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On Quasi-class (Q) Operator

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Abstract: In this paper we introduce the new classes of operator namely quasi-class (Q) operator acting on a complex Hilbert space H. An operator $T \in$ quasi-class (Q) if $T\left({T^*}^2T^2\right) = (T^*T)^2\, T$ where T^* is the adjoint of the operator T. We investigate some basic properties of this operator.

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1. Introduction and Preliminaries

Throughout this paper H is a complex Hilbert Space and $B(H)$ is the algebra of all bounded linear operators acting on H. If $T \in B(H)$ then T^* is its adjoint. An operator T is unitary if $T^*T = TT^* = I$, T is isometry if $T^*T = I$, T is normal if $T^*T = TT^*$, T is quasi normal if $TT^*T = T^*T^2$. An operator $T \in B(H)$ is called class (Q) if ${T^*}^2T^2 = (T^*T)^2$ [1]. Let $T = U + iV$, where $U = Re\,T = \frac{T+T^*}{2}$ and $V = Im\,T = \frac{T-T^*}{2i}$ are the real and imaginary parts of T. We shall write $B^2 = {T^*}^2T^2$ and $C^2 = (T^*T)^2$, where B and C are non - negative definite [6]. In this paper we will study some properties of quasi-class (Q) operators. Exactly we will give conditions under which an operator T is quasi-class (Q). Also, we shall show that of T and S are quasi-class (Q) operators, we shall obtain conditions under which their sum and product are quasi-class (Q).

Definition 1.1. *An operator $T \in B(H)$ is called class (Q) if ${T^*}^2T^2 = (T^*T)^2$.*

Definition 1.2. *An operator $T \in B(H)$ is called quasi-class (Q) if $T\left({T^*}^2T^2\right) = (T^*T)^2\, T$.*

2. Properties of Class (Q) Operator

Theorem 2.1. *If $T \in$ quasi-class (Q) then*

(1). λT for any real number 'λ'.

(2). Any $S \in B(H)$ that is unitarily equivalent to T.

(3). The restriction $T/_M$ of T to any closed subspace M of H that reduces to T.

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Proof.

(1). It is obvious from the definition of quasi-class (Q).

(2). Let $S \in B(\mathrm{H})$ be unitarily equivalent to T then there is a unitary operator $U \in B(H)$ such that $S = U^*TU$ which implies that $S^* = U^*T^*U$. Thus

$$\begin{aligned} S\left(S^{*2}S^2\right) &= U^*TU\left(U^*T^*UU^*T^*US^2\right) \\ &= U^*TUU^*T^*UU^*T^*UU^*TUU^*TU \\ &= U^*TT^*T^*TTU \\ &= U^*TT^{*2}T^2U \\ &= U^*\left(T\left(T^{*2}T^2\right)\right)U \end{aligned}$$

and

$$\begin{aligned} (S^*S)^2 S &= (U^*T^*UU^*TU)^2 S \\ &= (U^*T^*TU)^2 U^*TU \\ &= U^*T^*TUU^*T^*TUU^*TU \\ &= U^*(T^*T)^2TU \end{aligned}$$

Since $T(T^{*2}T^2) = (T^*T)^2 T$. We have $S(S^{*2}S^2) = (S^*S)^2 S$. Thus $S \in$ quasi-class (Q).

(3). The restriction $T/_M$ of T to any closed subspace M of H that reduces to T. By [1] we have

$$\begin{aligned} \left(T/_M\right)\left(\left(T/_M\right)^{*2}\left(T/_M\right)^2\right) &= \left(T/_M\right)\left(T^{*2}T^2/_M\right) \\ &= \left(T\left(T^{*2}T^2\right)/_M\right) \\ &= \left((T^*T)^2 T/_M\right) \\ &= \left((T^*T)^2/_M\right)\left(T/_M\right) \\ &= \left(\left(T/_M\right)^*\left(T/_M\right)\right)^2\left(T/_M\right) \end{aligned}$$

