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Fuzzy Filters of Meet-semilattices

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Abstract: The notion of fuzzy filter of a meet-semilattice with truth values in a general frame is introduced and proved certain properties of these. In particular, it is prove that the fuzzy filters form an algebraic fuzzy system. Also, we have established a procedure to construct any fuzzy filter form a given family of filters with certain conditions. Dually, in this paper the notion of fuzzy ideal of a join-semilattice is introduced and discussed certain properties of these, which are analogues to those of fuzzy filters of meet-semilattices.

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1. Introduction

The Introduction of the concept of a fuzzy subset of a set X as a function from X into the closed interval [0.1] by Zadeh in his pioneering paper [19]. After the notion of fuzzy sets, Rosenfield [12] defined the notion of a fuzzy subgroup of a group and since then several researchers have applied this concept to abstract algebras such as semigroup, ring, semiring, field, near-ring, lattice etc. For example, Kuroki [7] investigated the properties of fuzzy ideals of a semigroup. Malik and Moderson [10] worked on fuzzy subrings and ideals of rings. Liu [9] introduced fuzzy invarient subgroups and fuzzy ideals. Attallah [1] and Lehmke [8] introduced fuzzy ideals of lattices. Jun, Kim and Oztirk [5] introduced fuzzy maximal ideals of Gamma near-rings. Katsaras and Liu [6] introduced fuzzy vector spaces and fuzzy topological vector spaces.

In most of the works mentioned above, the fuzzy statements take truth values in the intervel $[0, 1]$ of real numbers. However, Gougen [3] realised that the unit intervel $[0, 1]$ is insufficient to have the truth values of general fuzzy statements and it is necessary to consider a more general class of lattices in place of $[0, 1]$ by means of a complete lattice in an attempt to make a generalised study of fuzzy set theory by studying L-fuzzy sets. Further, to make an abstract study, Swamy and other researchers in [13–18] consider a general complete lattice satasfying the infinite meet distributivity to have truth values of fuzzy statements. This type of lattice is called a frame. In this paper, we introduce the notion of fuzzy filter of a meet-semilattice $(S, \wedge)$ $\big($dually, fuzzy ideal of a join-semilattice $(S, \vee)$ $\big)$, having truth values in a general frame L and prove certain properties of these.

Throughout this paper, L stands for a frame $(L, \wedge, \vee, 0, 1)$; i.e., L is a non-trivial complete lattice in which the infinite meet distributive law is satisfied. That is;

$$\alpha \wedge \Big(\bigvee_{\beta \in M} \beta \Big) = \bigvee_{\beta \in M} (\alpha \wedge \beta)$$

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for all $\alpha \in L$ and $M \subseteq L$. Here the operations $\vee$ and $\wedge$ are, respectively, supremum and infimum in the lattice L and $0, 1$ are respectively, greatest and smallest elements in L. Also S stands for a meet-semilattice $(S, \wedge)$ $\big($ join-semilattice $(S, \vee)$ $\big)$, unless otherwise stated. As usual, by an L-fuzzy subset of S, we mean a mapping of S into L. For the sake of convenience, we write fuzzy subset instead of L-fuzzy subset. A fuzzy subset $A : S \to L$ is said to be non-empty if it is not the constant map which assumes the value 0 of L. For any fuzzy subset A of S and $\alpha \in L$, the set

$$A_\alpha = A^{-1}[\alpha, 1] = \{x \in S : \alpha \le A(x)\}.$$

is called the α-cut of A.

2. Preliminaries

In his section, some basic definitions, results and notations which will be needed later on are presented.

Definition 2.1. *For any non-empty set X, any subset R of $X \times X$ is called a binary relation on X. A binary relation R on X is said to be a partial order on X if*

(1). xRx for all $x \in X$ (reflexive)

(2). xRy and $yRx \Rightarrow x = y$ (antisymmetric)

(3). xRy and $yRz \Rightarrow xRz$ (transitive)

The partial orders are usually denote by the symbols $\le$, $\ge$, $\subseteq$, $\supseteq$ etc. A non-empty set X together with a partial order $\le$ is called a partial ordered set or poset and we simply denote it by $(X, \le)$.

Definition 2.2. *Let $(X, \le)$ be a poset. Then the relation*

$$\le^{-1} = \{(x, y) \in X \times X : \ y \le x\}$$

is also a partial order on X and it is denoted by $\ge$. $\ge$ is called the dual order of $\le$. That is $a \ge b$ if and only if $b \le a$.

