose $|N^+(X)| > |N^-(X)|$ and $|N^+(Y)| > |N^-(Y)|$. Then, define $f : V(S_{k_1,k_2}) \to \{-1, 0, 1\}$ as follows,

$$f(s) = \begin{cases} 0 & \text{if } s \in \{u, v\} \\ 1 & \text{otherwise.} \end{cases}$$

Claim: f is an MDF of S_{k_1,k_2}.

Let $u_i's$ and $v_j's$ be the pendent vertex associated with u and v respectively for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2 - 1$. Now for any pendent vertex associated with u it is easy to see that $u_i's$ are saturated under f. Similarly, we can see that pendent vertex associated with v are also saturated. Now it remains to show that both u and v are saturated under f. We have,

$$f(N[u]) = f(u) + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i)f(u_i) \tag{7}$$

Using the definition of f we can write the above equation as follows, $f(N[u]) = \sum_{i=1}^{k_1-1} \sigma(uu_i) = |N^+(X)| - |N^-(X)|$. Since $|N^+(X)| > |N^-(X)|$ this implies that $|N^+(X)| - |N^-(X)| \geq 1$. Therefore, $f(N[u]) \geq 1$. Similarly, we can show that $f(N[u]) \geq 1$. Hence, claim holds.

Case (c): Suppose $|N^+(X)| = |N^-(X)|$ and $|N^+(Y)| > |N^-(Y)|$,

Define $f : V(S_{k_1,k_2}) \to \{1, 0, -1\}$ by letting,

$$f(s) = \begin{cases} 0 & \text{if } s = u \\ 1 & \text{otherwise.} \end{cases}$$

Any pendent vertex associated with u is trivially saturated by the definition. Further, any pendent vertex associated with v is also saturated as $|N^-(Y)| = \emptyset$. For central vertex u, we have,

$$f(N[u]) = f(u) + \sigma(uv)f(v) + \sum_{i=1}^{k_1-1} \sigma(uu_i)f(u_i)$$

By definition of f we can write the above equation as $f(N[u]) = 0 + \sigma(uv) + \sum_{i=1}^{k_1-1} \sigma(uu_i)$, but we have $\sigma(uv) = +1$ and $|N^+(X)| = |N^-(X)|$ in the hypothesis. Therefore, $f(N[u]) = 1 \geq 1$. Hence, u is saturated under f. Further, for central vertex v, we have,

$$f(N[v]) = f(v) + \sigma(uv)f(u) + \sum_{j=1}^{k_2-1} \sigma(vv_j)f(v_j)$$

Again using the definition above equation yields, $f(N[v]) = 1 + \sum_{j=1}^{k_2-1} \sigma(vv_j)$. Now, by hypothesis we know that $|N^-(Y)| = \emptyset$. Therefore, $f(N[v]) \geq 1$. Hence, $f(N[s]) \geq 1$ for all $s \in S_{k_1,k_2}$. Claim holds and Theorem follows.

Case (d): $|N^+(X)| > |N^-(X)|$ and $|N^+(Y)| = |N^-(Y)|$.

It is easy to see that, the proof of case (d) is very similar to the previous case. Therefore, we leave to the reader. In all the above cases we have seen that there exist at least one minus dominating function for any S_{k_1,k_2} satisfying the necessary conditions claimed in hypothesis. By this we can write that $\mathcal{M}(S_{k_1,k_2}) \neq \emptyset$. □

4. Classes of Signed Graphs G with $\mathcal{M}(G) \neq \emptyset$

In this section we will see some of the sufficient conditions for any graph G to admit an MDF.

Theorem 4.1. *Let G any signed graph of order n. If $d^+(v) = n-1$ for some vertex $v \in V(G)$. Then, $\mathcal{M}(G) \neq \emptyset$.*

Proof. Let v be a vertex of G such that $d^+(v) = n-1$, where n is the order of G.

Define $f : V(G) \to \{1, 0, -1\}$ by letting,

$$f(u) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{If } u \neq v \end{cases}$$

Claim: f is an MDF of G.

In order to the prove the claim, we will show that $f(N[s]) \geq 1$, $\forall\ s \in V(G)$.

Case 1: Assume that $s = v$, we have

$$f(N[v]) = f(v) + \sum_{i=1}^{n-1} \sigma(vv_i)f(v_i) = 1 + \sum_{i=1}^{n-1} \sigma(vv_i).0 = 1.$$

Therefore, $f(N[v]) \geq 1$.

