Further Study on $\omega$-closed Sets

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Abstract: The aim of this paper is to prove the notion called semi-$\omega_0$-open sets which is weaker than $\alpha - \omega_0$-open sets and stronger than $\beta - \omega_0$-open sets. Also we introduce and investigate some new generalized classes of $\tau_\omega$.

MSC: 47A63.

Keywords: $\omega$-closed Sets, Semi-$\omega_0$-open sets, $\beta - \omega_0$-open sets

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1. Introduction

In this paper, we prove the notion called semi-$\omega_0$-open sets which is weaker than $\alpha - \omega_0$-open sets and stronger than $\beta - \omega_0$-open sets and investigate some new generalized classes of $\tau_\omega$.

2. Preliminaries

By a space $(X, \tau)$, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $H \subset X$, $cl(H)$ and $int(H)$ will, respectively, denote the closure and interior of $H$ in $(X, \tau)$.

Definition 2.1 ([14]). A subset $H$ of a space $(X, \tau)$ is called

(1) $\alpha$-closed if $cl(int(cl(H))) \subset H$,

(2) $\alpha$-open if $X \setminus H$ is $\alpha$-closed, or equivalently, if $H \subset int(cl(int(H)))$.

For a subset $H$ of $(X, \tau)$, the intersection of all $\alpha$-closed subsets of $X$ containing $H$ is called the $\alpha$-closure of $H$ and is denoted by $cl_\alpha(H)$. It is known that $cl_\alpha(H) = H \cup cl(int(cl(H)))$ and $cl_\alpha(H) \subset cl(H)$. The union of all $\alpha$-open subsets of $X$ contained in $H$ is called the $\alpha$-interior of $H$ and is denoted by $int_\alpha(H)$.

In 1982, the notions of $\omega$-closed sets and $\omega$-open sets were introduced and studied by Hdeib [8]. In 2009, Noiri et al [15] introduced some generalizations of $\omega$-open sets and investigated some properties of the sets. Moreover, they used them to obtain decompositions of continuity. Throughout this paper, $R$ (resp. $N, Q, Q^*, Z$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers, the set of all integers). $\tau_u$ denotes the usual topology.

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Definition 2.2 ([17]). A space $(X, \tau)$ is called submaximal if every dense subset is open.

Definition 2.3 ([17]). Let $H$ be a subset of a space $(X, \tau)$, a point $p$ in $X$ is called a condensation point of $H$ if for each open set $U$ containing $p$, $U \cap H$ is uncountable.

Definition 2.4 ([8]). A subset $H$ of a space $(X, \tau)$ is called $\omega$-closed if it contains all its condensation points. The complement of an $\omega$-closed set is called $\omega$-open.

It is well known that a subset $W$ of a space $(X, \tau)$ is $\omega$-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U-W$ is countable. The family of all $\omega$-open sets, denoted by $\tau_\omega$, is a topology on $X$, which is finer than $\tau$. The interior and closure operator in $(X, \tau_\omega)$ are denoted by $int_\omega$ and $cl_\omega$ respectively.

Definition 2.5 ([16]). A subset $H$ of a space $(X, \tau)$ is said to be semi-$\omega$-open if $H \subset cl(int_\omega(H))$.

Definition 2.6 ([16]). A subset $H$ of a space $(X, \tau)$ is said to be

1. semi$^*$-$\omega$-open if $H \subset cl_\omega(int(H))$.
2. semi$^*$-$\omega$-closed if $int(cl(H)) \subset H$.

Definition 2.7. A subset $H$ of a space $(X, \tau)$ is said to be

1. $\alpha$-$\omega$-open if $H \subset int_\omega(cl_\alpha(int(H)))$,
2. semi-$\alpha$-$\omega$-open if $H \subset cl_\omega(int_\alpha(int(H)))$,
3. pre-$\omega$-open if $H \subset int_\omega(cl_\alpha(H))$,
4. $\beta$-$\omega$-open if $H \subset cl_\omega(int_\alpha(cl_\alpha(H)))$,
5. $b$-$\omega$-open if $H \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int(H))$.

An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies

1. $A \in I$ and $B \subseteq A \Rightarrow B \in I$ and
2. $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

If $I$ is an ideal on $X$ and $X \notin I$, then $F = X : G \in I$ is a filter [11]. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(.)^* : \varphi(X) \rightarrow \varphi(X)$, called a local function [17] of $A$ with respect to $\tau$ and $I$, is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = x \in X \mid U \cap A \notin I$ for every $U \in \tau(x)$ where $\tau(x) = U \in \tau \mid x \in U$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $^*$-topology, finer than $\tau$ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [11]. When there is no chance for confusion, we will simply write $A^*$ for $A^*(I, \tau)$ and $\tau^*$ for $\tau^*(I, \tau)$. $int^*(A)$ will denote the interior of $A$ in $(X, \tau^*)$. $(X, \tau, I)$ is called an ideal topological space or an ideal space.

Definition 2.8 ([11]). A subset $H$ of an ideal topological space $(X, \tau, I)$ is said to be $^*$-closed if $H^* \subset H$ or $cl^*(H) = H$.

The complement of an $^*$-closed set is called $^*$-open.

Proposition 2.9 ([4]). If $(X, \tau)$ is a door space, then every pre-$\omega$-$\alpha$-open set in $(X, \tau, I)$ is $\omega$-open.

Lemma 2.10 ([11]). Let $(X, \tau, I)$ be an ideal topological space and $A$, $B$ subsets of $X$. Then the following properties hold:

1. $A \subset B \Rightarrow A^* \subset B^*$,
(2). $A^* = cl(A^*) \subset cl(A)$,

(3). $A^* \cup B^* = (A \cup B)^*$,

(4). $(A^*)^* \subset A^*$,

(5). $A^*$ is closed in $(X, \tau)$,

(6). If $H \in \tau$, then $H \cap A^* \subset (H \cap A)^*$.

3. Properties of Semi-$\omega\alpha$-open Sets

Definition 3.1. A subset $H$ of a space $(X, \tau)$ is said to be

(1). semi-$\omega\alpha$-open if $H \subset cl(\alpha(int \omega(H)))$.

