Generalized Method to Find the Generators of Matrix Algebras when its Dimension 2 and 3

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Abstract: Let A be an algebraically closed field of characteristic zero and consider a set of $2 \times 2$ or $3 \times 3$ matrices. Using a theorem of Shemesh, we give conditions for when the matrices in the set generate the full matrix algebra.

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1. Introduction

Let $A$ be an algebraically closed field of characteristic zero, and let $N_m = N_m(A)$ be the algebra of $m \times m$ matrices over $A$. Given a set $K = \{B_1, \ldots, B_t\}$ of $m \times m$ matrices, we would like to have conditions for when the $A_i$ generate the algebra $M_n$. In other words, determine whether every matrix in $N_m$ can be written in the form $T(B_1, \ldots, B_t)$, where $T$ is a non-commutative polynomial. (We identify scalars with scalar matrices so the constant polynomials give the scalar matrices.)

The case $m = 1$ is of course trivial, and when $t = 1$, the single matrix $B_1$ generates a commutative sub algebra. We therefore assume that $m, t \geq 2$. This question has been studied by many authors, see for example the extensive bibliography in [7]. We will give some generalize in the case of $m = 2$ or 3.

2. General Observations

Let $G$ be the algebra generated by $K$. If we could show that the dimension of $G$ as a vector space is $m^2$, it would follow that $G = N_m$. This can sometimes be done when we know a linear spanning set $H = \{H_1, \ldots, H_q\}$ of $G$. Let $N$ be the $m^2 \times q$ matrix obtained by writing the matrices in $H$ as column vectors. We would like to show that rank $N = m^2$. Since $N$ is an $m^2 \times m^2$ matrix and rank $N = \text{rank}(NN^*)$, it sufficient to show that $\text{det}(NN^*) \neq 0$. Unfortunately, the size of $H$ may be big [4]. In this paper we will combine this method with results of Shemesh and Spencer, Rivlin, Aslaksen and Sletsjoe to get some simple results for $m = 2$ or 3.

Lemma 2.1. Let $\{B_1, \ldots, B_t\}$ be a set of matrices in $N_m$ where $m = 2$ or 3. The $b_i$’s generate $N_m$ if and only if they do not have a common eigenvector.

We can therefore use the following theorem due to Shemesh [5].

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Theorem 2.2. Two $m \times m$ matrices, $B$ and $H$, have a common eigenvector if and only if

$$\sum_{u,v=1}^{m-1} [B^u, H^v]^* [B^u, H^v]$$

is singular.

Adding scalar matrices to the $B_i$'s does not change the subalgebra they generate, so we sometimes assume that our matrices lie in $W_M = \{ N \in N_m | \text{trace } N = 0 \}$. We also sometimes identify matrices in $N_m$ with vectors in $A_m^2$, and if $M_1, \ldots, M_m \in N_m$, then $\det(M_1, \ldots, M_m)$ denotes the determinant of the $m^2 \times m^2$ matrix whose $j^{th}$ column is $M_j$, written as $(M_{j1}, \ldots, M_{jn})^t$, where $M_{jk}$ is the $k^{th}$ row of $M_j$ for $k = 1, 2, \ldots, n$. We write the scalar matrix $aI$ as $a$. When we say that a set of matrices generate $N_m$, we are talking about $N_m$ as an algebra, while when we say that a set of matrices form a basis of $N_m$, we are talking about $N_m$ as a vector space.

3. The $2 \times 2$ Case

The following theorem is well-known, but we include a proof since it illustrated a technique we will use in the $3 \times 3$ case. Notice that the proof gives us an explicit basis for $N_2$.

Theorem 3.1. Let $B, H \in N_2$. $B$ and $H$ generate $N_2$ if and only if $[B, H]$ is invertible.

Proof. We know that in matrix $BH = -HB$, then a direct computation shows that

$$\det(I, B, H, BH) = -\det(I, B, H, HB) = \det[B, H].$$

Hence

$$\det(I, B, H, [B, H]) = 2 \det[B, H]$$

(1)

But if $I, B, H, [B, H]$ are linearly independent, then the dimension of $G$ as a vector space is 4, so $B$ and $H$ generate $N_2$. We call $[N, M, T] = [N, [M, T]]$ a double commutator. The characteristic polynomial of $A$ can be written as

$$\lambda^2 - (\text{trace } B)\lambda + ((\text{trace } B)^2 - \text{trace } B^2)/2.$$

It follows that the discriminant of the characteristic polynomial of $A$ can be written as discriminant $(B) = 2 \text{ trace } B^2 - (\text{trace } B)^2$. □

Lemma 3.2. Let $B, H, G \in N_2$ and suppose that no two of them generate $N_2$. Then $B, H, G$ generate $N_2$ if and only if the double commutator $[B, H, G] = [B, [H, G]]$ is invertible.