Thus $T/_M \in$ quasi-class (Q). □

Remark 2.2. *If $T \in$ quasi-class (Q) such that $T^2 = 0$ then it is not necessarily that $T = 0$, for a counter example* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *acting on R^2.*

Theorem 2.3. *If T is quasi-class (Q) operator which is a self adjoint operator if and only if T^* is quasi-class (Q) operator.*

Proof. **Case (i):** T is quasi-class (Q) we have

$$T(T^{*2}T^2) = (T^*T)^2T$$

Since T is a self adjoint we have $T^* = T$. Replace T^* by T, we get

$$(T^*)\left(((T^*)^*)^2(T^*)^2\right) = (T^*)\left(T^2T^{*2}\right) = T\left(T^{*2}T^2\right)$$

and

$$\left((T^*)^* T^*\right)^2 (T^*) = (TT^*)^2 (T^*) = (T^*T)^2 T$$

From combing above T^* is quasi-class (Q) operator.

Case (ii): Since T^* is a self adjoint of T, we have $T^* = T$. Now

$$T(T^{*^2}T^2) = T\left(T^2T^2\right) = T^5$$
$$(T^*T)^2T = \left(T^2\right)^2(T) = T^5$$

and hence $T(T^{*2}T^2) = (T^*T)^2T$. Therefore T is quasi-class (Q) operator. □

Theorem 2.4. *If $T \in$ quasi-class (Q) then $\left(T^2T^{*2}\right)T^* = T^*\left(T^*T\right)^2$.*

Proof. Since $T \in$ quasi-class (Q), $T^* \in$ quasi-class (Q). Thus we have

$$T^*\left((T^*)^{*2}(T^*)^2\right) = \left((T^*)^*(T^*)\right)^2 T^*$$

which implies that $T^*\left(T^2T^{*2}\right) = (T^*T)^2\,T^*$. □

Theorem 2.5. *Let T be any operator on a Hilbert space H. Then*

(1). $(T + T^*)$ is quasi-class (Q)

(2). TT^* is quasi-class (Q)

(3). T^*T is quasi-class (Q)

(4). $(I + T^*T), (I + TT^*)$ are quasi-class (Q)

Theorem 2.6. *If $T \in$ quasi-class (Q) then $T\left(T^2T^{*^2}\right) = (TT^*)^2\,T$.*

Proof. If $T \in$ quasi-class (Q) then if $T^* \in$ quasi-class (Q). Thus we have $T^*\left(T^{*2}T^{**2}\right) = (T^{**}T^*)^2\,T^*$. Which implies that $T\left(T^2T^{*2}\right) = (TT^*)^2\,T$ (since $T^* = T$). □

Theorem 2.7. *If T is a self adjoint operator and $T \in$ quasi-class (Q) and T^{-1} exists, then $T^{-1}$is a quasi-class (Q) operator.*

Proof. Since T is a self adjoint operator, we have $T^* = T$ (i.e.) $\left(T^{-1}\right)^* = (T^*)^{-1} = (T)^{-1}$. From the above we have T^{-1} is self adjoint operator. Further we have

$$\begin{aligned}
\left(T^{-1}\right)\left(\left(\left(T^{-1}\right)^*\right)^2\left(T^{-1}\right)^2\right) &= \left(T^{-1}\right)\left(\left((T^*)^{-1}\right)^2\left(T^{-1}\right)^2\right)\\
&= \left(T^{-1}\right)\left(\left(T^{-1}\right)^2\left(T^{-1}\right)^2\right)\\
&= \left(T^{-1}\right)^5\\
\left(\left(\left(T^{-1}\right)^*\right)\left(T^{-1}\right)\right)^2\left(T^{-1}\right) &= \left((T^*)^{-1}\left(T^{-1}\right)\right)^2\left(T^{-1}\right)\\
&= \left(\left(T^{-1}\right)\left(T^{-1}\right)\right)^2\left(T^{-1}\right)\\
&= \left(T^{-1}\right)^5 \quad (\text{Since } T^* = T)
\end{aligned}$$