(2). semi-$\omega\alpha$-closed if $int(\alpha(cl(\omega(H)))) \subset H$.

The complement of a semi-$\omega\alpha$-open set is called semi-$\omega\alpha$-closed.

Example 3.2. In $(\mathbb{R}, \tau_u), H = Q^*$ is semi-$\omega\alpha$-open for $cl(\alpha(int \omega(H))) = cl(\alpha(H)) = \mathbb{R} \supset H$.

Example 3.3. In $(\mathbb{R}, \tau_u), H = Q$ is not semi-$\omega\alpha$-open for $cl(\alpha(int \omega(H))) = cl(\alpha(\phi)) = \phi \not\supset H$.

Proposition 3.4. In a space $(X, \tau)$, every semi-$\omega\alpha$-open subset is semi-$\omega\alpha$-open. Let $H$ be semi-$\omega\alpha$-open in $(X, \tau)$. Then $H \subset cl(\alpha(int \omega(H))) \subset cl(int(\omega(H)))$. This proves that $H$ is semi-$\omega\alpha$-open.

Remark 3.5. The converse of Proposition 3.4 is not true.

Example 3.6. In $\mathbb{R}$ with the topology $\tau = \phi, R, Q$, $H = Q \cup \sqrt{2}$ is semi-$\omega\alpha$-open for $cl(\alpha(int(\omega(H)))) = cl(Q) \subset R \supset H$. But $H$ is not semi-$\omega\alpha$-open for $cl(\alpha(int(\omega(H)))) = \phi \not\subset H$.

Proposition 3.7. Let $H$ be a subset of an ideal topological space $(X, \tau, I)$. Then $H$ is $\alpha$-$I\omega$-open if and only if it is semi-$I\omega$-open.

Theorem 3.8. For a subset of a space $(X, \tau)$, the following properties hold:

(1). Every $\omega$-open set is semi-$\omega\alpha$-open.

(2). Every open set is semi-$\omega\alpha$-open.

(3). Every $\alpha$-$\omega\alpha$-open set is semi-$\omega\alpha$-open.

(4). Every semi-$\omega\alpha$-open set is $\beta$-$\omega\alpha$-open.

(5). Every semi-$\omega\alpha$-open set is $b$-$\omega\alpha$-open.

Proof.

(1). If $H$ is $\omega$-open, then $H \subset cl(\alpha(H)) = cl(\alpha(int(\omega(H))))$. Therefore $H$ is semi-$\omega\alpha$-open.

(2). If $H$ is open, then $H \subset cl(\alpha(H)) = cl(\alpha(int(H))) \subset cl(\alpha(int(\omega(H))))$. Therefore $H$ is semi-$\omega\alpha$-open.

(3). If $H$ is $\alpha$-$\omega\alpha$-open, then $H \subset int(\omega(cl(\alpha(int(\omega(H)))))) \subset cl(\alpha(int(\omega(H))))$. Therefore $H$ is semi-$\omega\alpha$-open.
(4). If H is semi-$\omega$-open, then $H \subset \text{cl} \alpha(int \omega(H)) \subset \text{cl} \alpha(int \omega(cl(H)))$. Therefore H is $\beta$-$\omega$-open.

(5). If H is semi-$\omega$-open, then $H \subset \text{cl} \alpha(int \omega(H)) \subset \text{int} \omega(cl(H)) \cup \text{cl} \alpha(int \omega(H))$. Therefore H is $b$-$\omega$-open. The following Examples support that the separate converses of Theorem 3.8 are not true in general.

Example 3.9. In X with the topology $\tau = \phi, R, N, Q^*, Q^* \cup N$,

(1). $H = Q$ is semi-$\omega$-open, since $\text{cl} \alpha(int \omega(R)) = \text{cl} \alpha(R) = R \supseteq Q = H$. But $H = Q$ is not $\omega$-open, since $int \omega(H) = N \notin H$.

(2). $H = Q$ is semi-$\omega$-open by (1), but not open.

Example 3.10. By (1) of Example 3.9, $H = Q$ is semi-$\omega$-open. But $int \omega(cl \alpha(int \omega(H))) = int \omega(cl \alpha(N)) = int \omega(Q) = N \gneq Q = H$.

Example 3.11. In $(R, \tau_u)$,

(1). $H = Q$ is $\beta$-$\omega$-open, since $\text{cl} \alpha(int \omega(cl(H))) = \text{cl} \alpha(int \omega(R)) = \text{cl} \alpha(R) = R \supseteq Q = H$. But $H = Q$ is not semi-$\omega$-open, since $\text{cl} \alpha(int \omega(H)) = \text{cl} \alpha(\phi) = \phi \notin Q = H$.

(2). $H = Q$ is $b$-$\omega$-open, since $int \omega(cl \alpha(H)) \cup cl \alpha(int \omega(H)) = int \omega(R) \cup cl \alpha(\phi) = R \cup \phi = R \supseteq H$. But $H = Q$ is not semi-$\omega$-open by (1).

Proposition 3.12. The intersection of a semi-$\omega$-open set and an open set is semi-$\omega$-open.

Proof. Let H be semi-$\omega$-open and U be open in X. Then $H \subset \text{cl} \alpha(int \omega(H))$ and $int \omega(U) = U$. By Lemma 2.10, we have $U \cap H \subset U \cap \text{cl} \alpha(int \omega(H)) = \text{cl} \alpha(U \cap int \omega(H)) = \text{cl} \alpha(int \omega(U) \cap int \omega(H)) = \text{cl} \alpha(int \omega(U \cap H))$ which proves that $U \cap H$ is semi-$\omega$-open.

Remark 3.13. The intersection of two semi-$\omega$-open sets need not be semi-$\omega$-open as can be seen from the following Example.

