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Dhage Iteration Method for IVPs of Nonlinear First Order Hybrid Functional Integrodifferential Equations of Neutral Type

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Abstract: In this paper we prove an existence and approximation result for a first order initial value problems of nonlinear hybrid functional integrodifferential equations of neutral type via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2015) and includes the existence and approximation theorems for several functional differential equations considered earlier in the literature. An example is also furnished to illustrate the hypotheses and the abstract result of this paper.

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Keywords: Hybrid neutral functional differential equation; Hybrid fixed point principle; Dhage iteration method; Existence and Approximation theorem.

1. Statement of the Problem

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the **history space** of the functional differential equation in question.

For a continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t+\theta),\ -r \leq \theta \leq 0. \tag{2}$$

The integrodifferential equations involving the history of the dynamic systems are called functional integrdifferential equations and the integrdifferential equations involving the derivative of history function are called neutral functional integrodifferential equations. It has been recognized long back the importance of such problems in the theory of differential and

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integral equations. Since then, several classes of nonlinear functional differential and integrodifferential equations of neutral type have been discussed in the literature for different qualitative properties of the solutions. Recently, the study of a special class of functional differential and integrodifferential equations is initiated by Dhage [8], Dhage and Dhage [17, 18] and Dhage and Otrocol [23] via a new Dhage iteration method and established the existence and approximation results along with algorithm for the solutions of such equations. Therefore, it is desirable to extend this new method to other classes of functional differential and integrodifferential equations involving delay in the arguments. Very recently, the present author in [10, 11] applied this new iteration method to IVPs of nonlinear first order functional differential equations involving a delay. The present paper is also an attempt in this direction and extends the Dhage iteration method to nonlinear first order functional integrodifferential equations of neutral type.

In this paper, we consider the following nonlinear first order hybrid functional integrodifferential equation (in short HFIDE) of neutral type

$$\left.\begin{aligned} \frac{d}{dt}\left[x(t)-f\big(t,x(t),x_t\big)\right] &= g\left(t,x_t,\int_0^t k(s,x_s)\,ds\right),\ t\in I,\\ x_0 &= \phi, \end{aligned}\right\} \tag{3}$$

where $\phi\in\mathcal{C}$ and $f : I\times\mathbb{R}\times\mathcal{C}\to\mathbb{R}$, $k : I\times\mathcal{C}\to\mathbb{R}$ and $g : I\times\mathcal{C}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Definition 1.1. *A function $x\in C(J,\mathbb{R})$ is said to be a solution of the HFIDE (3) if*

(1). $x_0\in\mathcal{C}$,

(2). $x_t\in\mathcal{C}$ for each $t\in I$, and

(3). the function $t\mapsto\left[x(t)-f\big(t,x(t),x_t\big)\right]$ is continuously differentiable on I and satisfies the equations in (3),

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The neutral HFIDE (3) is a new linear perturbation of second type of functional differential equation and includes several functional differential equations studied earlier in Dhage [12, 13], Dhage *et.al* [20] and Dhage *et.al* [22] as the special cases. It is known that such nonlinear equations can be handled with the hybrid operator theoretic technique involving the sum of two operators in a Banach space. See Dhage [10, 11] and the references therein. It has been discussed in Ntouyas *et al.* [27] with usual known method of Lerray-Schauder fixed point principle and established the existence theorem. The special cases of it are well-known and extensively discussed in the literature for different aspects of the solutions. See Hale [25], Dhage and Jadhav [21], Dhage [11] and the references therein. There is a vast literature on nonlinear functional differential equations of neutral type for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence and approximation theorem for the hybrid functional differential equations neutral type (3) via a new Dhage iteration method which does not require the existence of both upper and lower solution as well the related monotonic inequality and also obtain the algorithm for the solutions under some natural conditions.

The rest of the paper is organized as follows. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main result and an illustrative example is given in Sections 3.

2. Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E,\preceq,\|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x\preceq y$ or $y\preceq x$ holds. A non-empty subset C

of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [24] and the references therein. Similarly a few details of a partially ordered normed linear space is given in Dhage [7] while orderings defined by different order cones are given in Deimling [1], Guo and Lakshmikantham [24], Heikkilaá and Lakshmikantham [26] and the references therein.