Already we have proved that every self adjoint operator is quasi-class (Q) and T^{-1} is also self adjoint operator. Therefore T^{-1} is quasi-class (Q) operator. □

Theorem 2.8. *Let T be a quasi-class (Q) on H. Let S be the self adjoint operator for which T & S commute, then ST is also quasi-class (Q) operator.*

Proof. Since S is a self adjoint operator, we have $S^* = S$. Since T & S commute, we get $ST = TS$. Also $(ST)^* = (TS)^*$. This implies that $T^*S^* = S^*T^*$ and $T^*S = ST^*$. Also $(ST)^* = T^*S = ST^*$. Since T is quasi-class (Q) operator, we get

$$T\left(T^2T^{*2}\right) = (TT^*)^2\, T$$

From $ST = TS \text{and} S^* = S$, we can easily prove that

$$\begin{aligned}
(ST)^* &= T^*S = ST^*;\\
ST^{*2} &= T^{*2}S;\\
TS^{*2} &= S^{*2}T;\\
ST^2 &= T^2S \text{ and}\\
(ST)^2 &= S^2T^2.
\end{aligned}$$

Now

$$\begin{aligned}
(ST)\left((ST)^{*2}\,(ST)^2\right) &= STT^{*2}S^{*2}S^2T^2\\
&= ST^{*2}TS^{*2}S^2T^2\\
&= T^{*2}SS^{*2}TS^2T^2\\
&= T^{*2}S^{*2}SS^2TT^2\\
&= T^{*2}S^{*2}S^2ST^2T\\
&= T^{*2}S^{*2}S^2T^2ST\\
&= ((ST)^*\,(ST))^2\,(ST)
\end{aligned}$$

Hence $ST \in$ quasi-class (Q) operator. □

Theorem 2.9. *If $T \in B(H)$ is quasi normal, then $T \in$ quasi-class (Q) operator.*

Proof. Since T is quasi-normal then $TT^*T = T^*TT$ (i.e.) $TT^*T = T^*T^2$. Multiply the above by TT^*, we get

$$\begin{aligned}
TT^*TT^*T &= TT^*T^*T^2\\
T(T^*T)^2 &= T(T^{*2}T^2)\\
T(T^{*2}T^2) &= T(T^*T)^2
\end{aligned}$$

Therefore $T \in$ quasi-class (Q) operator. □

Theorem 2.10. *If $T \in B(H)$ is isometry, then $T \in$ quasi-class (Q).*

Proof. Since T is isometry, $T^*T = I$

$$(T^*T)^2 = I \quad \text{and} \quad T^{*^2}T^2 = I$$

From the above $T^{*2}T^2 = (T^*T)^2$. Now multiply (??) by T

$$T(T^{*2}T^2) = (T^*T)^2\, T$$

Therefore $T \in$ quasi-class (Q) operator. □

Theorem 2.11. *Let T be a self adjoint operator on a Hilbert space H and S be any operator on H, then S^*TS is a quasi-class (Q) operator.*

Proof. Since T is self adjoint, then $T^* = T$. Consider

$$(S^*TS)^* = S^*T^*S^{**} = S^*T^*S = S^*TS$$

S^*TS is a self adjoint operator then by Theorem 2.7, S^*TS is quasi-class (Q).

$$\begin{aligned}
i.e.,\ (S^*TS)\left((S^*TS)^{*2}(S^*TS)^2\right) &= (S^*TS)\left((S^*T^*S^{**})^2(S^*TS)^2\right)\\
&= (S^*TS)\left((S^*T^*S)^2(S^*TS)^2\right)\\
&= (S^*TS)(S^*T^*S)^4\\
&= (S^*T^*S)^5\\
\left((S^*TS)^*(S^*TS)\right)^2(S^*TS) &= \left((S^*T^*S^{**})(S^*TS)\right)^2(S^*TS)\\
&= \left((S^*T^*S)(S^*TS)\right)^2(S^*TS)\\
&= \left((S^*TS)(S^*TS)\right)^2(S^*TS)\\
&= (S^*TS)^4(S^*TS)\\
&= (S^*TS)^5
\end{aligned}$$

It is proved that S^*TS is quasi-class (Q) operator. □

Theorem 2.12. *If $T \in$ quasi-class (Q), then*

(1). If T and S are of quasi-class (Q) such that $ST = TS = T^*S = ST^* = 0$. Then TS is of quasi-class (Q).

