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Centralizing Properties of $(\alpha, 1)$ Derivations in Semiprime Rings

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Abstract: Let R be a semiprime ring with center Z, S be a non-empty subset of R, α be an endomorphism on R and d be an $(\alpha, 1)$ derivation of R. A mapping f from R into itself is called centralizing on S if $[f(x), x] \in Z$, for all $x \in S$. In the present paper, we study some centralizing properties of $(\alpha, 1)$ derivations in semiprime rings one of the following conditions holds: $(i)\, d([x,y]) = [x,y]_{\alpha,1}$, for all $x, y \in R$. $(ii)\, d([x,y]) = -[x,y]_{\alpha,1}$, for all $x, y \in R$. $(iii)\, d(x)\, d(y) \mp xy \in Z$, for all $x, y \in R$. $(iv)\, d(xoy) = (xoy)_{\alpha,1}$, for all $x, y \in R$. $(v)\, d(xoy) = -(xoy)_{\alpha,1}$, for all $x, y \in R$. Also we prove that d is centralizing on R if d acts as a homomorphism on R and d is centralizing on S if d acts as an antihomomorphism on R.

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Keywords: Semiprime rings, $(\alpha, 1)$ derivation, centralizing mappings, homomorphism and antihomomorphism.

1. Introduction

Throughout this paper, R will represent an associative ring with center Z. A ring R is said to be prime if $xRy = 0$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = 0$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime for any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol (xoy) stands for the anti-commutator $xy+yx$. A derivation d on R is determined to be an additive endomorphism satisfying the product rule $d(xy) = d(x)y+xd(y)$, for all $x, y \in R$. Let α be an endomorphism on R. An additive mapping d from R into itself to be an $(\alpha, 1)$ derivation if $d(xy) = d(x)\alpha(y) + xd(y)$. Let S be a non-empty subset of R. A mapping f from R into itself is called centralizing on S if $[f(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$, for all $x \in S$. If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all x,y $\in$ R, then d is said to act as homomorphism or anti-homomorphism on R respectively. The study of centralizing mappings was initiated by E.C.Posner [8]. He proved that the existence of a non-zero centralizing derivation on a prime ring forces the ring to be commutative(Posner second Theorem). Yenigul and Argac [9] studied prime and semiprime rings with α derivations. Then in 2004, Argac [1] obtained some results on near-rings with two sided α derivations, that is, $(\alpha, 1)$ derivations and $(1, \alpha)$ derivations. Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on appropriate subsets of R (see [2, 4, 8]). The purpose of this paper is to study some centralizing properties of $(\alpha, 1)$ derivations in semiprime rings. Also we prove that d is centralizing on R if d acts as a homomorphism on R and d is centralizing on S if d acts as an antihomomorphism

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on R. Through out the present paper, we will make extensive use of the following basic commutator identities [7]:

$$[x, yz] = y[x, z] + [x, y]z,$$

$$[xy, z] = [x, z]y + x[y, z],$$

$$[xy, z]_{\alpha,1} = x[y, z]_{\alpha,1} + [x, z]y = x[y, \alpha(z)] + [x, z]_{\alpha,1} y,$$

$$[x, yz]_{\alpha,1} = y[x, z]_{\alpha,1} + [x, y]_{\alpha,1} \alpha(z),$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z],$$

$$(xo(yz))_{\alpha,1} = (xoy)_{\alpha,1} \alpha(z) - y[x, z]_{\alpha,1} = y(xoz)_{\alpha,1} + [x, y]_{\alpha,1} \alpha(z),$$

$$((xy)oz)_{\alpha,1} = x(yoz)_{\alpha,1} - [x, z]y = (xoz)_{\alpha,1} y + x[y, \alpha(z)].$$

2. Results

Lemma 2.1 ([?]). *Let R be a semiprime ring and suppose that $a \in R$ centralizes all commutators $xy - yx$, $x, y \in R$. Then $a \in Z$.*

Theorem 2.2. *Let R be a semiprime ring and d be an $(\alpha, 1)$ derivation of R. If d satisfies one of the following conditions, then d is centralizing.*

(i). $d([x, y]) = [x, y]_{\alpha,1}$, for all $x, y \in R$.