We need the following definitions (see Dhage [3–7, 9] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either nondecreasing or nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a compact subset of E. Similarly, a non-empty subset S of the partially ordered Banach space E is called **partially relatively compact** if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. *Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.2 (Dhage [6, 7])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\leq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.*

Definition 2.3. *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-**function** provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially **nonlinear $\mathcal{D}$-contraction** if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{4}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k. Moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

The procedure of applying the hybrid fixed point principles of Dhage to nonlinear equations in a prartially ordered or simply ordered Banach space is called the Dhage iteration method. The **Dhage iteration method** embodied in the following applicable hybrid fixed point principle of Dhage [9] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of other hybrid fixed point theorems involving the **Dhage iteration principle** and method are given in Dhage [9, 10], Dhage and Dhage [18], Dhage *et.al* [19] and the references therein.

Theorem 2.4. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a). $\mathcal{A}$ isa partially bounded and partial nonlinear $\mathcal{D}$-contraction,

(b). $\mathcal{B}$ is partially continuous and partially compact, and

(c). there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$ or $\alpha_0 \succeq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_0 = \alpha_0,\ x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n,\ n = 0, 1, \ldots;$ converges monotonically to x^*.

Remark 2.5. *The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence and approximation results of this paper.*

Remark 2.6. *The regularity of E in above Theorem 2.4 may be replaced with a stronger continuity condition of the operator $\mathcal{A}$ and $\mathcal{B}$ on E which is a result proved in Dhage [7].*

3. Main Results

In this section, we prove an existence and approximation result for the neutral HFIDE (3) on a closed and bounded interval $J = [-r, T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the neutral HFIDE (3) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\preceq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{5}$$

and

$$x \preceq y \iff x(t) \leq y(t) \quad \text{for all} \quad t \in J. \tag{6}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\preceq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [7, 9, 10] and the references therein. The following useful lemma concerning the Janhavi subsets of $C(J, \mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \preceq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\preceq$ defined by (5) and (6) respectively. Then every partially compact subset of $C(J, \mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [10], Dhage and Dhage [16] and so we omit the details. □

We introduce an order relation $\preceq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\preceq$ defined in $C(J,\mathbb{R})$. Thus, for any $x,y \in \mathcal{C}$, $x \preceq_{\mathcal{C}} y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Moreover, if $x,y \in C(J,\mathbb{R})$ and $x \preceq y$, then $x_t \preceq_{\mathcal{C}} y_t$ for all $t \in I$.

We need the following definition in what follows.

Definition 3.2. *A function $u \in C(J,\mathbb{R})$ is said to be a solution of the HFIDE (3) if*

(i). $u_t \in \mathcal{C}$ for each $t \in I$, and

(ii). the function $t \mapsto \big[u(t) - f\big(t,u(t),u_t\big)\big]$ is continuously differentiable on I and satisfies

$$\left.\begin{aligned} &\frac{d}{dt}\big[u(t) - f\big(t,u(t),u_t\big)\big] \leq g\left(t,u_t,\int_0^t k(s,u_s)\,ds\right),\ t \in I, \\ &u_0 \preceq_{\mathcal{C}} \phi, \end{aligned}\right\} \tag{$*$}$$

Similarly, a differentiable function $v \in C(J,\mathbb{R})$ is called an upper solution of the neutral HFIDE (3) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows.

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x,y)| \leq M_f$ for all $t \in I$ and $x \in \mathbb{R}$ and $y \in \mathcal{C}$;

(H_2) There exists $\mathcal{D}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \varphi\Big(\max\big\{x_1 - y_1\,,\ \|x_2 - y_2\|_{\mathcal{C}}\big\}\Big)$$

for all $t \in I$ and $x_1,y_1 \in \mathbb{R}$ and $x_2,y_2 \in \mathcal{C}$ with $x_1 \geq y_1$, $x_2 \succeq_{\mathcal{C}} y_2$. Moreover, $\varphi(r) < r$ for $r > 0$.

(H_3) The function $k(t,x)$ is nondecreasing in x for each $t \in I$.

(H_4) The function g is bounded on $I \times \mathcal{C} \times \mathbb{R}$ with bound M_g.

(H_5) The function $g(t,x,y)$ is nondecreasing in x and y for each $t \in I$.

(H_6) neutral HFIDE (3) has a lower solution $u \in C(J,\mathbb{R})$.

