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On N-Metric Equivalence of Operators

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Abstract: In this paper, we introduce a new equivalence relation, the class of n-metrically equivalent operators and examine some of the properties they get to enjoy. We also study their relation to other classes of operators like quasinormal, k-quasinormal and metrically equivalent operators.

Keywords: n-metric equivalence, metric equivalence, quasinormal and k-quasinormal operators.

1. Introduction

We consider the properties of certain classes of operators. Some of the concerned properties of n-normal operators have been covered in [2] and a lot of research has been done on both quasi-normal operators in [1] and [5] and k-quasi-normal operators in [3].

2. Preliminaries

Definition 2.1. *Two operators $S \in B(H)$ and $T \in B(H)$ are said to be n-metrically equivalent, denoted by $S \sim_{n-m} T$, provided $S^*S^n = T^*T^n$ for any positive integer $n \in \mathbb{N}$.*

Theorem 2.2 ([4])**.** *If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.*

3. Main Results

Theorem 3.1. *If S is an n-normal operator and $T \in B(H)$ is unitarily equivalent to S, then T is an n-normal.*

Proof. Since $T = U^*SU$ with U being unitary and S n-normal, we have:

$$\begin{aligned} T^*T^n &= U^*S^*UU^*S^nU \\ &= U^*S^*S^nU \\ &= U^*S^nS^*U \\ &= T^nU^*S^*U \end{aligned}$$

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$$= T^nU^*UT^*$$
$$= T^nT^*$$

Which proves the claim. □

Corollary 3.2. *An operator $T \in B(H)$ is an n-normal if and only if T and T^* are n-metrically equivalent.*

Proof. The proof follows from Theorem 3.1. □

Proposition 3.3. *Let S and T be n-metrically equivalent, then S^* and T^* are co-n-metrically equivalent.*

Proof. Since S and T are n-metrically equivalent, we have;

$$S^*S^n = T^*T^n, \text{ taking adjoint on both sides we obtain;}$$
$$= (S^*S^n)^* = (T^*T^n)^*$$
$$= (S^*)^*(S^n)^* = (T^*)^*(T^n)^*$$
$$= S(S^n)^* = T(T^n)^*.$$

Hence S^* and T^* are co-n-metrically equivalent. □

Proposition 3.4. *If S and T are p-metrically equivalent, then S and T are $p+2$-metrically equivalent operators and hence S and T are n-metrically equivalent for every $n \geq p$.*

Proof. Since S and T are p-metrically equivalent, we have;

$$S^*S^p = T^*T^p \tag{k'}$$

Pre-multiplying and post-multiplying (k') by S on the left hand side and by T on the right hand side;

$$SS^*S^pS = TT^*T^pT$$
$$SS^*S^{p+1} = TT^*T^{p+1}$$
$$S^*S^{p+2} = T^*T^{p+2}$$

Hence S and T are $p+2$ metrically equivalent operators. □

4. Relationship Between n-metric Equivalence and Other Classes of Operators

Proposition 4.1. *If S and T are unitarily $k+1$-metrically equivalent operators, then S and T are n-metrically equivalent for $n \geq k$, and if S is k-quasinormal, then T is k-quasinormal.*

Proof. Since S and T are unitarily $k+1$-metrically equivalent, we have; $S^*S^{k+1} = UT^*T^{k+1}U^*$; and by Proposition 3.4, S and T are n-metrically equivalent since they are $k+1$-metrically equivalent. Now $S^*S^{k+1} = UT^*T^{k+1}U^*$ gives us;

$$S^*S^{k+1} = S^*SS^k = UT^*TT^kU^* \tag{1}$$

$$= S^k S^* S = UT^k T^* TU^* \tag{2}$$

From (1) and (2) we have;

$$UT^k T^* TU^* = UT^* TT^k U^*$$
$$T^k T^* T = T^* TT^k$$
$$T^k (T^* T) = (T^* T)T^k;$$

hence T is k-quasinormal. □

Theorem 4.2. *If S and T are unitarily 2-metrically equivalent operators and S is quasinormal, then T is quasinormal.*

Proof.

$$S^* S^2 = UT^* T^2 U^*$$
$$= UT^* T^2 U^* = T^* T^2$$
$$= UT^* TTU^* = T^* TT$$
$$= (T^* T)T = (T^* T)T$$
$$= T(T^* T) = (T^* T)T$$

hence the proof. □

Theorem 4.3. *If S and T are unitarily 2-metrically equivalent operators then they are metrically equivalent provided they are idempotent.*

Proof. Since S and T are 2-metrically equivalent operators, we have; $S^* S^2 = T^* T^2$; and since S and T are idempotent, we have that; $S^2 = S$ and $T^2 = T$ giving us; $S^* S = T^* T$. □

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References

[1] A. Bala, *A note on quasi normal operator*, University of Delhi, June (1976).

[2] S. A. Alzuraiqi, *On n-normal operators*, General math. Notes, 1(2)(2010), 61-73.

[3] D. Senthilkumar, P. Masheswari Naik and R. Santhi, *k-quasi-normal operators*, International Journal of Mathematics and Computation, 15(2)(2012), 0974-5718.

[4] B. M. Nzimbi, G. P. Pokhariyal and S. K. Moindi, *A note on Metric Equivalence of some operators*, Far East Journal of Mathematical Sciences, 75(2)(2013), 301-318.

[5] K. L. Shaakir and Marai Saad, *Quasi-normal operator of order n*, Tikrit Journal of Pure Sciences, 20(4)(2015), 167-169.