Example 3.14. In $(R, \tau_u)$, $A = (0, 1]$ is semi-$\omega$-open for cl $\alpha(int \omega(A)) = cl \alpha((0, 1]) = [0, 1] \supset A$. Similarly $B = [1, 2)$ is also semi-$\omega$-open. But $A \cap B = 1$ is not semi-$\omega$-open for $cl \alpha(int \omega(A \cap B)) = cl \alpha(int \omega(1)) = cl \alpha(\phi) = \phi \notin 1 = A \cap B$.

Theorem 3.15. If a subset H of a space $(X, \tau)$ is both $\alpha$-closed and $\beta$-$\omega$-open, then H is semi-$\omega$-open.

Proof. Since H is $\beta$-$\omega$-open, $H \subset \text{cl} \alpha(int \omega(cl(H))) = cl \alpha(int \omega(H))$, H being $\alpha$-closed. Therefore H is semi-$\omega$-open.

Theorem 3.16. If a subset H of a space $(X, \tau)$ is both $\beta$-$\omega$-open and a t-$\omega$-set, then H is semi-$\omega$-open.

Proof. Since H is a t-$\omega$-set, $int(H) = int \omega(cl \alpha(H))$. Also H is $\beta$-$\omega$-open implies $H \subset \text{cl} \alpha(int \omega(cl \alpha(H))) \subset \text{cl} \alpha(int \omega(H)) \subset \text{cl} \alpha(int \omega(H))$. Therefore H is semi-$\omega$-open.

Theorem 3.17. If a subset H of a space $(X, \tau)$ is both $b$-$\omega$-open and a t-$\omega$-set, then H is semi-$\omega$-open.

Proof. Since H is a t-$\omega$-set, $int \omega(cl \alpha(H)) = int(H) \subset int \omega(H)$. Also H is $b$-$\omega$-open implies $H \subset int \omega(cl \alpha(H)) \cup \text{cl} \alpha(int \omega(H)) \subset \text{int} \omega(H) \cup \text{cl} \alpha(int \omega(H)) = cl \alpha(int \omega(H))$. Therefore H is semi-$\omega$-open.

Proposition 3.18. A subset H of a space $(X, \tau)$ is semi-$\omega$-open if and only if $cl \alpha(H) = cl \alpha(int \omega(H))$. 

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3.12. Let $H$ be semi-$\omega$-open. Then $H \subseteq \text{cl } \alpha(\text{int } H)$ and $\text{cl } \alpha(H) \subseteq \text{cl } (\text{int } \omega(H))$. But $\text{cl } \alpha(\text{int } \omega(H)) \subseteq \text{cl } \alpha(H)$. Thus $\text{cl } \alpha(H) = \text{cl } \alpha(\text{int } \omega(H))$. Conversely, let the condition hold. We have $H \subseteq \text{cl } \alpha(H) = \text{cl } (\text{int } \omega(H))$, by assumption. Thus $H \subseteq \text{cl } (\text{int } \omega(H))$ and hence $H$ is semi-$\omega$-open. \hfill $\Box$

**Proposition 3.19.** In $(X, \tau)$ if $H$ is a b-$\omega$-open set such that $\text{cl } \alpha(H) = \phi$, then $H$ is semi-$\omega$-open.

**Definition 3.20.** A subset $H$ of a space $(X, \tau)$ is called $\alpha$-dense if $\text{cl } \alpha(H) = X$.

**Definition 3.21.** A space $(X, \tau)$ is called $\alpha$-submaximal if every $\alpha$-dense subset of $X$ is open.

**Theorem 3.22.** For a subset $H$ of an $\alpha$-submaximal space $(X, \tau)$, the following are equivalent.

(1). $H$ is semi-$\omega$-open,

(2). $H$ is $\beta$-$\omega$-open.

**Proof.** (1) $\Rightarrow$ (2): From (4) of Theorem 3.8.

(2) $\Rightarrow$ (1): Let $H$ be a $\beta$-$\omega$-open set in $X$. Then $H \subseteq \text{cl } \alpha(\text{int } \omega(\text{cl } \alpha(H)))$ and $\text{cl } \alpha(H) \subseteq \text{cl } (\text{int } \omega(\text{cl } \alpha(H)))$. Thus, $\text{cl } \alpha(H)$ is semi-$\omega$-open. Put $A = \text{cl } \alpha(H)$ and $K = H \cup (X \alpha(H))$. We have $H = \text{cl } \alpha(H) \cup K$ and $\text{cl } \alpha(K) = X$. This implies that $H = A \cap K$, where $A$ is semi-$\omega$-open and $K$ is $\alpha$-dense. Since $X$ is $\alpha$-submaximal, $K$ is open. By Proposition 3.12, $H = A \cap K$ is semi-$\omega$-open. \hfill $\Box$

**Theorem 3.23.** A subset $H$ of a space $(X, \tau)$ is semi-$\omega$-open if and only if there exists $U \in \tau_\omega$ such that $U \subseteq H \subseteq \text{cl } \alpha(U)$.

**Proof.** Let $H$ be semi-$\omega$-open. Then $H \subseteq \text{cl } \alpha(\text{int } \omega(\text{cl } \alpha(H)))$. Take $\text{int } \omega(H) = U$. Then $U \subseteq H \subseteq \text{cl } \alpha(U)$. Conversely, let $U \subseteq H \subseteq \text{cl } \alpha(U)$ for some $U \in \tau_\omega$. Since $U \subseteq H$, $U \subseteq \text{int } \omega(H)$ and $H \subseteq \text{cl } \alpha(U) \subseteq \text{cl } (\text{int } \omega(H))$ which implies $H$ is semi-$\omega$-open. \hfill $\Box$

**Proposition 3.24.** If $A$ is a semi-$\omega$-open set in a space $(X, \tau)$ and $A \subseteq B \subseteq \text{cl } \alpha(A)$, then $B$ is semi-$\omega$-open.