Definition 2.3. *A partial order $\le$ on as set X is called a total order or linear order on X, if for any $x, y \in X$, either $x \le y$ or $y \le x$ and, in this case, $(X, \le)$ is called a total ordered set or a chain.*

Definition 2.4. *Let $(X, \le)$ be a poset, $Y \subseteq X$ and $a \in X$.*

(1). If $x \le a$ for all $x \in X$, then a is called the largest element in X.

(2). If $a \le x$ for all $x \in X$, then a is called the smallest element in X.

(3). If $x \le a$ for all $x \in Y$, then a is called an upper bound of Y.

(4). If $a \le x$ for all $x \in Y$, then a is called a lower bound of Y.

(5). A lower bound a of Y is called the greatest lowerbound, which will be denoted by g.l.b Y or inf Y, if $b \le a$ for all lower bounds b of Y. Similarly, an upperbound a of Y is called the least upper bound, which will be denote by l.u.b Y or sup Y, if $a \le b$ for all upper bounds b of Y.

Definition 2.5. *A poset $(X, \le)$ is called a meet-semilattice (dually, join-semilattice) if $inf\{a, b\}$ (dully, $sup\{a, b\}$) exists in X for any $a, b \in X$*

Lemma 2.6. *The dual of a meet-semilattice is a join-semilattice, and conversely.*

Proposition 2.7. *The poset $(X, \leq)$ is a lattice if and only if it is meet and join-semilattice.*

Definition 2.8. *A semilattice is an algebra $S = (S, \circ)$ with one binary operation o satisfying the identities*

$$(x \circ y) \circ z = x \circ (y \circ z)$$

$$x \circ y = y \circ x$$

$$x \circ x = x$$

i.e, the operation $\circ$ is associative, commutative and idempotent.

Proposition 2.9. *Let $(X, \leq)$ be a meet-semilattice (join-semilattice). Then the algebra $X = (X, \wedge)$ $\Big((X, \vee)\Big)$ is a semilattice, when $a \wedge b = inf\{a, b\}\Big(a \vee b = sup\{a, b\}\Big)$ for any $a, b \in X$.*

Theorem 2.10. *Let $S = (S, \circ)$ be a semilattice. For any a and $b \in S$, define binary relations $\leq_\wedge$ and $\leq_\vee$ on S by*

$$a \leq_\wedge b \quad \text{if and if only} \quad a \circ b = a \quad \text{and} \quad a \leq_\vee b \quad \text{if and if only} \quad a \circ b = b.$$

Then $\leq_\wedge$ and $\leq_\vee$ are partial orders on S and consequently, $(S, \leq_\wedge)$ is a meet-semilattice in which $a \wedge b = a \circ b$ for all $a, b \in S$, and $(S, \leq_\vee)$ is a join-semilattice in which $a \vee b = a \circ b$ for all $a, b \in S$.

In the light of the previous theorem, semilattices can be alternatively considered as meet or join-semilattices, respectively.

Theorem 2.11. *Let $(S, \wedge)$ be a meet-semilattice with greatest element 1. Let $F(S)$ be the set of all filters of S. Then $(F(S), \subseteq)$ is a complete lattice. $F(S)$ called the lattice of filters of S.*

Definition 2.12. *Let $\mathscr{A}$ be a non-empty class of subsets of a non-empty set X. A subclass $\mathscr{B}$ of $\mathscr{A}$ is said to be directed above if, for any B and C $\in \mathscr{B}$, there exists $D \in \mathscr{B}$ such that $B \subseteq D$ and $C \subseteq D$.*

(1). $\mathscr{A}$ is said to be closed under unions of directed above subclasses if, for any directed above subclass $\mathscr{D}$ of $\mathscr{A}$, $\bigcup\limits_{D \in \mathscr{D}} D \in \mathscr{A}$.

(2). $\mathscr{A}$ is said to be closure set system on X if $\mathscr{A}$ is closed under arbitrary intersection; that is $\mathscr{B} \subseteq \mathscr{A} \Rightarrow \bigcup\limits_{B \in \mathscr{B}} B \in \mathscr{A}$.

