Int. J. Math. And Appl., 7(3)(2019), 95–102
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Eta-Directional Derivative

Susanta Kumar Mohanta[1,*]

1 Department of Mathematics, Government College Sundargarh, Rangadhipa, Sundargarh-770002, Odisha, India.

Abstract: The primary objective of this article is to introduced generalized directional derivative(η-directional derivative) of a function in the direction of a certain function in Linear spaces, Hilbert spaces and Banach spaces. This will be the generalization of Frechet derivative, Gauteaux derivative and Hadamard derivative under certain conditions. Some properties of η-directional derivative with there examples have been studied.

MSC: 49J50; 58C20.

Keywords: Frechet derivative, Gauteaux derivative, Hadamard derivative, η-invex set, Normalized vector function.

1. Introduction

In past, Frechet derivative [1], Gauteaux derivative [1] and Hadamard derivative [3] has been introduced and proved some fascinating results on Differential calculus, Optimization engineering, Banach spaces and Linear spaces. In this review we contrast on to define the generalized the definition of derivative of a function in Linear spaces, Hilbert spaces and Banach spaces. This is the generalization of Frechet derivative [1], Gauteaux derivative [1] and Hadamard derivative [3]. Some known definitions and results are recalled for our need in section 2; η-directional derivative and its properties defined newly along with there examples in section 3; higher order η-directional derivative and a Theorem have been studied in section 4; the article end with a conclusion in section 5.

2. Preliminaries

To make the article self contained some known definitions and results are recited for our requirement.

Definition 2.1 ([1])**.** *Suppose* $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* K *be an open set. The function* f *is classically differentiable at* $x_0 \in K$ *if,*

(a). The partial derivative of f, $\frac{\partial f_i}{\partial x_j}$ *for* $i = 1, 2, ...m$ *and* $j = 1, 2, ...n$ *exist at* x_0,

(b). The Jacobian matrix $J(x_0) = \frac{\partial f_i}{\partial x_j}(x_0) \in \mathbb{R}^{m\times n}$ *satisfies the following*

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - J(x_0)(x - x_0)\|}{\|x - x_0\|}.$$

---

[*] *E-mail: smohanta129@gmail.com*

We say that the Jacobian matrix $J(x_0)$ is the derivative of f at x_0, that is called total derivative.

Definition 2.2 ([1]). *Let X, Y are Banach spaces, the directional derivative of $f : X \to Y$ at x, $K \subset X$ in the direction $h \in X$, denoted by the symbol $f'(x; h)$, is defined by the equation*

$$f'(x;h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

whenever the limit on the right exists.

Definition 2.3 ([1]). *Let f be a function on an open subset K of a Banach space X into the Banach space Y. We say f is Gauteaux differentiable at $x \in K$. If there is bounded and linear operator $T_x : X \to Y$ such that*

$$T_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

for every $h \in X$. The operator T_x is called the Gauteaux derivative of f at x.

Definition 2.4 ([1]). *Let f be a function on an open subset K of a Banach space X into the Banach space Y. we say f is Frechet differentiable at $x \in K$. If there is bounded and linear operator $T : X \to Y$ such that*

$$T_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

is uniform for every $h \in S_X$. Where $S_X = \{x \in X : \|x\| = 1\}$. The operator T_x is called the Frechet derivative of f at x.

Definition 2.5 ([1]). *Let f be a real-valued function on an open subset K of a Banach space X. we say that f is uniformly Gauteaux differentiable on K if for every $h \in S_X$*

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = T_x(h)$$

is uniform $x \in K$. Where $S_X = \{x \in X : \|x\| = 1\}$.

Definition 2.6 ([1]). *Let f be a real-valued function on an open subset K of a Banach space X. We say that f is uniformly Frechet differentiable on K if*

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = T_x(h)$$

is uniform for every $h \in S_X$ and $x \in K$. Where $S_X = \{x \in X : \|x\| = 1\}$.