**Proof.** By assumption $B \subseteq \text{cl } \alpha(A) \subseteq \text{cl } (\text{cl } \alpha(\text{int } \omega(A)))$ (for $A$ is semi-$\omega$-open) $= \text{cl } \alpha(\text{int } \omega(A)) \subseteq \text{cl } (\text{int } \omega(B))$ by assumption. This implies $B$ is semi-$\omega$-open. \hfill $\Box$

## 4. Properties of $\delta$-$\omega$-$\alpha$-open Sets

**Definition 4.1.** A subset $H$ of a space $(X, \tau)$ is said to be

(1). $\delta$-$\omega$-open if $\text{int } \omega(\text{cl } \alpha(H)) \subset \text{cl } \alpha(\text{int } \omega(H))$.

(2). $\delta$-$\omega$-closed if $\text{int } \alpha(\text{cl } \omega(H)) \subset \omega(\text{int } \alpha(H))$. The complement of a $\delta$-$\omega$-open set is called $\delta$-$\omega$-closed.

**Example 4.2.** In $(R, \tau_u)$, for the subset $Q$, $\text{int } \omega(\text{cl } \alpha(Q)) = \text{int } \omega(R) = R$ and $\text{cl } \alpha(\text{int } \omega(Q)) = \text{cl } \alpha(\phi) = \phi$. Thus $\text{int } (\text{cl } \alpha(Q) \not\subseteq \text{cl } \alpha(\text{int } \omega(Q)))$ which proves that $Q$ is not $\delta$-$\omega$-open.

**Example 4.3.** In $(R, \tau_u)$, for the subset $H = 1$, $\text{int } \omega(\text{cl } \alpha(H)) = \text{int } \omega(H) = \phi$ and $\text{cl } \alpha(\text{int } \omega(H)) = \text{cl } \alpha(\phi) = \phi$. Thus $\text{int } (\text{cl } \alpha(H) \not\subseteq \text{cl } \alpha(\text{int } \omega(H)))$ which proves that $H$ is $\delta$-$\omega$-open.

**Proposition 4.4.** For a subset $H$ of a space $(X, \tau)$, the following properties hold:

(1). Every $\alpha$-$\omega$-open set is $\delta$-$\omega$-open.

(2). Every $t$-$\omega$-set is $\delta$-$\omega$-open.
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Proof.

(1). Since $H$ is $\omega$-open, $H \subseteq \text{int} \omega(cl \alpha(int \omega(H))) \subseteq cl \alpha(int \omega(H))$. So $cl \alpha(H) \subseteq cl \alpha(int \omega(H))$ and $int \omega(cl \alpha(H)) \subseteq cl \alpha(H) \subseteq cl \alpha(int \omega(H)))$. Therefore $H$ is $\delta$-$\omega$-open.

(2). Since $H$ is a $\omega$-set, $int \omega(cl \alpha(H)) = int(H) \subseteq H$. Then $int \omega(cl \alpha(H)) \subseteq int \omega(H) \subseteq cl \alpha(int \omega(H))$. Therefore $H$ is $\delta$-$\omega$-open. The converses of (1) and (2) in Proposition 4.4 are not true in general as seen from the following Example.

Example 4.5.

(1). In $(R, \tau_u)$, the subset $H = 1$ is $\delta$-$\omega$-open by Example 4.3. But $H$ is not $\omega$-$\omega$-open for $int \omega(cl \alpha(int \omega(H))) = int \omega(cl \alpha(\phi)) = int \omega(\phi) = \phi \not\subseteq 1 = H$.

(2). In $(R, \tau_u)$, for the subset $H = Q^*$, $int \omega(cl \alpha(H)) = int \omega(R) = R$ and $cl \alpha(int \omega(H)) = cl \alpha(H) = R$. Thus $int \omega(cl \alpha(H)) \subseteq cl \alpha(int \omega(H))$ and hence $H$ is $\delta$-$\omega$-open. But $H$ is not a $\omega$-set for $int \omega(cl \alpha(H)) = int \omega(R) = R \not\neq \phi = int(H)$.

Definition 4.6. A subset $H$ of a space $(X, \tau)$ is said to be $\omega$-$\omega$-closed if $int \alpha(cl \omega(int \alpha(H))) \subseteq H$. The complement of a $\omega$-$\omega$-open set is called $\omega$-$\omega$-closed.

Proposition 4.7. A subset $H$ of a space $(X, \tau)$ is $\omega$-$\omega$-closed if and only if $int \alpha(cl \omega(int \alpha(H))) = int \alpha(H)$.

Proof. Since $H$ is $\omega$-$\omega$-closed set, $int \alpha(cl \omega(int \alpha(H))) \subset H$ and hence $int \alpha(cl \omega(int \alpha(H))) \subseteq int \alpha(H)$. Also $int \alpha(H) \subseteq cl \omega(int \alpha(H))$ and hence $int \alpha(H) \subseteq int \alpha(cl \omega(int \alpha(H)))$. Thus $int \alpha(cl \omega(int \alpha(H))) = int \alpha(H)$. Conversely, let the condition hold. We have $int \alpha(cl \omega(int \alpha(H))) = int \alpha(H) \subset H$. Therefore $H$ is $\omega$-$\omega$-closed.

Theorem 4.8. For a subset $H$ of a space $(X, \tau)$, the following properties are equivalent.

(1). $H$ is semi-$\omega$-closed.

(2). $H$ is $\omega$-$\omega$-closed and $\delta$-$\omega$-closed.

Proof. (1) $\Rightarrow$ (2): Let $H$ be semi-$\omega$-closed. By (4) of Theorem 3.8, $H$ is $\omega$-$\omega$-closed. Since $H$ is semi-$\omega$-closed, $int \alpha(cl \omega(H)) \subset H$ and so $int \alpha(cl \omega(H)) \subseteq int \alpha(H)$ which implies $cl \omega(int \alpha(cl \omega(H))) \subseteq cl \omega(int \alpha(H))$. Thus $int \alpha(cl \omega(H)) \subseteq cl \omega(int \alpha(cl \omega(H))) \subseteq cl \omega(int \alpha(H))$ and so $H$ is $\delta$-$\omega$-closed.