Case 2: Assume that $s \neq v$, we have the following sub-cases.

Case 2.1: Suppose s is any pendent vertex.

Then, the only vertex in G for which s is adjacent is with v. This implies that $f(N[s]) = f(s) + \sigma(sv)f(v) = 0 + 1$. (By definition). Therefore, $f(N[s]) \geq 1$.

Case 2.2: Suppose s is not a pendent vertex.

Then, let $v_1, v_2, ..., v_t$ be the adjacent vertices of s other than v, where $1 \leq t \leq n-2$. Hence,

$$\begin{aligned} & f(N[s]) = f(s) + \sigma(sv_1)f(v_1) + ... + \sigma(sv_t)f(v_t) + \sigma(sv)f(v) \\ \implies & f(N[s]) = f(s) + \sum_{i=1}^{t} \sigma(sv_t)f(v_t) + \sigma(sv)f(v) \\ \implies & f(N[s]) = 0 + 0 + 1 \ \ \text{(By definition)}. \end{aligned}$$

Therefore, $f(N[s]) \geq 1$. Claim holds and the proof is complete. □

Theorem 4.2. *Let G any signed graph of order n. If $d^{\pm}(v) \geq 0$ for every vertex $v \in V(G)$. Then, $\mathcal{M}(G) \neq \emptyset$.*

Proof. Define $f : V(G) \to \{1, 0, -1\}$ by letting, $f(v) = 1$, $\forall\ v \in V(G)$.

Claim : f is an MDF of G.

In order to the prove the claim, we will show that $f(N[s]) \geq 1$, $\forall\ s \in V(G)$. Let v be any vertex of G. Suppose $v_1, v_2, ..., v_t$

are the vertices adjacent to v, for $1 \leq t \leq n-1$. We have, $f(N[v]) = f(v) + \sum_{i=1}^{t} \sigma(vv_i)f(v_i)$, for $1 \leq t \leq n-1$. By definition of f we can write,

$$f(N[v]) = 1 + \sum_{i=1}^{t} \sigma(vv_i) \qquad for \;\; 1 \leq t \leq n-1 \tag{8}$$

Now, we consider the following cases.

Case 1: Suppose v is a vertex of odd degree.

Then, $degree(v) = 2k+1$, for some $k \in \{0,1,2,...\}$. From (5), we can write $t = 2k+1$. Since $d^{\pm}(v) \geq 0$, it is easy to see that

$$|N^{-}(v)| \leq \lfloor \frac{t}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor = k.$$

Therefore $|N^{+}(v)| \geq k+1$, these observations with (5) leads to the fact that $f(N[v]) \geq 2$.

Case 2: Assume v is a vertex of even degree.

Then, $degree(v) = 2k$ for some $k \in \{1,2,...\}$. From (5), we can write $t = 2k$. Since $d^{\pm}(v) \geq 0$. It is obvious that $f(N[v]) \geq 1$.

Therefore, Claim holds and hence the proof. □

In what follows for any signed graph G, $E^{+}(G)$ and $E^{-}(G)$ will denote the sets of positive and negative edges of G, respectively.

Theorem 4.3. *Let G be any signed graph of order n, If $\delta(G) \geq 2$ and $|E^{+}(G)| = \emptyset$. Then, G admits an MDF.*

Proof. Let G be any graph satisfying the hypothesis. Then,

Define $f : V(G) \to \{1,0,-1\}$ by letting, $f(v) = -1, \forall\, v \in V(G)$.

Claim : f is an MDF of G.

Let v be any vertex in $V(G)$. We will show that $f(N[v]) \geq 1$, we have, $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(vu)f(u)$. For any positive integer t, Let $v_1, v_2, ..., v_t$ be adjacent vertices of v and since $\delta(G) \geq 2$, this implies that, $2 \leq t \leq n-1$. We have, $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(vu)f(u)$

$$\implies f(N[v]) = -1 + \sum_{i=1}^{t} \sigma(vv_t)f(v_t), \qquad for \;\; 2 \leq t \leq n-1.$$

Since $|E^{+}(G)| = \emptyset$ and by the definition of f we can write,

$$f(N[v]) = -1 + \sum_{i=1}^{t} (-1).(-1), \;\; \text{for } 2 \leq t \leq n-1.$$

$$\implies f(N[v]) \geq 1 \;\; (\text{Since, } t \geq 2).$$

As v was an arbitrary vertex in G. We have $f(N[v]) \geq 1, \forall\, v \in V(G)$. Hence the proof. □

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