Definition 2.7 ([3]). *The set K is said to be η-invex set if there exist a vector function $\eta : K \times K \to X$ such that for all $u, v \in K$ and $t \in (0,1)$, we have $x + t\eta(u,v) \in K$.*

Definition 2.8. *The definition of "normalized vector function" is familiar from analysis. To say that a vector function $\eta : K \times K \to X, \forall u, v \in K$ is normalized vector function if $\|\eta(u,v)\| = 1$ for $K \subset X$.*

3. Generalized Directional Derivative

Let us define the generalized directional derivative of a function f at x in the direction of the normalized vector function $\eta(u,v)$ (i.e., η-directional derivative) with some examples and its properties as follows.

Definition 3.1. *Let $f : K \to Y$ be a function on an open η-invex subset K of a Banach space X into the Banach space Y and a normalized vector function η defined over the neighborhood of x with radius $\epsilon > 0$ such that $\eta : N_\epsilon \times N_\epsilon \to X, \forall u, v \in N_\epsilon$. We say f has η-directional derivative of f at $x \in K$ in the direction of a normalized vector function η. If there is bounded and continuous linear operator $D : X \to Y$ such that*

$$D_\eta f(x) = \lim_{t\to 0} \frac{f(x + t\eta(u, v)) - f(x)}{t}$$

for every $u, v \in K$. The operator "D_η" is called the η-directional derivative of f at $x \in K$ in the direction of a vector function η. Whenever K is η-invex compact set and the limit is uniform for η in this case, we write $D_\eta f(x) = f'_\eta(x)$.

For the existence of the definition, as sequences x_n and t_n convergence to x and t for $n \to \infty$ and now by the continuity of f and Mean value theorem, their exist x^*_n between x_n and $x_n + t_n\eta(u, v)$. For the existence, we have to show $\theta \to 0$, whenever $t \to 0$,

$$\theta = \frac{f(x_n + t_n\eta(u, v)) - f(x_n)}{t_n} - f'_\eta(x^*)$$

with the following limit exist $\lim\limits_{n\to\infty} \sup\limits_{\|\theta\|_Y} \|\theta\| = 0$, implies that

$$\lim_{n\to\infty} \sup_{\|\theta\|_Y} \left\| \frac{f(x_n + t_n\eta(u, v)) - f(x_n)}{t_n} - f'_\eta(x^*) \right\| = 0,$$

$$\Rightarrow D_\eta f(x) - f'_\eta(x^*) = 0.$$

Remark 3.2.

(a). When $\eta(u, 0) = \eta(u)$ for $v = 0$ or $\eta(0, v) = \eta(v)$ for $u = 0$. When $\eta(u) = u$ and $\eta(v) = v$, in this case η-directional derivative of f at x in the direction of normalized vector function η coincided with Gauteaux derivative and Frechet derivative respectively.

(b). When $\eta(u, 0) = u$ for $v = 0$ or $\eta(0, v) = v$ for $u = 0$. Let f be a real-valued function on an open set K of Banach spaces X and f is uniformly η-directional derivative at x in the direction of normalized vector function η if the limit is uniformly on η and $x \in K$.

(c). When $\eta(u, v) \to u$ for $v \to 0$ or $\eta(u, v) \to v$ for $u \to 0$. Let f be a continuous and bounded a real-valued function on an open set K subset of Banach spaces X. In this case η-directional derivative of f at x in the direction of normalized vector function η coincides with Hadamard derivative.

(d). $D_{-\eta}f(x) = -D_\eta f(x)$, immediately follows from the definition, when $\eta(u, v)$ is skew symmetric i.e., $\eta(u, v) = -\eta(v, u)$. Again when η is symmetric i.e., $\eta(u, v) = \eta(v, u)$, we have $D_\eta f(x) = D_{-\eta}f(x)$.

(e). In one dimension, there are two η-directional derivative of a function for every point: one directed "forward", i.e., $\eta(u, v) = u - v$ and other directed "backward", i.e., $\eta(u, v) = -(u - v) = v - u$.

(f). In two or more dimension, there are infinitely many η-directional derivative of a function for every point. But according to our requirement we can fix a certain direction by defining the vector function η.

(g). The η-directional derivative is a one directional calculation of derivative in the specified direction defined by the normalized vector function η. If the vector function η is one dimensional along a specific direction u, then η-directional derivative coincides with Gauteaux differential.