Lemma 3.3. *A function $x \in C(J,\mathbb{R})$ is a solution of the neutral HFIDE (3) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \phi(0) - f(0,\phi(0),\phi) \\ \quad + f(t,x(t),x_t) + \displaystyle\int_0^t g\left(s,x_s,\int_0^s k(\tau,x_\tau)\,d\tau\right) ds, & \textit{if } t \in I, \\ \phi(t), \quad \textit{if } t \in I_0. \end{cases} \tag{7}$$

Theorem 3.4. *Suppose that hypotheses (H_1) through (H_6) hold. Then the neutral HFIDE (3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \phi(0) - f(0,\phi(0),\phi) \\ \quad + f\big(t,x_n(t),x_t^n\big) + \displaystyle\int_0^t g\left(s,x_s^n,\int_0^s k(\tau,x_\tau^n)\,d\tau\right) ds, & \textit{if } t \in I, \\ \phi(t), \quad \textit{if } t \in I_0, \end{cases} \tag{8}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} -f(0, \phi(0), \phi) + f(t, x(t), x_t) & \text{if } t \in I, \\ 0, \quad \text{if } t \in I_0, \end{cases} \tag{9}$$

and

$$\mathcal{B}x(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t g\left(s, x_s, \int_0^s k(\tau, x_\tau)\, d\tau\right) ds, & \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{10}$$

From the continuity of the functions f, g and the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.3, the neutral HFIDE (3) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{11}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.4 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq_C y_t$ for all $t \in I$ and by hypothesis (H_2), we get

$$\begin{aligned} \mathcal{A}x(t) &= \begin{cases} -f(0, \phi(0), \phi) + f(t, x(t), x_t), & \text{if } t \in I, \\ 0, \quad \text{if } t \in I_0, \end{cases} \\ &\geq \begin{cases} -f(0, \phi(0), \phi) + f(t, y(t), y_t), & \text{if } t \in I, \\ 0, \quad \text{if } t \in I_0, \end{cases} \\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{A}$ is nondecreasing on E.

Similarly, by hypothesis (H_4), we get

$$\begin{aligned} \mathcal{B}x(t) &= \begin{cases} \phi(0) + \displaystyle\int_0^t g\left(s, x_s, \int_0^s k(\tau, x_\tau)\, d\tau\right) ds, & \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0, \end{cases} \\ &\geq \begin{cases} \phi(0) + \displaystyle\int_0^t g\left(s, y_s, \int_0^s k(\tau, y_\tau)\, d\tau\right) ds, & \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0, \end{cases} \\ &= \mathcal{B}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{B}$ is also nondecreasing on E.

Step II: *$\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on E.*

Let $x, y \in E$ be any two elements such that $x \succeq y$. Then, by hypothesis (H$_2$),

$$\begin{aligned}|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq |f(t, x(t), x_t) - f(t, y(t), y_t)| \\ &\leq \varphi\left(\max\{|x(t) - y(t)|\,,\, \|x_t - y_t\|_{\mathcal{C}}\}\right) \\ &\leq \varphi(\|x - y\|)\end{aligned} \tag{12}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x - y\|)$$

for all $x, y \in E$, $x \geq y$, where $\psi(r) = \varphi(r) < r$ for $r > 0$. As a result $\mathcal{A}$ is a partially nonlinear $\mathcal{D}$-contraction on E in view of Remark 2.6.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}x_n(t) &= \begin{cases} \phi(0) + \displaystyle\int_0^t \left[\lim_{n\to\infty} g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right)\right] ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\ &= \begin{cases} \phi(0) + \displaystyle\int_0^t g\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\ &= \mathcal{B}x(t)\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$\begin{aligned}&\Big|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)\Big| \\ &\quad \leq \left|\int_0^{t_2} g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right) ds - \int_0^{t_1} g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right) ds\right| \\ &\quad \leq \left|\int_{t_1}^{t_2} \left|g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right)\right| ds\right| \\ &\quad \leq M_g|t_2 - t_1| \\ &\quad \to 0 \quad \text{as} \quad t_2 \to t_1,\end{aligned}$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(0)| + |\mathcal{B}x_n(0) - \mathcal{B}x_n(t_1)| \to 0 \ \text{ as } \ t_2 \to t_1.$$

Thus in all three cases, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the sequence $\{\mathcal{B}x_n\}$ is an equicontinuous set of functions on J. Therefore, the pointwise convergence of $\{\mathcal{B}x_n\}$ on J implies the uniform convergence and hence $\mathcal{B}x_n$ converges to $\mathcal{B}x$ uniformly on J. Consequently, $\mathcal{B}$ is a partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H$_2$),

$$\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&\leq \begin{cases} |\phi(0)| + \displaystyle\int_0^t \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) \right| ds, & \text{if } t \in I, \\ |\phi(t)|, \quad \text{if } t \in I_0, \end{cases} \\
&\leq \|\phi\| + M_f T \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E.