3. Fuzzy Filters

A filter of a meet-semilattice $(S, \wedge)$ is a non-empty subset F of S such that, for all $a, b \in S$, $a \wedge b \in F$ if and only if $a, b \in F$. A filter can also be characterised by:

(1). $a, b \in F \Rightarrow a \wedge b \in F$ (F is closed under $\wedge$);

(2). $a \in F$ and $a \leqslant x \Rightarrow x \in F$ (F is final segment)

Now, we introduce the notion of fuzzy filter of a meet-semilattice $S = (S, \wedge)$ with truth values in a general frame L. Here after L stands for a general frame.

Definition 3.1. *A fuzzy subset A of S is said to be an L-fuzzy filter (simply, fuzzy filter) of S if $A(x_0) = 1$ for some $x_0 \in S$ and $A(x \wedge y) = A(x) \wedge A(y)$ for all $x, y \in S$.*

The following is a characterisation of fuzzy filters.

Theorem 3.2. *The following are equivalent to each other for any fuzzy subset A of S.*

(i). A is a fuzzy filter of S.

(ii). $A(x_0) = 1$ for some $x_0 \in S$,

(iii). $A(x \wedge y) \geq A(x) \wedge A(y)$ and $x \leq y \Rightarrow A(y) \geq A(x)$ (i.e., A is an isotone) for all $x, y \in S$.

(iv). A_α is a filter of S for all $\alpha \in L$.

Proof. $(i) \Rightarrow (ii)$: It is clear.
$(ii) \Rightarrow (iii)$: Let $\alpha \in L$. By (ii), $A(x_0) = 1 \geq \alpha$ for some $x_0 \in S$ and hence $x_0 \in A_\alpha$. Therefore A_α is non-empty. Let $a, b \in A_\alpha$. Then $\alpha \leq A(a)$ *and* $\alpha \leq A(b)$. Again by (ii), $\alpha \leq A(a) \wedge A(b) \leq A(a \wedge b)$ and hence $a \wedge b \in A_\alpha$. Further, if $a \in A_\alpha$ and $a \leq x$, then $\alpha \leq A(a) \leq A(x)$, since A is an isotone. Therefore $x \in A_\alpha$. Thus A_α is filter of S.
$(iii) \Rightarrow (ii)$: It is a simple consequence of the transfer principle for fuzzy sets. □

Let X be a non-empty subset of S, and let $[X)$ denote the smallest filter containing X in S. It is well known that

$$[X) = \{a \in S : \bigwedge_{i=1}^{n} x_i \leq a \text{ for some } x_i \in X\} \text{ and } [a) = \{x \in S : a \leq x\} \textit{ for any } a \in S$$

Lemma 3.3. *Let A be a fuzzy filter of S and X a non-empty subset of S, and $x, y \in S$. We have*

(i). $x \in [X) \Rightarrow A(x) \geq \bigwedge_{i=1}^{m} A(a_i)$ for some $a_1, a_2, \ldots a_m \in X$

(ii). $x \in [y) \Rightarrow A(x) \geq A(y)$

(iii). If S has the greatest element 1, then $A(1) = 1$.

Proof.

(i). Let $x \in [X)$. Then $\bigwedge_{i=1}^{n} a_i \leq x$ for some $a_i \in X$. Therefore, $A(x) \geq A(\bigwedge_{i=1}^{n} a_i) = \bigwedge_{i=1}^{n} A(a_i)$ (since A is an isotone).

(ii). It is clear from the fact that A is an isotone.

(iii). Suppose that $A(x_0) = 1$ for some $x_0 \in S$. If S has the greatest element 1, then $x_0 \leq 1$ and hence $A(x_0) = 1 \leq A(1)$, and hence $A(1) = 1$. □

Let $\mathcal{F}F(S)$ denote the set of all fuzzy filters of a meet-semilattice $(S, \wedge)$ with greatest element 1. For any A and $B \in \mathcal{F}F(S)$, we define $A \leq B$ if and only if $A(x) \leq B(x)$ for all $x \in S$. Then $(\mathcal{F}F(S), \leq)$ is a poset.
Now the following is straight forward verification.

Theorem 3.4. *$(\mathcal{F}F(S), \leq)$ is a complete lattice in which, for any family $\{A_i : i \in \Delta\}$ of fuzzy filters of S, the g.l.b and l.u.b are given by*

$$\bigwedge_{i \in \Delta} A_i = \textit{ The point-wise infimum of } A_i's \textit{ and } \bigvee_{i \in \Delta} A_i = \textit{ The point-wise infimum of } \{A \in \mathcal{F}F(S) : A_i \leq A \textit{ for all } i \in \Delta\}.$$

If A is any non-empty fuzzy subset of S, then the point-wise infimum of fuzzy filters containing A is non-empty and hence a fuzzy filter which becomes the fuzzy filter generated by A and is denoted by $\overline{A}$. For more description of $\bar{A}$, we prove the following.