3.1. Examples on η-Directional Derivative

Example 3.3 (Inner product space)**.** *Let y and x are two vectors, K be a η-invex subset of inner product space and define a linear function $f(x) = x^T y$. Now the η-directional derivative of f at x in the direction of a normalized vector function η:*

$$\begin{aligned} D_\eta f(x) &= \lim_{t\to 0}\left[\frac{(x+t\eta(u,v))^T y - x^T y}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{x^T y + t\eta^T(u,v)y - x^T y}{t}\right] \\ &= \eta^T(u,v)y. \end{aligned}$$

Again, Let x be a vector, K be a η-invex subset of inner product space and define a quadratic function $f(x) = x^T x$. Now the η-directional derivative of f at x in the direction of a normalized vector function η:

$$\begin{aligned} D_\eta f(x) &= \lim_{t\to 0}\left[\frac{(x+t\eta(u,v))^T x - x^T x}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{x^T x + t\eta^T(u,v)x - x^T x}{t}\right] \\ &= \eta^T(u,v)x. \end{aligned}$$

Now, let A be a symmetric metrics, and define a quadratic function $f = 2x^T y + x^T Ax$. The η-directional derivative of f at x in the direction of a normalized vector function η:

$$\begin{aligned} D_\eta f(x) &= \lim_{t\to 0}\left[\frac{2(x+t\eta)^T y - 2x^T y}{t}\right] + \lim_{t\to 0}\left[\frac{(x+t\eta)^T A(x+t\eta) - x^T Ax}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{2x^T y + 2t\eta^T y - 2x^T y}{t}\right] + \lim_{t\to 0}\left[\frac{(x^T + t\eta^T)A(x+t\eta) - x^T Ax}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{2t\eta^T y}{t}\right] + \lim_{t\to 0}\left[\frac{x^T A(x+t\eta) - x^T Ax}{t}\right] + \lim_{t\to 0}\left[\frac{t\eta^T A(x+t\eta)}{t}\right] \\ &= \eta^T(u,v)(2y + Ax) + x^T A\eta(u,v). \end{aligned}$$

Example 3.4 (Infinite-dimensional linear space)**.** *In the case of an infinite-dimensional linear space V whose elements are real-valued functions and define $e^u : V \to V$ simply $u(x)$ maps point wise to its exponential function $e^{u(x)}$. Let x be a vector, K be a η-invex subset of infinite-dimensional linear space. Now the η-directional derivative of e^x at x in the direction of a normalized vector function η:*

$$\begin{aligned} D_\eta(e^x) &= \lim_{t\to 0}\left[\frac{e^{(x+t\eta(u,v))} - e^x}{t}\right] = \lim_{t\to 0}\left[\frac{e^x e^{t\eta(u,v)} - e^x}{t}\right] \\ &= e^x \lim_{t\to 0}\left[\frac{e^{t\eta(u,v)} - 1}{t}\right] = \eta(u,v)e^x, \ \textit{using series of } e^{t\eta(u,v)}. \end{aligned}$$

Example 3.5 (The absolute value function in $\mathbb{R}$)**.** *Let $f(x) = |x|$ absolute value mapping, K be a η-invex subset of $\mathbb{R}$. Now the η-directional derivative of f at x in the direction of a normalized vector function η:*

(a). When $x = 0$, $\quad D_\eta(f) = \lim_{t\to 0}\left(\frac{|x+t\eta(u,v)|-|x|}{t}\right) = |\eta(u,v)|\,;$

(b). When $x < 0$, $\quad D_\eta(f) = \lim_{t\to 0}\left(\frac{|x+t\eta(u,v)|-|x|}{t}\right) = -\eta(u,v),$

(c). When $x > 0$, $\quad D_\eta(f) = \lim_{t\to 0}\left(\frac{|x+t\eta(u,v)|-|x|}{t}\right) = \eta(u,v).$

Remark 3.6. *When mapping $\eta(u,v)$ is symmetric i.e., $\eta(u,v) = \eta(v,u)$, the η-directional derivative of $|x|$ at x in the direction of a normalized vector function η exits and depending on the values of normalized vector function $\eta(u,v)$ for all values of $x \in \mathbb{R}$.*

