

Solitary Wave Solutions to the Korteweg-de Vries (KdV) and the Modified Regularized Long Wave (MRLW) Equations

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Abstract: The main aim of this paper is to demonstrate the use of cosine function method for nonlinear partial differential equations such as Korteweg-de Vries (kdV) equation and Modified Regularized Long Wave (MRLW) equation. The cosine function method provides a mathematical powerful tool for obtaining exact travelling wave solution of these equations. The significance of obtained solutions gives credence to the explanation and understanding of related physical phenomena.

Keywords: Korteweg-de Vries (kdV) equation and the Modified Regularized Long Wave (MRLW) equation, Cosine-Function method.

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1. Introduction

Many physical, mechanical, chemical, biological, engineering and some economic laws and relations appear mathematically in the form of differential equations which are linear or nonlinear, homogeneous or nonhomogeneous. Almost all differential equations relating physical phenomena are nonlinear. Methods of solutions of linear differential equations are reasonably easy and well avowed. In contrast, the techniques of solutions of nonlinear differential equations are less obtainable and in general, approximations are generally used. Nonlinearity is a fascinating element of nature, today; many scientists observe nonlinear science as the most important frontier or the fundamental understanding of nature. The analytical solutions of such equations are of fundamental importance to reveal the inner structure of the phenomena. The world around us is inherently nonlinear. Nonlinear Partial Differential Equations (NPDEs) are able to designate the real applications in a plethora of areas in science, technology and engineering. Some analytic methods have been utilized to investigate NPDEs, such as the Hirota bilinear method [1–4], the Bäcklund transformation method [5–7], Cosine-function method [8], Darboux transformation [9], the inverse scattering method [10] and the modified sub-equation method [11], the unified method (UM) and its generalized form (GUM) [12, 13] and so on [14–20] are rather heuristic and the most frequently used in the literature. These approaches have powerful properties that make it conceivable for the purpose of constructing solitary wave solutions for variety nonlinear partial differential equations.

Our goal in this paper is to present an application the cosine-function method to the Korteweg-de Vries (KdV) equation and the Modified Regularized Long Wave (MRLW) equation to be solved by this method. The rest of the paper is organized as follows: In Section 2, we give the description the cosine-function method. In Section 3, we apply this method to the

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Korteweg-de Vries (KdV) equation and the Modified Regularized Long Wave (MRLW) equation and conclusions are given at last section.

2. The Cosine-Function Method

Consider the nonlinear partial differential equation in the form

$$P(u_t, u_x, u^n u_x, u_{xxx}, u_{xxt}, \dots) = 0, \quad (1)$$

where $u(x, t)$ is the solution of Equation (1); u_t and u_x etc. are the partial derivatives of u with respect to t and x , respectively. Suppose Equation (1) admits travelling wave solution. Consider the transformation $u(x, t) = g(\eta)$, where $\eta = x - ct - d$, c is the speed of the traveling wave and d is a constant. This enables us to use the following changes:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -c \frac{\partial g}{\partial \eta}, \\ \frac{\partial u}{\partial x} &= \frac{\partial g}{\partial \eta}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 g}{\partial \eta^2}, \\ \frac{\partial^3 u}{\partial x^3} &= \frac{\partial^3 g}{\partial \eta^3}, \end{aligned} \quad (2)$$

Using the above transformation the nonlinear partial differential equation (1) is transformed to nonlinear ordinary differential equation:

$$Q(g', g^n g', g''', \dots) = 0. \quad (3)$$

Now integrating (3) with respect to η , we get

$$R(g, g^{n+1}, g'', \dots) = 0. \quad (4)$$

We assume the solution of equation (4) as

$$g(\eta) = \kappa \cos^\alpha(\xi\eta), \quad (5)$$

where κ , α and ξ are the unknown parameters. Since Equation (5) is the solution of Equation (4), we have

$$\begin{aligned} g'(\eta) &= -\kappa\alpha\xi \cos^{\alpha-1}(\xi\eta) \cdot \sin(\xi\eta), \\ g''(\eta) &= -\kappa\xi^2\alpha \cos^\alpha(\xi\eta) + \kappa\xi^2\alpha(\alpha-1) \cos^{\alpha-2}(\xi\eta) - \kappa\xi^2\alpha(\alpha-1) \cos^\alpha(\xi\eta), \quad \text{etc.} \end{aligned} \quad (6)$$

Inserting Equation (5) and Equation (6) into Equation (4), then we obtain an equation in different powers of cosine functions. Now equating the coefficients of the same powers of cosine functions we obtain a system of algebraic equations in the parameters κ , α and ξ . This system can be solved to obtain the values of κ , α and ξ . The exact analytical solution of NLPDE (1) is then obtained by substituting the values of the unknown parameters in Equation (5).