Theorem 3.5. *Let A be a fuzzy subset of S. Then the fuzzy filter $\bar{A}$ generated by A is given by*

$$\overline{A}(x_0) = 1 \quad \textit{for some } x_0 \in S \textit{ and } \bar{A}(x) = \bigvee \Big\{ \bigwedge_{i=1}^{n} A(a_i) : a_1, a_2, \ldots a_n \in S, \quad \bigwedge_{i=1}^{n} a_i \leq x \Big\} \textit{ for any } x_0 \neq x \in S.$$

Proof. Define $B(x) = \bigvee\Big\{\bigwedge_{i=1}^{n} A(a_i) : a_1, a_2, \dots a_n \in S \text{ and } \bigwedge_{i=1}^{n} a_i \le x\Big\}$. Clearly $A(x) \le B(x)$ for all $x \in S$ and hence $A \le B$. Let $x, y \in S$ and $x \le y$. Then, for any $a_1, a_2, \dots a_n \in S$,

$$\bigwedge_{i=1}^{n} a_i \le x \Rightarrow \bigwedge_{i=1}^{n} a_i \le y \Rightarrow \bigwedge_{i=1}^{n} A(a_i) \le B(y)$$

which implies that $B(x) \le B(y)$ and hence B is an isotone and it follows that $B(x \wedge y) \le B(x) \wedge B(y)$ for all $x, y \in S$. Now, by the infinite meet distributivity in L, we have

$$\begin{aligned} B(x) \wedge B(y) &= \Big(\bigvee\Big\{\bigwedge_{i=1}^{n} A(a_i) :\ a_i \in S \text{ and } \bigwedge_{i=1}^{n} a_i \le x\Big\}\Big) \wedge \Big(\bigvee\Big\{\bigwedge_{j=1}^{m} A(b_j) :\ b_j \in S \text{ and } \bigwedge_{j=1}^{m} b_j \le y\Big\}\Big) \\ &= \bigvee\Big\{\bigwedge_{i=1}^{n} A(a_i) \wedge \bigwedge_{j=1}^{m} A(b_j) :\ \bigwedge_{i=1}^{n} a_i \wedge \bigwedge_{j=1}^{m} b_j \le x \wedge y\Big\} \\ &\le B(x \wedge y). \end{aligned}$$

By Theorem 3.2, B is a fuzzy filter of S. If C is a fuzzy filter of S and $A \le C$, then, for any $x \in S$ and $a_1, a_2, \dots\dots a_n \in S$ with $\bigwedge_{i=1}^{n} a_i \le x$,

$$\bigwedge_{i=1}^{n} A(a_i) \le \bigwedge_{i=1}^{n} C(a_i) = C\Big(\bigwedge_{i=1}^{n} a_i\Big) \le C(x)$$

it follows that $B(x) \le C(x)$ for all $x \in S$, so that $B \le C$. Thus $B = \overline{A}$. □

Corollary 3.6. *Let $\{A_i\}_{i\in\Delta}$ be a class of fuzzy filters of S. Then the supremum $\bigvee_{i\in\Delta} A_i$ of $\{A_i\}_{i\in\Delta}$ in $\mathcal{F}F(S)$ is given by $\Big(\bigvee_{i\in\Delta} A_i\Big)(x) = \bigvee\Big\{\bigwedge_{a\in X} B(a) : x \in [X),\ X$ is a non-empty finite subset of $S\Big\}$, where $B(x) = \bigvee\Big\{A_i(x) : i \in \Delta\Big\}$ (i.e., the point-wise supremum of A_i's).*

Corollary 3.7. *For any fuzzy filters A and B of S, the supremum $A \vee B$ is given by*

$$(A \vee B)(x) = \bigvee\Big\{\bigwedge_{a\in X} \big(A(a) \vee B(a)\big) : x \in [X),\ X \text{ is a non-empty finite subset of } S\Big\}.$$