Proof. A direct computation shows that

$$\det(I, B, H, G)^2 = -\det[B, [H, G]] - \text{discriminant}(B) \det[H, G]$$

(2)

But if $I, B, H, G$ are linearly independent, then $B, H$ and $G$ generate $N_2$. □

Notice that the above proof gives us an explicit basis for $N_2$. We can now give a complete solution for the case $m = 2$.

Theorem 3.3. The matrices $B_1, \ldots, B_t \in N_2$ generate $N_2$ if and only if at least one of the commutators $[B_i, B_j]$ or double commutators $[B_i, B_j, B_k] = [B_i, [B_j, B_k]]$ is invertible.
Proof. If \( t > 4 \), the matrices are linearly dependent, so we can assume that \( t \leq 4 \). Suppose that \( B_1, B_2, B_3, B_4 \) generate \( N_2 \), but that no proper subset of them generates \( N_2 \). Then the four matrices are linearly independent, and we can write the identity \( I \) as a linear combination of them. If the coefficient of \( B_4 \) in this expression is nonzero, then \( B_1, B_2, B_3 \) span and therefore generate \( N_2 \), so \( B_1, B_2, B_3 \) generate \( N_2 \). Thus, if \( B_1, \ldots, B_t \) generate \( N_2 \), we can always find a subset of three of these matrices that generate \( N_2 \). \( \square \)

4. Two \( 3 \times 3 \) Matrices

In the case of two \( 3 \times 3 \) matrices, we have the following well-known theorem.

**Theorem 4.1.** Let \( B, H \in N_3 \). If \([B, H]\) is invertible, then \( B \) and \( H \) generate \( N_3 \).

For \( N \in N_3 \), we define \( L(N) \) to be the linear term in the characteristic polynomial of \( N \). Hence \( L(N) = ((\text{trace } N)^2 - \text{trace } N^2)/2 \), which is equal to the sum of the three principal minors of degree two of \( N \). Notice that \( L(N) \) is invariant under conjugation, and that if \([B, H]\) is singular, then \([B, H]\) is nilpotent if and only if \( L([B, H]) = 0 \). The following theorem shows that if \([B, H]\) is invertible and \( L([B, H]) \neq 0 \), then we can give an explicit basis for \( N_3 \).

**Theorem 4.2.** Let \( B, H \in N_3 \). Then

\[
\]

so if \( \det[B, H] \neq 0 \) and \( L([B, H]) \neq 0 \), then \( \{I, B, B^2, H, H^2, BH, HB, [B, [B, H]], [H, [H, B]]\} \) form a basis for \( N_3 \).

The proof of (3) is by direct computation. Notice that this can be thought of as a generalization of (1) and (2). We can also use Shemesh’s Theorem to characterize pairs of generators for \( N_3 \).

**Theorem 4.3.** The two \( 3 \times 3 \) matrices \( B \) and \( H \) generate \( N_3 \) if and only if both

\[
\sum_{u, v=1}^{m-1} [B^u, H^v][B^u, H^v]^* \quad \text{and} \quad \sum_{u, v=1}^{m-1} [B^u, H^v][B^u, H^v]^*
\]

are invertible.

5. Three or More \( 3 \times 3 \) Matrices

We start with the following theorem due to Laffey [6].

**Theorem 5.1.** Let \( K \) be a set of generators for \( N_3 \). If \( K \) has more than four elements, then \( N_3 \) can be generated by a proper subset of \( K \).

It is therefore sufficient to consider the cases \( t = 3 \) or 4. Following the approach outlined earlier, we start by finding a linear spanning set. Using the polarized Cayley-Hamilton Theorem, Spencer and Rivlin [1, 2] deduced the following theorem.