(ii). $d([x, y]) = -[x, y]_{\alpha,1}$, for all $x, y \in R$.

Proof. (i). Assume that $d([x, y]) = [x, y]_{\alpha,1}$, for all $x, y \in R$. Replacing y by yx, we get

$$d(y[x, x] + [x, y]x) = y[x, x]_{\alpha,1} + [x, y]_{\alpha,1} \alpha(x),$$

and so $d(y)\alpha([x, x]) + yd([x, x]) + d([x, y])\alpha(x) + [x, y]d(x) = y[x, x]_{\alpha,1} + [x, y]_{\alpha,1} \alpha(x)$. Using the hypothesis, we obtain

$$[x, y]d(x) = 0, for\ all\ x, y \in R. \tag{1}$$

Substituting $d(x)y$ for y in (1) and using (1) we have

$$[x, d(x)]yd(x) = 0, for\ all\ x, y \in R. \tag{2}$$

Replacing y by yx in (2) we get

$$[x, d(x)]yxd(x) = 0, for\ all\ x, y \in R. \tag{3}$$

Multiplying (2) on the right of x, we have

$$[x, d(x)]yd(x)x = 0, for\ all\ x, y \in R. \tag{4}$$

Subtracting (4) from (3), we arrive at $[x, d(x)]y[x, d(x)] = 0$, for all $x, y \in R$. By the semiprimeness of R, we conclude that $[x, d(x)] = 0$, for all $x \in R$ and so $[x, d(x)] \in Z$.

(ii). If d is an $(\alpha, 1)$ derivation satisfying the property $d([x, y]) = -[x, y]_{\alpha,1}$, for all $x, y \in R$, then $(-d)$ satisfies the condition $(-d)([x, y]) = -[x, y]_{\alpha,1}$, for all $x, y \in R$. Hence d is centralizing by (i). □

Corollary 2.3. *Let R be a prime ring and d be an $(\alpha, 1)$ derivation of R. If d satisfies one of the following conditions, then R is commutative integral domain.*

(i). $d([x, y]) = [x, y]_{\alpha,1}$, *for all* $x, y \in R$.

(ii). $d([x, y]) = -[x, y]_{\alpha,1}$, *for all* $x, y \in R$.

Theorem 2.4. *Let R be a semiprime ring and d be an $(\alpha, 1)$ derivation of R. If d acts as a homomorphism on R, then d is centralizing.*

Proof. Assume that d acts as a homomorphism on R. Now we have $d(xy) = d(x)\alpha(y) + xd(y) = d(x)d(y)$, for all $x, y \in R$. Replacing y by yz, $z \in R$ in the above equation, we get

$$d(x)\alpha(y)\alpha(z) + xd(y)\alpha(z) + xyd(z) = d(x)d(y)\alpha(z) + d(x)y\alpha(z), for\ all\ x, y, z \in R.$$

Using the hypothesis and d is derivation on R in the last relation gives $xyd(z) = d(x)yd(z)$, and so

$$(d(x) - x)yd(z) = 0, for\ all\ x, y, z \in R. \tag{5}$$

Writing y by $d(y)$ in (5) we get $(d(x) - x)d(y)d(z) = 0$, for all $x, y, z \in R$. By the hypothesis, we obtain

$$(d(x) - x)d(yz) = (d(x) - x)d(y)\alpha(z) + (d(x) - x)d(yz) = 0.$$

Using (5), we have

$$(d(x) - x)d(y)\alpha(z) = 0,$$

and so

$$d(x)d(y)\alpha(z) = xd(y)\alpha(z),$$
$$d(xy)\alpha(z) = d(x)\alpha(y)\alpha(z) + xd(y)\alpha(z) = xd(y)\alpha(z).$$

That is $d(x)\alpha(y)\alpha(z) = 0$ for all $x, y, z \in R$. Explain to this part, we can show that $[x, d(x)]\alpha(y)[x, d(x)] = 0$, for all $x, y \in R$. Since R is semiprime, we get $[x, d(x)] = 0$, for all $x \in R$. Hence d is commuting, and so d is centralizing. $\square$