(2) $\Rightarrow$ (1): Since $H$ is $\delta$-$\omega$-closed, $int \alpha(cl \omega(H)) \subseteq cl \omega(int \alpha(H))$ and so $int \alpha(cl \omega(H)) \subseteq int \alpha(cl \omega(int \alpha(H))) \subseteq H$ since $H$ is $\omega$-$\omega$-closed. Thus $H$ is semi-$\omega$-closed.

Remark 4.9. The following Examples show that the concepts of $\omega$-$\omega$-closedness and $\omega$-$\omega$-closedness are independent.

Example 4.10. In $(R, \tau_u)$, the subset $H = R - 1$ is $\delta$-$\omega$-closed by Example 4.3. But $int \alpha(cl \omega(int \alpha(H))) = int \alpha(cl \omega(int(H))) = int \alpha(cl \omega(H)) = int \alpha(R) = int(R) = R \not\subseteq R - 1 = H$ which proves that $H$ is not $\omega$-$\omega$-closed.

Example 4.11. In $(R, \tau_u)$, the subset $H = Q^*$ is not $\omega$-$\omega$-closed for $Q$ is not $\omega$-$\omega$-open by Example 4.2. But $int \alpha(cl \omega(int \alpha(H))) = int \alpha(cl \omega(int(H))) = int \alpha(cl \omega(\phi)) = int \alpha(\phi) = \phi \subset H$. Thus $H = Q^*$ is $\omega$-$\omega$-closed.

Theorem 4.12. Let $(X, \tau)$ be a space. Then a subset of $X$ is $\omega$-$\omega$-open if and only if it is both $\delta$-$\omega$-open and pre-$\omega$-open.
Proof.  Necessity: Let $H$ be an $\alpha$-$\omega$-open set. Then $H \subset \text{int}(\text{cl}(\text{int}(H)))$. It implies that $\text{cl}(\alpha(H)) \subset \text{cl}(\text{int}(\text{int}(H)))$ and $\text{int}(\text{cl}(\alpha(H))) \subset \text{int}(\text{cl}(\text{int}(\text{int}(H)))) \subset \text{cl}(\text{int}(\text{int}(H))).$ Hence, $H$ is a $\delta$-$\omega$-open set. On the other hand, since $H$ is an $\alpha$-$\omega$-open set, $H$ is a pre-$\omega$-open set by Proposition 3.7.

Sufficiency: Let $H$ be both a $\delta$-$\omega$-open and pre-$\omega$-open. Since $H$ is a $\delta$-$\omega$-open, $\text{int}(\text{cl}(\alpha(H))) \subset \text{cl}(\text{int}(\text{int}(H)))$ and hence $\text{int}(\text{cl}(\alpha(H))) \subset \text{int}(\text{cl}(\text{int}(\text{int}(H))))$. Since $H$ is pre-$\omega$-open, $H \subset \text{int}(\text{cl}(\alpha(H))) \subset \text{int}(\text{cl}(\text{int}(\text{int}(H))))$ which proves that $H$ is an $\alpha$-$\omega$-open set.

Remark 4.13. The following Examples show that the concepts of $\delta$-$\omega$-open-ness and pre-$\alpha$-openness are independent.

Example 4.14. In $(R, \tauu)$, $H = (0, 1]$ is $\delta$-$\omega$-open, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega([0, 1])) = (0, 1)$ and $\text{cl}(\text{int}(\omega([0, 1]))) = \text{cl}(\alpha((0, 1))) = [0, 1]$. But $H = (0, 1]$ is not pre-$\omega$-open, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega([0, 1])) = (0, 1) \nsubseteq H$.

Example 4.15. In $(R, \tauu)$, $H = Q$ is pre-$\omega$-open, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(R)) = R \supset Q = H$. But $\text{cl}(\alpha(\text{int}(\omega(H)))) = \text{cl}(\alpha(\phi)) = \phi$ and $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(R)) \nsubseteq \text{cl}(\text{int}(\omega(H)))$. Thus $H$ is not $\delta$-$\omega$-open.

Proposition 4.16. Let $A$ and $B$ be subsets of a space $(X, \tau)$. If $A \subset B \subset \text{cl}(\alpha(A))$ and $A$ is $\delta$-$\omega$-open in $X$, then $B$ is $\delta$-$\omega$-open in $X$.

Proof. Suppose that $A \subset B \subset \text{cl}(\alpha(A))$ and $A$ is $\delta$-$\omega$-open in $X$. Then $\text{int}(\text{cl}(\alpha(A))) \subset \text{cl}(\text{int}(\alpha(A))) \subset \text{cl}(\text{int}(\text{int}(B)))$. Since $B \subset \text{cl}(\alpha(A))$, $\text{cl}(\alpha(B)) \subset \text{cl}(\text{cl}(\alpha(A))) = \text{cl}(\alpha(A))$ and $\text{int}(\text{cl}(\alpha(B))) \subset \text{int}(\text{cl}(\text{int}(\alpha(A))))$. Therefore $\text{int}(\text{cl}(\alpha(B))) \subset \text{cl}(\text{int}(\text{int}(B))).$ This shows that $B$ is a $\delta$-$\omega$-open set.

Corollary 4.17. Let $(X, \tau)$ be a space. If $A \subset X$ is $\delta$-$\omega$-open and $\alpha$-dense in $(X, \tau)$, then every subset of $X$ containing $A$ is $\delta$-$\omega$-open.

Proof. It is obvious by Proposition 4.16.

5. Properties of Semi-$\omega$-open Sets

Definition 5.1. A subset $H$ of a space $(X, \tau)$ is said to be

(1) semi-$\omega$-open if $H \subset \text{cl}(\text{int}(\alpha(H)))$.

(2) semi-$\omega$-closed if $\text{int}(\text{cl}(\alpha(H))) \subset H$. The complement of a semi-$\omega$-open set is called semi-$\omega$-closed.

Example 5.2. In $(R, \tauu)$, $H = R\{0\}$ is not semi-$\omega$-closed, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(R)) = R \nsubseteq H$.

Example 5.3. In $R$ with the topology $\tau = \phi, R, Q, Q^*$, $H = Q$ is semi-$\omega$-closed, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(H)) = H \subset H$.