For any subset X of S, the characteristic map $\chi_X : S \mapsto L$ is defined by

$$\chi_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$

it can be easily observed that χ_X is a fuzzy filter of S if and only if X is a filter of S. Now, one can easily seen that the correspondence $X \mapsto \chi_X$ establishes a monomorphism from the complete lattice $(F(S), \subseteq)$ of all filters of S in to the complete lattice $(\mathcal{F}F(S), \le)$ of all fuzzy filters of S. Also, for any filter F of S, $\overline{\chi_F} = \chi_{[F)}$. In Theorem 3.2, we have proved that the $\alpha-$cuts of any fuzzy filter A of a meet-semilattice $(S, \wedge)$ are filters of S. Infact these $\alpha-$cuts completely determine the fuzzy filter in the sense of the following.

Theorem 3.8. *Let $(S, \wedge)$ be a semilattice with greatest element 1 and $\{F_\alpha\}_{\alpha\in L}$ a class of filters of S such that $\bigcap_{\alpha\in M} F_\alpha = F_{\bigvee_{\alpha\in M}\alpha}$, for any $M \subseteq L$. For any $x \in S$ define $A(x) = \vee\{\alpha \in L : x \in F_\alpha\}$. Then A is a fuzzy filter of S such that $A_\alpha = F_\alpha$, for any $\alpha \in L$.*

Proof. Clearly, $\alpha \le \beta \Rightarrow F_\beta \subseteq F_\alpha$, for any $\alpha, \beta \in L$. By the definition of A, we have $x \in F_\beta \Rightarrow \beta \le A(x) \Rightarrow x \in A_\beta$ for any $x \in S$ and $\beta \in L$. Therefore $F_\beta \subseteq A_\beta$ for all $\beta \in L$ On the otherhand,

$$x \in A_\beta \Rightarrow \beta \le A(x) = \wedge\Big\{\alpha \in L : x \in F_\alpha\Big\}$$

$$
\begin{aligned}
&\Rightarrow \beta = \beta \wedge \Big(\vee \{\alpha \in L : x \in F_\alpha\}\Big) \\
&\Rightarrow \beta = \vee\{\beta \wedge \alpha : x \in F_\alpha\} && \text{(by the infinite meet distributivity in } L) \\
&\Rightarrow F_\beta = \bigcap_{x \in F_\alpha} F_{\beta\wedge\alpha} \\
&\Rightarrow x \in F_\beta && \text{(since } \alpha \mapsto F_\alpha \text{ is an antitone)}
\end{aligned}
$$

Therefore $A_\beta = F_\beta$ for all $\beta \in L$. By Theorem 3.2, A is a fuzzy filter of S. The converse of above theorem is true, since, for any fuzzy filter A of S, the α-cuts A_α's are filter of S and for any $M \subseteq L$, $\bigcap_{\alpha\in M} A_\alpha = A_{\bigvee_{\alpha\in M}\alpha}$ and $A(x) = \vee\{\alpha \in L : x \in A_\alpha\}$, for any $x \in S$. □

It is well knows that any closure set system forms a complete lattice with respect of the inclusion ordering ($\subseteq$) and, conversely, any complete lattice is isomorphic to a closure set system. Also, it is known that a closure set system $\mathscr{A}$ is an algebric lattice if and only if $\mathscr{A}$ is closed under unions of directed above subclasses. Further, if $(S, \wedge)$ be a semilattice with greatest element 1 and $F(S)$ the class of all filters of S, then $F(S)$ is a closure set system which is closed under unions of directed above subclasses and hence $F(S)$ is an algebraic lattice.

In view of the above, we define the following

Definition 3.9. *Let $\mathscr{C}$ be a class of fuzzy subsets of a set X. A subclass $\{A_i\}_{i\in\Delta}$ of $\mathscr{C}$ is called as directed above if, for any $i, j \in \Delta$ there is $K \in \Delta$ such that $A_i \leq A_K$ and $A_j \leq A_K$. $\mathscr{C}$ is said to be an algebric fuzzy system if, $\mathscr{C}$ is ciosed under point-wise infimums and point-wise supremums of directed subclasses.*

Theorem 3.10. *Let $(S, \wedge)$ be a semilattice with greatest element 1. Then the class $\mathcal{F}F(S)$ of all fuzzy filter of S is an algebraic.*