Corollary 2.5. *Let R be a prime ring and d be an $(\alpha, 1)$ derivation of R. If d acts as homomorphism on R, then R is commutative integral domain.*

Theorem 2.6. *Let R be a semiprime ring and S be a non-empty subset of R. Let d be an $(\alpha, 1)$ derivation of R such that $\alpha(x) = x$, for all $x \in S$. If d acts as an anti-homomorphism on R, then d is centralizing on S.*

Proof. Assume that d acts as an anti- homomorphism on R, Now by the hypothesis we have

$$d(xy) = d(x)\alpha(y) + xd(y) = d(y)d(x),\ for\ all\ x, y \in R.$$

Replacing y by xy,in the last relation and using d is a $(\alpha, 1)$ derivation of R, we arrive at

$$d(x)\alpha(x)\alpha(y) + xd(x)\alpha(y) + xxd(y) = d(x)\alpha(y)d(x) + xd(y)d(y), for\ all\ x, y, z \in R.$$

By the hypothesis, we get $d(x)\alpha(x)\alpha(y) + xd(x)\alpha(y) + xxd(y) = d(x)\alpha(y)d(x) + xd(x)\alpha(y) + xxd(y)$. That is

$$d(x)\alpha(x)\alpha(y) = d(x)\alpha(y)d(x),\ \ for\ all\ x, y, z \in R. \tag{6}$$

Writing yx by y in (6), we have $d(x)\alpha(x)\alpha(yx) = d(x)\alpha(yx)d(x)$. Using (6), we arrive at $d(x)\alpha(x)d(x)\alpha(x) = d(x)\alpha(y)\alpha(x)d(x)$, and so $d(x)\alpha(y)[d(x), \alpha(x)] = 0$, for all $x, y \in R$. Using the same arguments in the proof of Theorem 2.1 (i), we obtain $[d(x), \alpha(x)] = 0$. Since $\alpha(x) = x$, for all $x \in$ S, then $[d(x), x] = 0$, for all $x \in S$. Hence d is commuting on S, and d is centralizing on S. □

Corollary 2.7. *Let R be a prime ring and d be an $(\alpha, 1)$ derivation of R. If d acts as anti-homomorphism on R, then R is commutative integral domain.*

Theorem 2.8. *Let R be a semiprime ring and d be an $(\alpha, 1)$ derivation of R. If R admits an $(\alpha, 1)$ derivation such that $d(x)d(y) - xy \in Z$, for all $x, y \in R$, then d is centralizing.*

Proof. Replacing x by xz in the hypothesis, we get

$$d(x)\alpha(z)d(y) + x(d(z)d(y) - zy) \in Z,\ \ for\ all\ x, y, z \in R. \tag{7}$$

Commuting (7) with x, we have $[d(x)\alpha(x)d(y), x] = 0$, for all $x, y, z \in R$ and so $[d(x)\alpha(z), x]d(y) + d(x)\alpha(z)[d(y), x] = 0$, for all $x, y, z \in R$. Writing $\alpha(z)$ by $zd(t)$, $t \in R$ in this equation and using this equation yields that $[d(x)zd(t), x]d(y) + d(x)zd(t)[d(y), x] = 0$. That is, $d(x)zd(t)[d(y), x] = 0$, for all $t, x, y, z \in R$. Taking x instead of y in the above equation, we find that