Next, we show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of the Banach space E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator on E into itself.

Step V: *u satisfies the operator inequality inequality $u \leq \mathcal{A}u + \mathcal{B}u$.*

By hypothesis (H$_6$), the neutral HFIDE (3) has a lower solution u defined on J. Then, by definition of the lower solution, we have

$$\left.\begin{aligned} \frac{d}{dt}\big[u(t) - f\big(t, u(t), u_t\big)\big] \leq g\left(t, u_t, \int_0^t k(s, u_s)\,ds\right),\ t \in I, \\ u_0 \preceq_C \phi. \end{aligned}\right\}$$

Integrating the above inequality from 0 to t, we get

$$u(t) \leq \begin{cases} \phi(0) - f(0, \phi(0), \phi) \\ \quad + f(t, u(t), u_t) + \displaystyle\int_0^t g\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds, & \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0, \end{cases}$$

$$= \mathcal{A}u(t) + \mathcal{B}u(t)$$

for all $t \in J$. As a result we have that $u \leq \mathcal{A}u + \mathcal{A}u$.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.4 and so the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the integral equation (7), and a fortiori the hybrid functional differential equation (3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotonically to x^*. This completes the proof. □

Remark 3.5. *The conclusion of Theorem 3.4 also remains true if we replace the hypothesis* (H_6) *with the following ones:*

(H_7) *The neutral HFIDE* (3) *has an upper solution* $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.4 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.6. *Given the closed and bounded intervals* $I_0 = [-1, 0]$ *and* $I = [0, 1]$, *consider the neutral HFIDE of neutral type*

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f_1(t, x(t), x_t)\big] &= g_1\left(t, x_t, \int_0^t k_1(s, x_s)\,ds\right), \ t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{13}$$

where $\phi \in \mathcal{C}$ *and* $k_1 : I \times \mathcal{C} \to \mathbb{R}$, $f_1 : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$, $g_1 : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions given by*

$$\phi(t) = \sin t, \ \ t \in [-1, 0],$$

$$k_1(t, x) = \begin{cases} 1, & \text{if } x \preceq_{\mathcal{C}} 0, \\ \ln(1 + \|x\|_{\mathcal{C}}) + 1, & \text{if } 0 \preceq_{\mathcal{C}} x, \ x \neq 0, \end{cases}$$

$$f_1(t, x, y) = \begin{cases} \dfrac{1}{2}\left[\dfrac{x}{x+1} + \dfrac{\|y\|_{\mathcal{C}}}{1 + \|y\|_{\mathcal{C}}}\right] + 1, & \text{if } 0 \leq x, \ 0 \leq_{\mathcal{C}} y, \ x \neq 0, \ y \neq 0, \\ \dfrac{1}{2}\dfrac{x}{x+1} + 1, & \text{if } 0 \leq x, \ x \neq 0, \ y \leq_{\mathcal{C}} 0, \\ \dfrac{1}{2}\dfrac{\|y\|_{\mathcal{C}}}{1 + \|y\|_{\mathcal{C}}} + 1, & \text{if } x \leq 0, \ 0 \leq_{\mathcal{C}} y, \ y \neq 0, \\ 1, & \text{if } x \leq 0, \ y \leq_{\mathcal{C}} 0, \end{cases}$$

and

$$g_1(t, x, y) = \begin{cases} \tanh(\|x\|_{\mathcal{C}}) + \tanh y + 1, & \text{if } x \geq_{\mathcal{C}} 0, \ x \neq 0, \ -\infty < y < \infty, \\ \tanh y + 1, & \text{if } x \leq_{\mathcal{C}} 0, \ -\infty < y < \infty, \end{cases}$$

for all $t \in I$.