Proposition 5.4. For a subset of a space $(X, \tau)$, every semi-$\omega$-open set is semi-$\omega$-open. If $H$ is semi-$\omega$-open, then $H \subset \text{cl}(\text{int}(H)) \subset \text{cl}(\text{int}(\alpha(H)))$. Therefore $H$ is semi-$\omega$-open.

Remark 5.5. The converse of Proposition 5.4 is not true.

Example 5.6. In $R$ with usual topology $\tauu$ and ideal $I = P(R)$, $H = Q$ is semi-$\omega$-closed, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(H)) = \phi \subset H$. But $H = Q$ is not semi-$\omega$-closed, since $\text{int}(\text{cl}(\alpha(H))) = \text{int}(\omega(R)) \nsubseteq R \nsubseteq H$.

Proposition 5.7. A subset $H$ of a space $(X, \tau)$ is semi-$\omega$-open if and only if $\text{cl}(\omega(H)) = \text{cl}(\text{int}(\alpha(H)))$. 
Proof. If $H$ is semi-$\omega$-open set, then $H \subset cl(\omega(int(\phi)))$ and $cl(\omega(H)) \subset cl(\omega(int(\phi)))$. But $cl(\omega(int(\phi))) \subset cl(\omega(H))$. Hence $cl(\omega(H)) = cl(\omega(int(\phi)))$. Conversely, $H \subset cl(\omega(H)) = cl(\omega(int(\phi)))$ by assumption. Therefore $H$ is semi-$\omega$-open.

Definition 5.8. A subset $H$ of a space $(X, \tau)$ is said to be a t-$\omega(\alpha^*)$-set if $int(\omega(cl(\phi))) = int(\omega(H))$.

Example 5.9. In $R$ with usual topology $\tau_u$ and ideal $I = \phi$,

(1). $H = (0, 1]$ is a t-$\omega(\alpha^*)$-set, since $int(\omega(H)) = (0, 1)$ and $int(\omega(cl(\phi))) = int(\omega((0, 1])) = (0, 1)$. 

(2). $H = Q^*$ is not a t-$\omega(\alpha^*)$-set, since $int(\omega(H)) = H$ and $int(\omega(cl(\phi))) = int(\omega(R)) = R$.

Proposition 5.10. In a space $(X, \tau)$, every $\alpha$-closed set is a t-$\omega(\alpha^*)$-set.

Let $H$ be an $\alpha$-closed set. Then $H = cl(\alpha(H))$ and $int(\omega(cl(\phi))) = int(\omega(H))$ which proves that $H$ is a t-$\omega(\alpha^*)$-set.

Remark 5.11. The converse of Proposition 5.10 is not true.

Example 5.12. In $(R, \tau_u)$, $H = (0, 1]$ is t-$\omega(\alpha^*)$-set by (1) of Example 5.9. But $H = (0, 1]$ is not $\alpha$-closed, since $cl(\alpha(H)) = [0, 1] \neq H$.

Proposition 5.13. In a space $(X, \tau)$, every t-$\omega(\alpha^*)$-set.

If $H$ is a t-$\omega$-set, then $int(\omega(cl(\phi))) = int(\omega(H)) \subset int(\omega(cl(\phi)))$. Thus $int(\omega(cl(\phi))) = int(\omega(H))$ and hence $H$ is a t-$\omega(\alpha^*)$-set.

Remark 5.14. The converse of Proposition 5.13 is not true.

Example 5.15. In $R$ with usual topology $\tau_u$ and ideal $I = P(R), H = (0, 1] \cap Q^*$ is a t-$\omega(\alpha^*)$-set since $int(\omega(cl(\phi))) = int(\omega(H))$. But $H$ is not a t-$\omega$-set since $int(\omega(cl(\phi))) = int(\omega(H)) = H \neq \phi = \phi$.

Theorem 5.16. A subset $H$ of a space $(X, \tau)$ is semi-$\omega$-closed if and only if $H$ is a t-$\omega(\alpha^*)$-set.

If $H$ is a semi-$\omega$-closed in $X$ then $X' = X'$ is semi-$\omega$-open set. Thus $\omega(X') = \omega(int(\alpha(X'))) = \omega(cl(\alpha(H)))$ by Proposition 5.7. Since $X(H) = \omega(cl(\alpha(H))) \Rightarrow int(\omega(H)) = int(\omega(cl(\alpha(H)))) \Rightarrow H$ is a t-$\omega(\alpha^*)$-set.

Proposition 5.17. If $A$ and $B$ are t-$\omega(\alpha^*)$-sets of a space $(X, \tau)$, then $A \cap B$ is a t-$\omega(\alpha^*)$-set.

Let $A$ and $B$ be t-$\omega(\alpha^*)$-sets. Then $\omega(A \cap B) \subset \omega(cl(\alpha(A \cap B))) \subset \omega(cl(\alpha(A)) \cap \omega(cl(\alpha(B))) = \omega(cl(\alpha(A)) \cap \omega(cl(\alpha(A)) \cap \omega(cl(\alpha(B))) = int(\omega(A) \cap \omega(B)) = int(\omega(A \cap B))$. Thus $int(\omega(A \cap B)) = int(\omega(cl(\alpha(A \cap B)))$ and hence $A \cap B$ is a t-$\omega(\alpha^*)$-set.

Definition 5.18. A subset $H$ of a space $(X, \tau)$ is said to be semi-$\omega$-regular if $H$ is semi-$\omega$-open and a t-$\omega(\alpha^*)$-set.

Example 5.19. In $R$ with usual topology $\tau_u$ and ideal $I = \phi$,

(1). $H = (0, 1]$ is a t-$\omega(\alpha^*)$-set by (1) of Example 5.9. Also $cl(\alpha(int(\phi))) = cl(\alpha((0, 1])) = [0, 1] \subset H$. Thus $H$ is semi-$\omega$-open. Hence $[0, 1]$ is semi-$\omega$-regular.

(2). $H = Q^*$ is not a t-$\omega(\alpha^*)$-set by (2) of Example 5.9. Hence $Q^*$ is not semi-$\omega$-regular.