Example 3.7 (η-directional derivative in $\mathbb{R}^3$). *Let $f(X) = X^2 = x_1^2 + x_2^2 + x_3^2$ be a mapping in $\mathbb{R}^3$, K be a η-invex subset of R^3. Now the η-directional derivative of f at x in the direction of a normalized vector function $\eta(u,v) = (a\cos\theta, a\sin\theta, c\theta)$ such that $a^2 + c^2 \cdot \theta^2 = 1$ and $u, v \in N_\epsilon$, neighborhood of x with radius $\epsilon > 0$.*

$$\begin{aligned}
D_\eta(f(X)) &= \lim_{t\to 0}\left[\frac{(X + t\eta(u,v))^2 - X^2}{t}\right] = \lim_{t\to 0}\left[\frac{(x_1 + t\eta)^2 + (x_2 + t\eta)^2 + (x_3 + t\eta)^2 - (x_1^2 + x_2^2 + x_3^2)^2}{t}\right] \\
&= \lim_{t\to 0}\left[\frac{(x_1 + ta\cos\theta)^2 - x_1^2}{t}\right] + \lim_{t\to 0}\left[\frac{(x_2 + ta\sin\theta)^2 - x_2^2}{t}\right] + \lim_{t\to 0}\left[\frac{(x_3 + tc\theta)^2 - x_3^2}{t}\right] \\
&= \lim_{t\to 0}\left[\frac{2x_1ta\cos\theta + (ta\cos\theta)^2}{t}\right] + \lim_{t\to 0}\left[\frac{2x_2ta\sin\theta + (ta\sin\theta)^2}{t}\right] + \lim_{t\to 0}\left[\frac{2x_3tc\theta + (tc\theta)^2}{t}\right] \\
&= 2\cdot x_1\cdot a\cos\theta + 2\cdot x_2\cdot a\sin\theta + 2\cdot x_3\cdot c\theta \\
&= 2X\cdot\eta(u,v).
\end{aligned}$$

3.2. Properties for η-Directional Derivative

a. Let $f(x) = C$, a constant map. Now the η-directional derivative of f at x in the direction of a normalized vector function η is always vanishes. The proof follows immediately from the definition.

b. η-directional derivative distributes over sums: $D_\eta(f \pm g) = D_\eta(f) \pm D_\eta(g)$.

The proof follows immediately from the definition.

c. Product rule: $D_\eta(fg) = fD_\eta(g) + gD_\eta(f)$ for element wise product.

$D_\eta(\langle f, g\rangle) = \langle f, D_\eta(g)\rangle + \langle D_\eta(f), g\rangle$ for inner product.

We have to begin with the definition of η-directional derivative:

$$\begin{aligned}
D_\eta(f\cdot g)(x) &= D_\eta(f(x)\cdot g(x)) = \lim_{t\to 0}\left[\frac{g(x + t\eta)f(x + t\eta) - f(x)\cdot g(x)}{t}\right] \\
&\quad g(x)\cdot f(x + t\eta(u,v)) \text{ add and subtract in numerator} \\
&= \lim_{t\to 0}\left[\frac{g(x + t\eta)f(x + t\eta) - g(x)\cdot f(x + t\eta(u,v)) + g(x)\cdot f(x + t\eta(u,v)) - f(x)\cdot g(x)}{t}\right] \\
&= \lim_{t\to 0} f(x + t\eta)\cdot\left[\frac{g(x + t\eta) - g(x)}{t}\right] + \lim_{t\to 0} g(x)\cdot\left[\frac{f(x + t\eta) - f(x)}{t}\right] \\
&= f\cdot D_\eta g(x) + g\cdot D_\eta f(x).
\end{aligned}$$

d. Quotient rule: $D_\eta\left(\frac{f}{g}\right) = D_\eta(f\cdot g^{-1}) = g^{-2}\{g\cdot D_\eta(f) - f\cdot D_\eta(g)\},\ g(x) \neq 0.$