**Theorem 5.2.** Let \( B, H, G \in N_3 \). Define

\[
K(B) = \{B, B^2\} \\
K(B_1, B_2) = S(B_1, B_2) \cup S(B_2, B_1)
\]
From this we deduce the following theorem.

The matrices

\[ W = B_1 B_2 B_3 \]

We next give a version of Shemesh’s Theorem for three 3 \times 3 words of length \( m \).

These spanning sets are not optimal. They include words of length 5. Paz [3] has proved that \( N_3 \) can be generated by words of length \( \lfloor m^2 + 2 \rfloor / 3 \). For \( N_3 \) this gives words of length 4. The general bound has been improved by Pappacena [4].

We next give a version of Shemesh’s Theorem for three 3 \times 3 matrices.

**Theorem 5.3.** The matrices \( B, H, G \in N_3 \) have a common eigenvector if and only the matrix

\[
N(B, H, G) = \sum_{N \in K(B), M \in K(H)} [N, M]^*[N, M] + \sum_{N \in K(B), M \in K(G)} [N, M]^*[N, M] + \sum_{N \in K(H), M \in K(G)} [N, M]^*[N, M] + \sum_{N \in K(B, H), M \in K(G)} [N, M]^*[N, M]
\]

is singular.

**Proof.** Let \( G \) be the algebra generated by \( B, H, G \). Set

\[
X = \bigcap_{N \in K(B), M \in K(H)} \ker[N, M] \bigcap_{N \in K(B), M \in K(G)} \ker[N, M] \bigcap_{N \in K(H), M \in K(G)} \ker[N, M]
\]

We claim that \( X \) is invariant under \( G \). Let \( x \in X \) and consider \( Gx \). We know from Theorem 5.1 that any element of \( G \) is a linear combination of terms of the form \( t(B, H)G^i u(B, H)G^j v(B, H) \) with \( t(B, H), u(B, H), v(B, H) \in I \cup K(B) \cup K(H) \cup K(B, H) \). Since \( x \in \ker[N(B, H), K(G)] \cap \ker[N(B), K(G)] \cap \ker[N(H), K(G)] \), we get

\[
t(B, H)G^i u(B, H)G^j v(B, H) x = t(B, H)G^i u(B, H) v(B, H)G^j x
\]

\[
= t(B, H)G^{i+j} u(B, H) v(B, H) x
\]

\[
= t(B, H)u(B, H) v(B, H)G^{i+j} x
\]

\[
= G^{i+j} t(B, H) u(B, H) v(B, H) x.
\]

In the same way we use the fact that \( x \in [K(B), K(H)] \) to sort the terms of the form \( t(B, H)u(B, H)v(B, H)x \), so that we finally get

\[
G x = \{ a_{ijk} G^i H^j B^k x | 0 \leq i, j, k \leq 2, a_{ijk} \in A \}
\]

Using the above technique, it follows easily that \( Gx \subset X \) and that \( X \) is \( G \) invariant. Hence we can restrict \( G \) to \( X \), but since the elements of \( G \) commute on \( X \), they have a common eigenvector, and we can finish as in the proof of Theorem 2.2.

From this we deduce the following theorem.

**Theorem 5.4.** Let \( B, H, G \in N_3 \). Then \( B, H, G \) generate \( N_3 \) if and only if both \( N(B, H, G) \) and \( N(B^i, H^i, G^i) \) are invertible.

For the case of four matrices, we can prove the following theorem.
Theorem 5.5. The matrices $B_1, B_2, B_3, B_4 \in N_3$ have a common eigenvector if and only the matrix

$$N(B_1, B_2, B_3, B_4) = \sum_{i,j=1}^{4} \left( \sum_{N \in K(B_i)} [N, M]^* [N, M] \right) + \sum_{i,j=1}^{3} \left( \sum_{N \in K(B_i, B_j)} [N, M]^* [N, M] \right) + \sum_{N \in K(B_1, B_2)} [N, M]^* [N, M] + \sum_{N \in K(B_1, B_2, B_3)} [N, M]^* [N, M]$$

is singular.

Proof. Similar to the proof of Theorem 5.3.

From this we deduce the following theorem.

Theorem 5.6. Let $B, H, G, J \in M_3$. Then $B, H, G, J$ generate $N_3$ if and only if both $N(B, H, G, J)$ and $N(B^t, H^t, G^t, J^t)$ are invertible.

References