Proof. Let $\{A_i\}_{i\in\Delta}$ be a directed above class of fuzzy filters of S and define $A : S \to L$ by

$$A(x) = \bigvee_{i\in\Delta} A_i(x) \text{ (the point-wise supremum)}$$

Clearly $A(1) = 1$. Now, let x and $y \in S$.

$$
\begin{aligned}
x \leq y \Rightarrow A_i(x) &\leq A_i(y) \;\; for\ all \;\; i \in \Delta \\
\Rightarrow \bigvee_{i\in\Delta} A_i(x) &\leq \bigvee_{i\in\Delta} A_i(y) \\
\Rightarrow A(x) &\leq A(y)
\end{aligned}
$$

it follows that A is an isotone and hence that $A(x \wedge y) \leq A(x) \wedge A(y)$. On the other hand, by the infinite meet distributivity in L,

$$
\begin{aligned}
A(x) \wedge A(y) &= \Big(\bigvee_{i\in\Delta} A_i(x)\Big) \wedge \Big(\bigvee_{j\in\Delta} A_j(y)\Big) \\
&= \bigvee_{i,j\in\Delta} \Big(A_i(x) \wedge A_j(y)\Big) \qquad (*)
\end{aligned}
$$

Now, for any $i, j \in \Delta$, there exists $K \in \Delta$ such that $A_i \leq A_K$ and $A_j \leq A_K$ and hence

$$
\begin{aligned}
A_i(x) \wedge A_j(y) &\leq A_K(x) \wedge A_K(y) \\
&= A_K(x \wedge y) \leq A(x \wedge y).
\end{aligned}
$$

Therefore, by (*) it follows that $A(x) \wedge A(y) \leq A(x \wedge y)$ and hence $A(x \wedge y) = A(x) \wedge A(y)$. Thus A is a fuzzy filter of S. □

Finally in this section we prove that the distributivity of a semilattice $(S, \wedge)$ can be extended to that of the lattice $\mathcal{F}F(S)$ of fuzzy filters of S. We recall that the map $x \mapsto [x)$ is an embedding of S into the lattice $\mathcal{F}(S)$ of filter of S. Also, the mapping $F \mapsto \chi_F$ is an embedding of $F(S)$ into $\mathcal{F}F(S)$. Thus S is isomorphic to a sublattice of $\mathcal{F}(S)$ and $\mathcal{F}(S)$ is isomorphic to a sublattice of $\mathcal{F}F(S)$.

Recall that a meet-semilattice $(S, \wedge)$ is said to be distributive if for any a, b and $c \in S$,

$$b \wedge c \leq a \Rightarrow \text{there exists } b_1, c_1 \in S \text{ such that } b_1 \geq b,\ c_1 \geq c \text{ and } a = b_1 \wedge c_1.$$

In view of the above, we prove the following.

Theorem 3.11. *Let $(S, \wedge)$ be a semilattice with greatest element 1. Then the following are equivalent to each other:*

(1). $\mathcal{F}F(S)$ is a distributive lattice.

(2). $F(S)$ is a distributive lattice.

(3). S is distributive.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) : Suppose that S is distributive. Let A, B and $C \in \mathcal{F}F(S)$. Clearly $(A \wedge B) \vee (A \wedge C) \leq A \wedge (B \vee C)$. On the otherhand, let $x \in S$. By the infinite meet distributivity in L, we have

$$\begin{aligned}(A \wedge (B \vee C))(x) &= A(x) \wedge (B \vee C)(x) \\ &= A(x) \wedge \Big[\bigvee_{\substack{F \Subset S \\ \wedge F \leq x}} \Big(\bigwedge_{a \in F} \Big(B(a) \vee C(a) \Big)\Big)\Big] \\ &= \bigvee_{\substack{F \Subset S \\ \wedge F \leq x}} \Big[\bigwedge_{a \in F} \Big(A(x) \wedge B(a) \Big) \vee \Big(A(x) \wedge C(a) \Big)\Big)\Big] \qquad (*)\end{aligned}$$

where $F \Subset S$ denotes that F is a finite subset of S and $\wedge F$ denotes the *inf* F. *Now, if $F = \{a_1, a_2, \ldots a_n\}$ and $\bigwedge F = \bigwedge_{i=1}^{n} a_i \leq x$ then, by the distributivity in S, there exists $b_1, b_2, \ldots b_n \in S$ such that each $b_i \geq a_i$ and $x = \bigwedge_{i=1}^{n} b_i$. Now consider*