Remark 5.20. In a space $(X, \tau)$,

(1). Every semi-$\omega$-regular set is semi-$\omega$-open.
(2). Every semi-ωα-regular set is t-ω(α*)-set.

The converses of (1) and (2) in Remark 5.20 are not true in general as illustrated in the following Examples.

Example 5.21. In R with usual topology τu and ideal I = φ, the subset H = Q* is semi-ω-open by Example 3.2. But H = Q* is not semi-ωα-regular by (2) of Example 5.19.

Example 5.22. In (R, τu), N = the set of all natural numbers is a t-ω(α*)-set for int ω(cl α(N)) = int ω(N). But N is not semi-ωα-regular for N is not semi-ω-open since cl α(int ω(N)) = cl α (φ) = φ ⊈ N.

Theorem 5.23. A subset of a space (X, τ) is semi-ωα-regular if and only if it is both β-ωα-open and semi*-ωα-closed.

Proof. If H is semi-ωα-regular, then H is both semi-ω-open and a t-ω(α*)-set. Since H is semi-ω-open, H is β-ωα-open by (4) of Theorem 3.8. Also H is a t-ω(α*)-set by assumption. Hence by Example 7.7, H is semi*-ωα-closed. Conversely, let H be semi*-ωα-closed and β-ωα-open. Since H is semi*-ωα-closed, by Theorem 5.16, H is a t-ω(α*)-set. Since H is β-ωα-open, H ⊂ cl α (ω(cl α(H))) = cl α (ω(H)). Therefore H is semi-ωα-open. Since H is both semi-ωα-open and a t-ω(α*)-set, H is semi-ωα-regular. □

Remark 5.24. The following Example shows that the concepts of β-ωα-open-ness and semi*-ωα-closedness are independent.

Example 5.25.

(1). In R with the topology τ = φ, R, Q*, H = Q is semi*-ωα-closed, since int ω(cl α(H)) = int ω(H) = φ ⊂ H. But H = Q is not β-ωα-open, since cl α(int ω(cl α(H))) = cl α(int ω(H)) = cl α (φ) = φ ⊈ H.

(2). In (R, τu), H = Q is β-ωα-open, since cl α (int ω(cl α(H))) = cl α (int ω(R)) = R ⊃ H. But H = Q is not semi*-ωα-closed, since int ω(cl α(H)) = int ω(R) = R ⊈ H.

6. Properties of ωα-R-closed Sets

Definition 6.1. A subset H of a space (X, τ) is called ωα-R-closed if H = cl α(int ω(H)).

Example 6.2. In R with the topology τ = φ, R, Q*, H = Q is not ωα-R-closed, since cl α (int ω(H)) = cl α (φ) = φ ⊈ H.

Example 6.3. In (R, τu), H = [0, 1] is ωα-R-closed for cl α(int ω(H)) = cl α ([0, 1]) = H.

Theorem 6.4. For a subset H of a space (X, τ), the following properties are equivalent.

(1). H (≠ φ) is ωα-R-closed.

(2). There exists a non-empty ω-open set G such that G ⊂ H = cl α(G).

(3). There exists a non-empty ω-open set G such that H = G ∪ (cl α(G) – G).

Proof. (1) ⇒ (2): Suppose H (≠ φ) is an ωα-R-closed set. Then H = cl α(int ω(H)). Let G = int ω(H). G is the required ω-open set such that G ⊂ H = cl α(G).

(2) ⇒ (3): Since H = cl α(G) = G ∪ (cl α(G) – G) where G is a nonempty ω-open set, (3) follows.

(3) ⇒ (1): H = G ∪ (cl α(G) – G) implies that H = cl α(G) = cl α(int ω(G)) ⊂ cl α(int ω(H)), since G is ω-open and G ⊂ H. Also cl α(int ω(H)) ⊂ cl α(H) = cl α(G) = H. Therefore H = cl α(int ω(H)) which implies that H is ωα-R-closed. □
Theorem 6.5. For each $\beta\omega$-open subset $H$ of a space $(X, \tau)$, $cl \alpha(H)$ is $\omega\alpha$-R-closed.

Proof. Suppose $H$ is $\beta\omega$-open. Then $H \subset cl \alpha(int \omega(cl \alpha(H)))$ and so $cl \alpha(H) \subset cl \alpha(int \omega(cl \alpha(H))) \subset cl \alpha(H)$ which implies that $cl \alpha(H) = cl \alpha(int \omega(cl \alpha(H)))$. Therefore $cl \alpha(H)$ is $\omega\alpha$-R-closed. \qed

Theorem 6.6. For a subset $H$ of a space $(X, \tau)$, the following properties are equivalent.

(1). $H$ is $\omega\alpha$-R-closed.

(2). $H$ is semi-$\omega\alpha$-open and $\alpha$-closed.

(3). $H$ is $\beta\omega$-open and $\alpha$-closed.

Proof. (1) $\Rightarrow$ (2): If $H$ is $\omega\alpha$-R-closed, then $H = cl \alpha(int \omega(H))$. Since $H \subset cl \alpha(int \omega(H))$, $H$ is semi-$\omega\alpha$-open. Also, $H = cl \alpha(H)$ and thus $H$ is $\alpha$-closed.

(2) $\Rightarrow$ (3): It follows from the fact that every semi-$\omega\alpha$-open set is a $\beta\omega$-open.

(3) $\Rightarrow$ (1): Suppose $H$ is $\beta\omega$-open and $\alpha$-closed. Then $H \subset cl \alpha(int \omega(cl \alpha(H)))$ and $H = cl \alpha(H)$. Now $cl \alpha(int \omega(H)) \subset cl \alpha(H) = H$. Also, $H \subset cl \alpha(int \omega(H))$. Therefore $H = cl \alpha(int \omega(H))$ which implies that $H$ is $\omega\alpha$-R-closed. \qed

Remark 6.7. The following Examples show that

(1). the concepts of semi-$\omega\alpha$-openness and $\alpha$-closedness are independent.