(2). If T and S are of quasi-class (Q). Then $T+S$ is of quasi-class (Q).

Proof.

(1). $$\begin{aligned}
(TS)\left((TS)^{*2}(TS)^2\right) &= TSS^{*2}T^{*2}T^2S^2\\
&= TT^{*2}T^2SS^{*2}S^2\\
&= T^{*2}T^3S^{*2}S^3\\
&= (TS)^{*2}(TS)^3
\end{aligned}$$

Hence TS is of quasi-class (Q).

(2). $$\begin{aligned}
(T+S)\left((T+S)^{*2}(T+S)^2\right) &= (T+S)\left(T^{*2}T^2 + S^{*2}S^2\right)\\
&= TT^{*2}T^2 + SS^{*2}S^2\\
&= T^{*2}T^3 + S^{*2}S^3
\end{aligned}$$

$$(T+S)\left((T+S)^{*2}(T+S)^2\right) = (T+S)^{*2}(T+S)^3$$

Which implies that $T+S$ is of quasi-class (Q). □

Theorem 2.13. *Let T be a quasi- class (Q) operator and $TB^2 = C^2T$. Then*

(1). B commutes with U and V.

(2). C commutes with U and V.

Proof. Since $TB^2 = C^2T$. We have $T\left(T^{*2}T^2\right) = (T^*T)^2 T$. Hence $\left(T^{*2}T^2\right)T^* = T^*(T^*T)^2$. Since T is quasi-class (Q) operator, we have

$$\begin{aligned}(1).\ B^2U &= \left(T^{*2}T^2\right)\left(\frac{T+T^*}{2}\right)\\ &= \frac{\left(T^{*2}T^2\right)T + \left(T^{*2}T^2\right)T^*}{2}\\ &= \frac{T^*\left(T^{*2}T^2\right) + T\left(T^{*2}T^2\right)}{2}\\ &= \left(\frac{T+T^*}{2}\right)\left(T^{*2}T^2\right)\\ &= UB^2\end{aligned}$$

Since B is non negative definite, it follows that $BU = UB$, similarly $BV = VB$.

$$\begin{aligned}(2).\ C^2U &= (T^*T)^2\left(\frac{T+T^*}{2}\right)\\ &= \frac{\left(T^{*2}T^2\right)T + \left(T^{*2}T^2\right)T^*}{2}\\ &= \frac{T^*\left(T^{*2}T^2\right) + T\left(T^{*2}T^2\right)}{2}\\ &= \left(\frac{T+T^*}{2}\right)(T^*T)^2\\ &= UC^2\end{aligned}$$

Since C is non negative definite, it follows that $CU = UC$, similarly $CV = VC$. □

Theorem 2.14. *If T be quasi-class (Q) operator and $TB^2 = C^2T$. Then*

(1). $C^2U = UC^2$.

(2). $C^2V = VC^2$.

Proof. Since $TB^2 = C^2T$

$$\begin{aligned}&\Rightarrow T\left(T^{*2}T^2\right) = (T^*T)^2 T\\ &\Rightarrow \left(T^{*2}T^2\right)T^* = T^*(T^*T)^2\end{aligned}$$

Since T is quasi-class (Q) operator, we have

$$\begin{aligned}(1).\ C^2U &= (T^*T)^2\left(\frac{T+T^*}{2}\right)\\ &= \frac{\left(T^{*2}T^2\right)T + \left(T^{*2}T^2\right)T^*}{2}\\ &= \frac{T^*\left(T^{*2}T^2\right) + T\left(T^{*2}T^2\right)}{2}\\ &= \left(\frac{T+T^*}{2}\right)(T^*T)^2\\ &= UC^2\end{aligned}$$

(2). Similarly, $C^2V = VC^2$. □

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