$$\begin{aligned}\bigwedge_{a \in F} \Big(\Big(A(x) \wedge B(a) \Big) \vee \Big(A(x) \wedge C(a) \Big) \Big) &= \bigwedge_{i=1}^{n} \Big(\Big(A(x) \wedge B(a_i) \Big) \vee \Big(A(x) \wedge C(a_i) \Big) \Big) \\ &\leq \bigwedge_{i=1}^{n} \Big(\Big(A(b_i) \wedge B(b_i) \Big) \vee \Big(A(b_i) \wedge C(b_i) \Big) \Big) \text{(since } A, B, C \text{ are isotones)} \\ &= \bigwedge_{i=1}^{n} \Big((A \wedge B)(b_i) \vee (A \wedge C)(b_i) \Big) \\ &\leq \Big((A \wedge B) \vee (A \wedge C) \Big)(x) \qquad \text{(since } x = \bigwedge_{i=1}^{n} b_i)\end{aligned}$$

Therefore, by (*), $A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)$ and hence $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$. Thus $\mathcal{F}F(S)$ is a distributive lattice. □

4. Fuzzy ideals of join-semilattices

Recall that, an ideal of a join-semilattice $S = (S, \vee)$ is a non-empty subset I of S such that, for all $a, b \in S$, $a \vee b \in I$ if and only if $a, b \in I$. In other words, I is an ideal of S if,

(1). $a, b \in I \Rightarrow a \vee b \in I$ (I is closed under $\vee$) and

(2). $a \in I$ and $x \leqslant a \Rightarrow x \in I$ (I is an initial segment)

As we have mentioned in the preliminaries, the partial order $\leq_\vee$ on a semilattice $(S, \circ)$ is precisely the inverse (or dual) of the partial order $\leq_\wedge$ on S. Also note that a subset I of S is an ideal of the join-semilattice $(S, \leq_\vee)$ if and only if I is a filter of the meet-semilattice $(S, \leq_\wedge)$. In this section, we introduce the notion of fuzzy ideal (or simply, fuzzy ideal) of a join-semilattice $(S, \vee)$ with truth values in a general frame L and discuss certain properties of these, which are analogues to those of fuzzy filters of a meet-semilattice $(S, \wedge)$. The proofs of most of the results are simply dual to the corresponding results on fuzzy filters. For this reason, we simply state the results and skip their proofs.

Definition 4.1. *Let $S = (S, \vee)$ be a join-semilattice. A fuzzy subset A of S is called an L-fuzzy ideal (or simply, a fuzzy ideal) of S if $A(x_0) = 1$ for some $x_0 \in S$ and $A(x \vee y) = A(x) \wedge A(y)$ for all x and $y \in S$.*

In the following, any ideal of S can be identified with a fuzzy ideal of S.

Theorem 4.2. *For any subset I of S, I is an ideal of S if and only if χ_I is a fuzzy ideal of S.*

The following is a characterization of fuzzy ideals.

Theorem 4.3. *The following are equivalent to each other for any fuzzy subset A of S*

(1). A is a fuzzy ideal of S.

(2). $A(x_0) = 1$ for some $x_0 \in S$, $A(x \vee y) \geq A(x) \wedge A(y)$, and $x \leq y \Rightarrow A(x) \geq A(y)$ (i.e., A is an antitone) for all $x, y \in S$.

(3). the α-cut A_α is an ideal of S, for all $\alpha \in L$.

Let us recall that, for any non-empty subset X of S, the ideal generated by X is

$$(X] = \{a \in S : a \leq \bigvee_{i=1}^{n} x_i \text{ for some } x_i \in X\}$$

and for any $a \in S$, the ideal generated by a is $(a] = \{x \in S : x \leq a\}$

Lemma 4.4. *Let A be a fuzzy ideal of S and X a non-empty subset of S. Then we have*

(1). $a \in (X] \Rightarrow A(a) \geq \bigwedge_{i=1}^{n} A(x_i)$ for some $x_i \in X$.

(2). $x \in (y] \Rightarrow A(x) \geq A(y)$.

(3). If S has the smallest element 0, then $A(0) = 1$.

It is well known that, the set $I(S)$ of all ideals of a join-semilattice $(S, \vee)$ with smallest element 0 is a complete lattice under the set inclusion ordering $\subseteq$. Let $\mathcal{FI}(S)$ denote the set of all fuzzy ideals of a join-semilattice $(S, \vee)$ with smallest element 0.