Clearly, f_1 *is continuous and bounded on* $I \times \mathbb{R} \times \mathcal{C}$ *with bound* $M_{f_1} = 2$ *and so the hypothesis* (H_1) *is satisfied. We show that* f_1 *satisfies the hypothesis* (H_2). *Let* $x_1, y_2 \in \mathbb{R}$ *and* $x_2, y_2 \in \mathcal{C}$ *be such that* $x_1 \geq y_1$ *and* $x_2 \succeq_{\mathcal{C}} y_2 \succeq_{\mathcal{C}} 0$. *Then*

$\|x\|_{\mathcal{C}} \geq \|y\|_{\mathcal{C}} > 0$ *and therefore, we have*

$$\begin{aligned} 0 &\leq f_1(t,x_1,x_2) - f_1(t,y_1,y_2) \\ &= \frac{1}{2}\left|\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1}\right| + \frac{1}{2}\left|\frac{\|x_2\|_{\mathcal{C}}}{1+\|x_2\|_{\mathcal{C}}} - \frac{\|y_2\|_{\mathcal{C}}}{1+\|y_2\|_{\mathcal{C}}}\right| \\ &\leq \frac{1}{2}\left[\frac{|x_1-y_1|}{1+|x_1-y_1|} + \frac{\|x_2-y_2\|_{\mathcal{C}}}{1+\|x_2-y_2\|_{\mathcal{C}}}\right] \\ &\leq \frac{\max\{|x_1-y_1|\,,\,\|x_2-y_2\|_{\mathcal{C}}\}}{1+\max\{|x_1-y_1|\,,\,\|x_2-y_2\|_{\mathcal{C}}\}} \\ &= \psi\Big(\max\left\{x_1-y_1\,,\,\|x_2-y_2\|_{\mathcal{C}}\right\}\Big) \end{aligned}$$

for all $t \in I$*, where* $\psi(r) = \dfrac{r}{1+r} < r$*,* $r > 0$*. Similarly, other cases of the variables* x, y *of* $f(t,x,y)$ *are treated and we obtain the final estimate as*

$$0 \leq f_1(t,x_1,x_2) - f_1(t,y_1,y_2) \leq \psi\Big(\max\{x_1-y_1\,,\,\|x_2-y_2\|_{\mathcal{C}}\}\Big)$$

for all comparable $x_1, y_2 \in \mathbb{R}$ *and* $x_2, y_2 \in \mathcal{C}$*. Thus, the function* f_1 *satisfies the hypothesis* (H_2)*.*

Next, $k_1(t,x)$ *is is nondcreasing in* x *for each* $t \in [0,1]$ *and so the hypothesis* (H_3) *is satisfied.. Again, the function* g *is bounded on* $I \times \mathcal{C} \times \mathbb{R}$ *with* $M_{f_1} = 3$ *and so the hypothesis* (H_4) *is satisfied. Again, it is easy to verify that the function* $g_1(t,x,y)$ *is nondecreasing in* x *as well as in* y *for each* $t \in [0,1]$ *and so the hypothesis* (H_5) *is satisfied. Finally, the function* $u \in C(J,\mathbb{R})$ *defined by*

$$u(t) = \begin{cases} 1-t, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0], \end{cases}$$

is a lower solution of the neutral HFIDE (13) *defined on* J*. Thus, the functions* k_1*,* f_1 *and* g_1 *satisfy all the hypotheses* (H_1) *through* (H_6)*. Hence we apply Theorem 3.4 and conclude that the neutral HFIDE (13) has a solution* x^* *on* J *and the sequence* $\{x_n\}$ *of successive approximation defined by*

$$x_0(t) = \begin{cases} -1-t, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0] \end{cases}$$

$$x_{n+1}(t) = \begin{cases} -f_1(0,0,\phi) + f_1(s,x_n(s),x_s^n) \\ \quad + \displaystyle\int_0^t g_1\left(s, x_s^n, \int_0^s k_1(\tau, x_\tau^n))\,d\tau\right) ds, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0], \end{cases}$$

for $n = 0, 1, \ldots,$ *converges monotonically to* x^**.*

Remark 3.7. *The conclusion given in Example 3.6 also remains true if we replace the lower solution* u *with the upper solution* v *of the neutral HFIDE (13) defined by*

$$v(t) = \begin{cases} 3t+2, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0]. \end{cases}$$

Remark 3.8. *We note that if the neutral HFIDE* (3) *has a lower solution u as well as an upper solution v such that $u \preceq v$, then under the given conditions of Theorem 3.4 it has corresponding solutions x_* and y^* and these solutions satisfy the inequality*

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* and y^* are the minimal and maximal solutions of the neutral HFIDE (3) *respectively in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C(J, \mathbb{R})$ defined by*

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because the order relation $\preceq$ defined by (6) *is equivalent to the order relation defined by the order cone*

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \textit{ for all } \ t \in J\},$$

which is a closed set in the Banach space $C(J, \mathbb{R})$. A few details concrning the order relation by the order cones and the Janhavi sets in an ordered banach space are given in Dhage [14, 15].

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