$$d(x)zd(t)[d(x), x] = 0,\ \ for\ all\ t, x, z \in R. \tag{8}$$

Multiplying (8) on the left by x, we have

$$xd(x)zd(t)[d(x), x] = 0,\ \ for\ all\ t, x, z \in R. \tag{9}$$

Again replacing z by xz in (8), we obtain

$$d(x)xzd(t)[d(x), x] = 0,\ \ for\ all\ t, x, z \in R. \tag{10}$$

Subtracting (9) from (10), we see that $[d(x), x]zd(t)[d(x), x] = 0$, for all $t, x, z \in R$. Again multiplying this equation on the left by $d(t)$, we have $d(t)[d(x), x]zd(t)[d(x), x] = 0$, for all $t, x, z \in R$. Since R is semiprime ring, we get $d(t)[d(x), x] = 0$, for all $t, x \in R$. Substituting xt for t in the last equation and using the last equation, we obtain $d(x)\alpha(t)[d(x), x] = 0$, for all $t, x \in R$. Using the same arguments in the proof of Theorem 2.2 (i), we conclude that $[d(x), x]\alpha(t)[d(x), x] = 0$, for all $t, x \in R$. Again using the semiprimeness of R, we get $[d(x), x] = 0$, for all $x \in R$. This yields that d is commuting, and so d is centralizing. □

Corollary 2.9. *Let R be a prime ring and d be an $(\alpha, 1)$ derivation of R. If R admits an $(\alpha, 1)$ derivation such that $d(x)d(y) - xy \in Z$, for all $x, y \in R$, then d is centralizing.*

In the similar manner of Theorem 2.5, we obtain the following theorem.

Theorem 2.10. *Let R be a semiprime ring and d be an $(\alpha, 1)$ derivation of R. If R admits an $(\alpha, 1)$ derivation such that $d(x)d(y) + xy \in Z$, for all $x, y \in R$, then d is centralizing.*

Corollary 2.11. *Let R be a prime ring and d be an $(\alpha,1)$ derivation of R. If R admits a $(\alpha,1)$ derivation such that $d(x)\,d(y)+xy\in Z$, for all $x,y\in R$, then d is centralizing.*

Theorem 2.12. *Let R be a semiprime ring and d be an $(\alpha,1)$ derivation of R. If d satisfies one of the following conditions, then d is centralizing.*

(i). $d(xoy)=(xoy)_{\alpha,1}$, *for all* $x,y\in R$.

(ii). $d(xoy)=-(xoy)_{\alpha,1}$, *for all* $x,y\in R$.

Proof. (i). Assume that $d(xoy)=(xoy)_{\alpha,1}$, for all $x,y\in R$. Writing y by yx in this equation yields that

$$d(xoy)\,\alpha(x)+(xoy)\,d(x)-d(y)\,\alpha\,[x,x]-yd\,[x,x]=(xoy)_{\alpha,1}\,\alpha(x)-y\,[x,x]_{\alpha,1}\,, for\ all\ x,y\in R.$$

Using the hypothesis, we get $(xoy)\,d(x)=0$, for all $x,y\in R$. Replacing y by zy in the above equation and using this equation, we get $z(xoy)\,d(x)+[x,z]\,yd(x)=0$, for all $x,y\in R$. That is $[x,z]\,yd(x)=0$. Again replacing z by *d(x)* in the above equation, we obtain

$$[x,d(x)]\,yd(x)=0,\ for\ all\ x,y\in R. \tag{11}$$

Replacing y by yx in (11), we get

$$[x,d(x)]\,yxd(x)=0,\ for\ all\ x,y\in R. \tag{12}$$

Multiplying (11) on the right by x, we have

$$[x,d(x)]\,yd(x)\,x=0,\ for\ all\ x,y\in R. \tag{13}$$

Subtracting (13) from (12), we arrive at

$$[x,d(x)]\,y\,[x,d(x)]=0,\ for\ all\ x,y\in R. \tag{14}$$

By the semiprimeness of R, we conclude that $[x,d(x)]=0$, for all $x\in R$ and so $[x,d(x)]\in Z$.
(ii). In the similar manner, we can prove that $d(xoy)=-(xoy)_{\alpha,1}$, for all $x,y\in R$. □

Corollary 2.13. *Let R be a prime ring and d be an $(\alpha,1)$ derivation of R. If d satisfies one of the following conditions, then d is centralizing.*

(i). $d(xoy)=(xoy)_{\alpha,1}$, *for all* $x,y\in R$.

(ii). $d(xoy)=-(xoy)_{\alpha,1}$, *for all* $x,y\in R$.

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