3. Applications

3.1. The Korteweg-de Vries (KdV) Equation

The Korteweg-de Vries Equation [17–19], also known as the KdV equation, arises in various areas of Nonlinear physics that includes Fluid Dynamics. Mathematically can be written as

$$u_t + u_x + \varepsilon(uu_x + u_{xxx}) = 0. \quad (7)$$

where ε is a small perturbation parameter and $u(x, t)$ is the solution of Equation (7), x is the space variable and t is the time. Using the above procedure we obtain

$$-cg' + g' + \varepsilon(gg' + g''') = 0. \tag{8}$$

Now integrating Equation (8) with respect to η and taking the integrating constant to zero, we obtain

$$-cg + g + \varepsilon\left(\frac{g^2}{2} + g''\right) = 0. \tag{9}$$

Using Equation (5) and Equation (6) into Equation (9), we obtain

$$-c\kappa \cos^\alpha(\xi\eta) + \kappa \cos^\alpha(\xi\eta) + \frac{\varepsilon\kappa^2 \cos^{2\alpha}(\xi\eta)}{2} - \kappa\varepsilon\xi^2 \alpha \cos^\alpha(\xi\eta) + \kappa\varepsilon\xi^2 \alpha (\alpha - 1) \cos^{\alpha-2}(\xi\eta) - \kappa\varepsilon\xi^2 \alpha (\alpha - 1) \cos^\alpha(\xi\eta) = 0. \tag{10}$$

Equation (10) is valid if and only if the following system of algebraic equations holds:

$$\begin{aligned} -\varepsilon\kappa\xi^2\alpha - \varepsilon\kappa\xi^2\alpha(\alpha - 1) + (1 - c)\kappa &= 0 \\ \varepsilon\kappa\xi^2\alpha(\alpha - 1) + \frac{\varepsilon}{2}\kappa^2 &= 0 \\ 2\alpha &= \alpha - 2 \end{aligned} \tag{11}$$

Now solving the system Equation (11), we obtain

$$\begin{aligned} \alpha &= -2, \\ \kappa &= -3(1 - c), \\ \xi &= \pm \frac{1}{2} \sqrt{\frac{1 - c}{\varepsilon}}. \end{aligned}$$

Thus, the exact solution of the kdV equation is

$$u(x, t) = -3(1 - c) \cos^{-2} \left(\pm \frac{1}{2} \sqrt{\frac{1 - c}{\varepsilon}} (x - ct - d) \right).$$

For $t = 0$, we have the solution,

$$u(x, 0) = -3(1 - c) \cos^{-2} \left(\pm \frac{1}{2} \sqrt{\frac{1 - c}{\varepsilon}} (x - d) \right).$$

3.2. The Modified Regularized Long Wave (MRLW) Equation

In this section, we take the modified regularized long wave equation [20] as the form

$$u_t + u_x + u^2 u_x - u_{xxt} = 0, \tag{12}$$

where $u(x, t)$ is the solution of Equation (12), x is the space variable and t is the time. Using the above procedure we obtain

$$-cg' + g' + g^2 g' + cg''' = 0. \tag{13}$$

By integrating Equation (13) with respect to η and taking the integrating constant zero, we obtain

$$-cg + g + \frac{g^3}{3} + cg'' = 0. \tag{14}$$

Using Equation (5) and Equation (6) into Equation (14), we obtain

$$-c\kappa \cos^\alpha(\xi\eta) + \kappa \cos^\alpha(\xi\eta) + \frac{\kappa^3 \cos^{3\alpha}(\xi\eta)}{3} - c\kappa\xi^2\alpha \cos^\alpha(\xi\eta) + c\kappa\xi^2\alpha(\alpha-1) \cos^{\alpha-2}(\xi\eta) - c\kappa\xi^2\alpha(\alpha-1) \cos^\alpha(\xi\eta) = 0 \quad (15)$$

Equation (15) is valid if and only if the following system of algebraic equations holds:

$$\begin{aligned} -c\kappa\xi^2\alpha - c\kappa\xi^2\alpha(\alpha-1) + (1-c)\kappa &= 0, \\ c\kappa\xi^2\alpha(\alpha-1) + \frac{1}{3}\kappa^3 &= 0, \\ 3\alpha &= \alpha - 2. \end{aligned} \quad (16)$$

Now solving the system (16), we obtain

$$\begin{aligned} \alpha &= -1, \\ \kappa &= \pm\sqrt{6(1-c)}, \\ \xi &= \pm\sqrt{\frac{(1-c)}{c}}. \end{aligned}$$

Thus, the exact soliton solution of the MRLW equation is

$$u(x, t) = \pm\sqrt{6(1-c)} \cos^{-1} \left(\pm\sqrt{\frac{(1-c)}{c}} (x - ct - d) \right).$$

4. Conclusion

This study shows that the Cosine-function method is quite efficient and practically well studied for use in finding exact travelling wave solutions to the Korteweg-de Vries (KdV) equation and the Modified Regularized Long Wave (MRLW) equation. Thus we can say that this method is a mathematical powerful tool for obtaining exact travelling wave solution of these equations. This method also to solve other nonlinear partial differential equations arising in various scientific real time application arenas.

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