We have to begin with the definition of η-directional derivative:

$$\begin{aligned}
D_\eta\left(\frac{f}{g}\right) &= D_\eta\left(\frac{f(x)}{g(x)}\right) = \lim_{t\to 0}\left[\frac{\frac{f(x + t\eta(u,v))}{g(x + t\eta(u,v))} - \frac{f(x)}{g(x)}}{t}\right] = \lim_{t\to 0}\left[\frac{g(x)f(x + t\eta) - f(x)\cdot g(x + t\eta)}{t\cdot g(x + t\eta)\cdot g(x)}\right] \\
&\quad g(x)\cdot f(x) \text{ add and subtract in numerator} \\
&= \lim_{t\to 0}\left[\frac{g(x)f(x + t\eta(u,v)) - g(x)\cdot f(x)}{t\cdot g(x + t\eta(u,v))\cdot g(x)}\right] - \lim_{t\to 0}\left[\frac{f(x)\cdot g(x + t\eta(u,v)) - g(x)\cdot f(x)}{t\cdot g(x + t\eta(u,v))\cdot g(x)}\right] \\
&= \lim_{t\to 0}\left[\frac{g(x)\{f(x + t\eta(u,v)) - f(x)\}}{t\cdot g(x + t\eta(u,v))\cdot g(x)}\right] - \lim_{t\to 0}\left[\frac{f(x)\{g(x + t\eta(u,v)) - g(x)\}}{t\cdot g(x + t\cdot\eta(u,v))\cdot g(x)}\right] \\
&= \frac{1}{g^2(x)}\left[g(x)\cdot D_\eta(f(x)) - f(x)\cdot D_\eta(g(x))\right].
\end{aligned}$$

e. η-directional derivative for composition of functions: Now, assume $g : Y \to V$ has η-directional derivative at $f(x) \in Y$, and that $f : X \to Y$ has η-directional derivative at $x \in X$. Now calculate the η-directional derivative their composition $(gof)(x) = g(f(x))$. We have to begin with the definition of η-directional derivative:

$$\begin{aligned}
D_\eta(gof)(x) &= \lim_{t\to 0}\left[\frac{(gof)(x+t\eta)-(gof)(x)}{t}\right] \\
&\quad \text{multiply and divide by } f(x+t\eta(u,v))-f(x) \\
&= \lim_{t\to 0}\left[\frac{g(f(x+t\eta))-g(f(x))}{f(x+t\eta)-f(x)}\cdot\frac{f(x+t\eta)-f(x)}{t}\right] \\
&= \lim_{t\to 0}\left[\frac{g(f(x+t\eta))-g(f(x))}{f(x+t\eta)-f(x)}\right]\cdot\lim_{t\to 0}\left[\frac{f(x+t\eta)-f(x)}{t}\right] \\
&= D_{f(x)}(g(f(x)))\cdot D_\eta f(x).
\end{aligned}$$

The following example illustrate the composition rule.

Example 3.8. *In the case of an infinite-dimensional linear space V whose elements are real-valued functions and define $e^w : V \to V$ simply $w(x)$ maps point wise to its exponential function $e^{w(x)}$. Let x be a vector, K be a η-invex subset of infinite-dimensional linear space. Now the η-directional derivative of e^{x^2} at x in the direction of a normalized vector function η, set $w(x) = x^2$*

$$\begin{aligned}
D_\eta(e^{w(x)}) &= \lim_{t\to 0}\left[\frac{e^{w(x+t\eta(u,v))}-e^{w(x)}}{t}\right] \\
&\quad \textit{multiply and divide by } w(x+t\eta(u,v))-w(x) \\
&= \lim_{t\to 0}\left[\frac{e^{w(x+t\eta(u,v))}-e^{w(x)}}{w(x+t\eta(u,v))-w(x)}\cdot\frac{w(x+t\eta(u,v))-w(x)}{t}\right] \\
&= \lim_{t\to 0}\left[\frac{e^{w(x+t\eta(u,v))}-e^{w(x)}}{w(x+t\eta(u,v))-w(x)}\right]\cdot\lim_{t\to 0}\left[\frac{w(x+t\eta(u,v))-w(x)}{t}\right] \\
&= D_{w(x)}(e^{w(x)})\cdot D_\eta w(x), \quad \because\ D_{w(x)}(e^{w(x)}) = e^{x^2},\ D_\eta w(x) = 2x\cdot\eta(u,v) \\
&= e^{x^2}\cdot 2x\cdot\eta(u,v) = 2x\cdot e^{x^2}\cdot\eta(u,v).
\end{aligned}$$