Theorem 4.5. *$\mathcal{FI}(S)$ is a complete lattice under point-wise ordering in which, for any $\{A_i\}_{i \in \Delta} \subseteq \mathcal{FI}(S)$, the g.l.b and l.u.b are given by*

$$\Big(\bigwedge_{i \in \Delta} A_i\Big)(x) = \bigwedge_{i \in \Delta} A_i(x) \text{ and } \Big(\bigvee_{i \in \Delta} A_i\Big)(x) = \bigwedge\{A(x) : A \in \mathcal{FI}(S) \text{ and } A_i \leq A \text{ for all } i \in \Delta\} \text{ for any } x \in S.$$

Theorem 4.6. *The smallest fuzzy ideal of S containing a non-empty fuzzy subset A of S is given by $\overline{A}(x) = \bigwedge\{B(x) : B \in \mathcal{F}I(S)$ and $A \leq B\}$.*

The following result gives a point-wise description of $\overline{A}$.

Theorem 4.7. *For any fuzzy subset A of S,*

$$\overline{A}(x_0) = 1 \text{ for some } x_0 \in S \text{ and } \overline{A}(x) = \bigvee\{\bigwedge_{i=1}^{n} A(a_i) : a_1, a_2, \dots a_n \in S \text{ and } x \leq \bigvee_{i=1}^{n} a_i\} \text{ for any } x_0 \neq x \in S.$$

Corollary 4.8. $\Big(\bigvee_{i\in\Delta} A_i\Big)(x) = \bigvee\Big\{\bigwedge_{i=1}^{n} B(a_j) : a_1, a_2 \dots a_n \in S \text{ and } x \leq \bigvee_{i=1}^{n} a_i\Big\}$, *where* $B(x) = \bigvee\{A_i(x) : i \in \Delta\}$.

Corollary 4.9. *For any $A, B \in \mathcal{FI}(S)$, the g.l.b $A \wedge B$ and l.u.b $A \vee B$ in $\mathcal{FI}(S)$ are respectively given by*

$$(A \wedge B)(x) = A(x) \wedge B(x) \text{ and } (A \vee B)(x) = \bigvee\Big\{\bigwedge_{j=1}^{n}\Big(A(a_j) \vee B(a_j)\Big) : a_1, a_2 \dots a_n \in S \text{ and } x \leq \bigvee_{j=1}^{n} a_j\Big\}.$$

Theorem 4.10. *Let $(S, \vee)$ be a join-semilattice with smallest element 0 and $\{I_\alpha\}_{\alpha\in L}$ a class of ideals of S such that $\bigcap_{\alpha\in M} I_\alpha = I_{\bigvee_{\alpha\in M}\alpha}$, For any $M \subseteq L$. For any $x \in S$ define $A(x) = \vee\{\alpha \in L : x \in I_\alpha\}$. Then A is a fuzzy ideal of S such that the α-cut $A_\alpha = I_\alpha$, for any $\alpha \in L$. Conversely every fuzzy ideal of S can be obtained as above.*

It is well known that the class $F(S)$ of all filters of a join-semilattice $(S, \vee)$ with smallest element 0 is an algebraic lattice. In view of this we prove the following.

Theorem 4.11. *Let $(S, \vee)$ be a join-semilattice with smallest element 0. Then the class $\mathcal{FI}(S)$ of all fuzzy ideals of S is an algebraic fuzzy system.*

Finally, we recall that a join-semilattice $(S, \vee)$ is said to be distributivity if, for any a, b *and* $c \in S$, $a \leq b \vee c \Rightarrow$ there exists $b_1, c_1 \in S$ such that $b_1 \leq b,\ c_1 \leq c$ and $a = b_1 \vee c_1$.

Theorem 4.12. *Let $(S, \vee)$ be a join-semilattice with smallest element 0. Then the following are equivalent to each other:*

(1). $\mathcal{FI}(S)$ is a distributive lattice.

(2). $\mathcal{I}(S)$ is a distributive lattice.

(3). S is distributive.

5. Conclusions

In this paper, we have studied the structural properties of fuzzy filters of a meet-semilattice $(S, \wedge)$, by introducing the notion of fuzzy filter of S with truth values in a general frame L. Further, we want to make an abstract study of the class of fuzzy filters and to investigate fuzzy ideals and congruences of a meet-semilattice.

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