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Introduction to Abelian Spaces: A Connection with Topology

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Abstract: An abelian group is an algebraic property of an algebraic structure, but it can also be considered as a metric space. This paper establishes a new concept "Abelian Space", a space containing a set and an operator which is not only an abelian group, but can also be viewed as a metric space. In this paper, with reference to the symmetric difference operator between two sets, it is proved that an abelian group can also be considered as an abelian "metric" space.

Keywords: Abelian Group, Symmetric Difference, Abelian Space.

1. Introduction

The consideration of an abelian group as an algebraic property is usual, but it can be considered as a topological space, having the same properties as a metric space, which further steps it into the topological properties of a metric space. This paper introduces a slight idea about a different view for the algebraic structures, apart from their algebraic geometries.

2. Main Results

Definition 2.1. *Let A and B be two sets such that the symmetric difference between them is defined as $A\Delta B = (A - B) \cup (B - A)$, or $A\Delta B = (A \cup B) - (A \cap B)$.*

Theorem 2.2. *Symmetric difference operator is a metric.*

Proof. Let $A\Delta B$, be the symmetric difference between two disjoint sets, such that

$$A\Delta B = (A \cup B) - (A \cap B)$$

Let the metric, d on the sets A and B is defined by $d(A, B) = A\Delta B = (A \cup B) - (A \cap B)$. For a metric, the following conditions need to be satisfied: for a defined metric d on a set X, $x, y \in X$,

(1). $d(x, y) = 0 \Leftrightarrow x = y$,

(2). $d(x, y) = d(y, x)$,

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(3). $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$.

So, let $P(X)$ be the power set of the set X, containing the disjoint sets A and B, therefore,

(1). $d(A, B) = 0 \Leftrightarrow A = B$.

$\Rightarrow$ Let $d(A, B) = 0$,

$$A \Delta B = 0$$
$$(A \cup B) - (A \cap B) = 0$$
$$(A \cup B) = (A \cap B)$$

When the two sets $(A \cup B)$ and $(A \cap B)$ are equal to each other, then $A = B$.

$\Leftarrow$ Let $A = B$, then

$$d(A, B) = A \Delta B$$
$$= (A \cup B) - (A \cap B)$$

But, since $A = B$, therefore, $A \cup B = A \cap B$, thus $d(A, B) = (A \cup B) - (A \cup B) = 0$.

(2). $d(A, B) = (A \cup B) - (A \cap B)$, since union and intersection of two sets are commutative, therefore, $(A \cup B) = (B \cup A)$ and $(A \cap B) = (B \cap A)$

$$d(A, B) = (B \cup A) - (B \cap A) = d(B, A)$$

Hence, symmetric difference is commutative.

(3). Let C be another disjoint set, such that

$$A \Delta C = (A \Delta B) \Delta (B \Delta C)$$
$$A \Delta B = (A \Delta C) \Delta (B \Delta C)$$
$$A \Delta B \subseteq (A \Delta C) \cup (B \Delta C)$$
$$A \Delta B \subseteq (A \Delta C) \cup (C \Delta B),$$

since the symmetric difference is commutative $\Rightarrow d(A, B) \leq d(A, C) + d(C, B)$. Hence, since all the three conditions are satisfied, therefore, the symmetric difference on two sets is a metric. Also, since $P(X)$ is the power set containing the two sets A and B, so the symmetric difference metric will also be on the power set $P(X)$, making $(P(X), \Delta)$, a metric space. But, since for X be any set, then $(P(X), \Delta)$ is also an abelian group with respect to the symmetric difference operation. Therefore, the metric space $(P(X), \Delta)$ can also be considered as an "Abelian metric space" or just an "Abelian Space". This space is a metric space as well as an abelian group. □

Theorem 2.3. *The metric space $(P(X), \Delta)$ is a T_0-space.*

Proof. Since every metric space satisfies the T_0 axiom of separation, therefore, the metric space $(P(X), \Delta)$ is a T_0 space. □

Theorem 2.4. *The metric space $(P(X), \Delta)$ is a T_1-space.*

Proof. Since every metric space satisfies the T_1 axiom of separation, therefore, the metric space $(P(X), \Delta)$ is a T_1 space. □

Theorem 2.5. *The metric space $(P(X), \Delta)$ is a T_2-space (Hausdorff space).*

Proof. Since every metric space is a T_2 space, hence the metric space $(P(X), \Delta)$ is a Hausdorff or a T_2 space. □

3. Conclusions

In this paper, an abelian group under the operation of symmetric difference is proved to be a metric space, hence, it further connects with the possibilities of being associated with the axioms of separation, therefore, a connection of an abelian group with the separation axioms proves a connection between two different aspects of the subject, leading to further new possibilities.

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