(2). the concepts of $\beta\omega$-openness and $\alpha$-closedness are independent.

Example 6.8. In $(\mathbb{R}, \tau_u)$,

(1). $H = Q^*$ is semi-$\omega\alpha$-open by Example 5.21. But $H$ is not $\alpha$-closed for $cl \alpha(H) = \mathbb{R} \neq H$.

(2). $N = \text{the set of all natural numbers}$ is not semi-$\omega\alpha$-open by Example 5.22. But $N$ is $\alpha$-closed for $cl \alpha(N) = N$.

Example 6.9. In $(\mathbb{R}, \tau_u)$,

(1). the subset $H = Q^*$ is semi-$\omega\alpha$-open by Example 5.21 and hence $\beta\omega$-open by (4) of Theorem 3.8. But $H = Q^*$ is not $\alpha$-closed for $cl \alpha(H) = \mathbb{R} \neq H$.

(2). $N = \text{the set of all natural numbers}$ is $\alpha$-closed by (2) of Example 6.8. But $N$ is not $\beta\omega$-open for $cl \alpha(int \omega(cl \alpha(N))) = cl \alpha(int \omega(N)) = cl \alpha(\phi) = \phi \nsubseteq N$.

7. Further Properties

Definition 7.1. A space $(X, \tau)$ is called $\omega\alpha$-submaximal if every $\alpha$-dense subset of $X$ is $\omega$-open.

Proposition 7.2.

(1). Every submaximal space is $\alpha$-submaximal.

(2). Every $\alpha$-submaximal space is $\omega\alpha$-submaximal.

Proof.

(1). If $(X, \tau)$ is submaximal and $H$ is $\alpha$-dense in the space $(X, \tau)$, then $cl \alpha(H) = X$. But $X = cl \alpha(H) \subset cl(H)$ implies $cl(H) = X$. Thus $H$ is dense in $X$ and by assumption $H$ is open in $X$. This shows that $(X, \tau)$ is $\alpha$-submaximal.
(2). Proof follows directly since any open set is $\omega$-open. The converses of (1) and (2) in Proposition 7.2 are not true in general as illustrated below. □

Example 7.3.

(1). In $(R, \tau_u)$, if $H$ is any $\alpha$-dense subset, then $cl \alpha(H) = R$ and so $H = R$ which is open. Thus $(R, \tau_u)$ is $\alpha$-submaximal.

But in $(R, \tau_u)$, $Q$ is dense in $R$ since $cl(Q) = R$. But $Q$ is not open. This shows that $(R, \tau_u)$ is not submaximal.

(2). In the space $(N, \tau)$, where $N$ is the set of all natural numbers, $\tau = \phi, N$, if $H$ is any $\alpha$-dense subset in $N$, then $H \subset N$.

Since $N$ is countable, $H$ is $\omega$-open. Thus $(N, \tau)$ is $\omega\alpha$-submaximal. The subset $A = 1$ is $\alpha$-dense in $N$ for $cl \alpha(A) = N$.

But $A = 1$ is not open in $N$. This shows that $(N, \tau)$ is not $\alpha$-submaximal.

Definition 7.4. A subset $H$ of a space $(X, \tau)$ is called $\alpha$-co-dense if $X'$ is $\alpha$-dense.

Theorem 7.5. For a space $(X, \tau)$, the following are equivalent.

(1). $X$ is $\omega\alpha$-submaximal,

(2). Every $\alpha$-co-dense subset of $X$ is $\omega$-closed.

$X$ is $\omega\alpha$-submaximal $\iff$ every $\alpha$-dense subset of $X$ is $\omega$-open $\iff$ every $\alpha$-co-dense subset of $X$ is $\omega$-closed since a subset $A$ is $\alpha$-dense in $X$ if and only if $X - A$ is $\alpha$-co-dense in $X$.

Example 7.6.

(1). In $R$ with usual topology $\tau_u$ and ideal $I = P(R)$, $Z$ is a $ta-I(\omega^*)$-set, since $cl^* (int \omega(cl^*(Z))) = cl^* (int \omega(Z)) = cl^*(\phi) = \phi = int(Z)$.

(2). In $R$ with usual topology $\tau_u$ and ideal $I = \phi$, $H = (0,1)$ is not a $ta-I(\omega^*)$-set, since $int(H) = (0,1)$ and $cl^* (int \omega(cl^*(H))) = cl^* (int \omega((0,1))) = cl^*((0,1)) = [0,1]$ which implies $int(H) \neq cl^* (int \omega(cl^*(H)))$.

(3). In $R$ with usual topology $\tau_u$ and ideal $I = \phi$, $H = Q$ is not a $ta-I(\omega^*)$-set, since $int(Q) = \phi$ and $cl^* (int \omega(cl^*(Q))) = cl^* (int \omega(cl(Q))) = cl^* (int \omega(R)) = cl^*(R) = R$ which implies $int(Q) \neq cl^* (int \omega(cl(Q)))$.

Example 7.7.

(1). In $R$ with usual topology $\tau_u$ and ideal $I = P(R)$, $Z$ is a $ta-I(\omega^*)$-set by (1) of Example 7.6 and hence a $Ba\alpha-I(\omega^*)$-set by (2) of Remark 7.8.

(2). In $R$ with usual topology $\tau_u$ and ideal $I = \phi$, $H = Q$ is not a $Ba\alpha-I(\omega^*)$-set. If $H = U \cap V$ where $U$ is open and $V$ is a $ta-I(\omega^*)$-set, then $H \subset U$. But $R$ is the only open set containing $H$. Hence $U = R$ and $H = R \cap V = V$ which means that $H$ is a $ta-I(\omega^*)$-set which is a contradiction by (3) of Example 7.6. Thus $H$ is not a $Ba\alpha-I(\omega^*)$-set.

Remark 7.8. In an ideal topological space $(X, \tau, I)$,

(1). Every open set is a $Ba\alpha-I(\omega^*)$-set.

(2). Every $ta-I(\omega^*)$-set is a $Ba\alpha-I(\omega^*)$-set.
References