Example 3.9. *In the case of an infinite-dimensional linear space V whose elements are real-valued functions and define $e^{z(w)} : V \to V$ simply $z = w(x)$ maps point wise to its exponential function $e^{z(w(x))}$. Let x be a vector, K be a η-invex subset of infinite-dimensional linear space. Now the η-directional derivative of $e^{\sin x^2}$ at x in the direction of a normalized vector function η, set $z(w) = \sin w$ and $w(x) = x^2$,*

$$\begin{aligned}
D_\eta(e^{z(w(x))}) &= \lim_{t\to 0}\left[\frac{e^{z(w(x+t\eta(u,v)))}-e^{z(w(x))}}{t}\right] \\
&\quad \textit{multiply and divide by } z(w(x+t\eta(u,v)))-z(w(x)) \textit{ and } w(x+t\eta(u,v))-w(x) \\
&= \lim_{t\to 0}\left[\frac{e^{z(w(x+t\eta(u,v)))}-e^{z(w(x))}}{z(w(x+t\eta(u,v)))-z(w(x))}\cdot\frac{z(w(x+t\eta(u,v)))-z(w(x))}{w(x+t\eta(u,v))-w(x)}\cdot\frac{w(x+t\eta(u,v))-w(x)}{t}\right] \\
&= \lim_{t\to 0}\left[\frac{e^{z(w(x+t\eta(u,v)))}-e^{z(w(x))}}{z(w(x+t\eta(u,v)))-z(w(x))}\right]\cdot\lim_{t\to 0}\left[\frac{z(w(x+t\eta(u,v)))-z(w(x))}{w(x+t\eta(u,v))-w(x)}\right] \\
&\cdot\ \lim_{t\to 0}\left[\frac{w(x+t\eta(u,v))-w(x)}{t}\right] \\
&= e^{\sin w}\cdot\cos w\cdot 2x\cdot\eta(u,v), \quad \because D_\eta w(x) = 2x\cdot\eta(u,v) \\
&= 2x\cdot\cos(x^2)\cdot e^{\sin(x^2)}\cdot\eta(u,v).
\end{aligned}$$

4. Higher Order Generalized Directional Derivative

Let X and Y be Banach spaces and K is η-invex subset of X and a normalized vector function $\eta : N_\epsilon \times N_\epsilon \to X, \forall u, v \in N_\epsilon$. The η-directional derivative of higher order can be state as follows:

Second order η-directional derivative:

$$\begin{aligned} D^2_\eta(f(x)) &= D_\eta(D_\eta(f(x))) = \lim_{t\to 0}\left[\frac{D_\eta f(x+t\eta(u,v)) - D_\eta f(x)}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{f(x+2t\eta) - 2f(x+t\eta) + f(x)}{t^2}\right]. \end{aligned}$$

Third order η-directional derivative:

$$\begin{aligned} D^3_\eta(f(x)) &= D_\eta(D^2_\eta(f(x))) = \lim_{t\to 0}\left[\frac{D^2_\eta f(x+t\eta(u,v)) - D^2_\eta f(x)}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{f(x+3t\eta) - 3f(x+2t\eta) + 3f(x+t\eta) - f(x)}{t^3}\right]. \end{aligned}$$

Fourth order η-directional derivative:

$$\begin{aligned} D^4_\eta(f(x)) &= D_\eta(D^3_\eta(f(x))) \;=\; \lim_{t\to 0}\left[\frac{D^3_\eta f(x+t\eta(u,v)) - D^3_\eta f(x)}{t}\right] \\ &= \lim_{t\to 0}\left[\frac{f(x+4t\eta) - 4f(x+3t\eta) + 6f(x+2t\eta) - 4f(x+t\eta) + f(x)}{t^4}\right]. \end{aligned}$$

and so on. Now from the above higher order η-directional derivative we can see a pattern of the coefficient of the numerator of the definitions in triangular form called **Palsu's Triangle** such as follows:

$$\begin{array}{lccccccccccccc}
n=1: & & & & & & 1 & & -1 & & & & & \\
n=2: & & & & & 1 & & -2 & & 1 & & & & \\
n=3: & & & & 1 & & -3 & & 3 & & -1 & & & \\
n=4: & & & 1 & & -4 & & 6 & & -4 & & 1 & & \\
n=5: & & 1 & & -5 & & 10 & & -10 & & 5 & & -1 & \\
n=6: & 1 & & -6 & & 15 & & -20 & & 15 & & -6 & & 1 \\
\vdots & & & & \vdots & & & & & \vdots & & & &
\end{array}$$

by continuing as such we can calculate the n^{th} order η-directional derivative of a function $f(x)$ in the direction of a normalized vector function η. The similar pattern can be generate by adding the previous coefficients ignoring negative sign and then using alternative sign. Sum of coefficient of any order η-directional derivative is zero.

Theorem 4.1. *Let X and Y be Banach spaces and K is η-invex subset of X and a normalized vector function $\eta : N_\epsilon \times N_\epsilon \to X, \forall u, v \in N_\epsilon$. If the η-directional derivative of $f : K \to Y$ is exist and bounded, i.e., $|D_\eta f(x)| \leq C$ iff f is Lipschitz near $x \in N_\epsilon$.*

Proof. Let us suppose that, the η-directional derivative of $f(x)$ is exist and bounded. Now by the definition, let $f(x)$ be a function on an open η-invex subset K of a Banach space X into the Banach space Y and a normalized vector function

$\eta : N_\epsilon \times N_\epsilon \to X, \forall u, v \in N_\epsilon$. We say $f(x)$ has η-directional derivative of $f(x)$ at $x \in N_\epsilon$ in the direction of a normalized vector function η. If there is bounded and continuous linear operator $D : X \to Y$ such that

$$D_\eta f(x) = \lim_{t\to 0} \frac{f(x + t\eta(u, v)) - f(x)}{t}$$

for every $u, v \in N_\epsilon$. Now the η-directional derivative of $f(x)$ is bounded, i.e., $|D_\eta f(x)| \leq C$.

$$\begin{aligned}
\Rightarrow \quad & \left| \lim_{t\to 0} \frac{f(x + t\eta(u, v)) - f(x)}{t} \right|_Y \leq C \\
\Rightarrow \quad & \lim_{t\to 0} \left\| \frac{f(x + t\eta(u, v)) - f(x)}{x + t\eta(u, v) - x} \right\|_Y \leq C \\
\Rightarrow \quad & \| f(x + t\eta) - f(x) \|_Y \leq C \| x + t\eta - x \|_X \\
& \text{set } z = x + t\eta \in K, \\
\Rightarrow \quad & \| f(z) - f(x) \|_Y \leq C \| z - x \|_X .
\end{aligned}$$

This implies f is Lipschitz near $x \in N_\epsilon$ and C is the Lipschitz constant. The converse of the theorem immediately follows from the above analysis. This proves the Theorem. □

Problem 4.1. ***Generalized Minimal η-Differential Inequality Problems***

(GM-η-DIP): Let X, Y be Banach spaces, K is η-invex subset of X and a normalized vector function $\eta : N_\epsilon \times N_\epsilon \to X, \forall u, v \in N_\epsilon$. If the η-direectional derivative of $f : K \to Y$ is exist. Finding $x_0 \in K$ such that $D_\eta f(x_0) \geq 0$ for $x_0 \in K$ $\forall u, v \in N_\epsilon$.

5. Conclusion

The η-directional derivative is the generalization all differentiation. According to the nature of its scope it will give a new edge in the field of nonlinear functional analysis and engineering applications.

References

[1] D. Behmardi and E. D. Nayeri, *Introduction to Frechet and Gateaux derivatives*, Applied Mathematical Sciences, 2(20)(2008), 975-980.

[2] M. A. Hanson, *On Sufficiency of the Kuhn-Tucker Conditions*, Journal of Mathematical Analysis and Applications, 80(1981), 545-550.

[3] L. Zajicek, *Hadamard differentiability via Gateaux differentiability*, Charles University, Faculty of Mathematics and Physics, Czech Republic, (2010), 1-9.