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A Note on a Line Graph of the Zero Divisor Graph of a Commutative Ring

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Abstract: The rings considered in this article are commutative with identity $1 \neq 0$. Recall that the zero divisor graph of a ring R is a simple undirected graph whose vertex set is the set of all nonzero zero divisors of the ring R and two distinct vertices x, y are adjacent in this graph if and only if $xy = 0$. In this article we studied the line graph of the zero divisor graph of a ring and we proved some results regarding the diameter of the line graph.

MSC: 13A15, 05C25.

Keywords: Line graph, zero divisor graph, diameter of a graph.

1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$. In 1988, Beck [6] defined the concept of zero divisor graph of a commutative ring R, where the vertices of this graph are all elements in the ring and two vertices x, y are adjacent in this graph if and only if $xy = 0$. Anderson and Livingston in [3] modified the definition of zero divisor graphs by restricting the vertices to the nonzero zero divisors of the ring R. The zero divisor graph was extensively studied in [1–3, 6]. The authors K. Nazzal and M. Ghanem in [8], studied the line graph of zero divisor graph. Let G be a simple undirected finite graph. Recall from [8] that line graph of G is denoted as $L(G)$ is defined to be the graph whose vertices are the edges of G, with two vertices being adjacent if the corresponding edges share a vertex in G. This article is motivated by the interesting theorem proved on line graph of zero divisor graph of ring R in [8, 9].

It is useful to recall the following definitions from graph theory before we describe the results that are proved in this article on $L(\Gamma(R))$. Let $G = (V, E)$ be a graph. Let $a, b \in V$ with $a \neq b$. Recall that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in in G if there exists such a path in G; otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. The *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) | a, b \in V\}$ [5]. A simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent in G [5, Definition 1.1.11]. Recall from [5, Definition 1.2.2], that a *clique* of G is a complete subgraph of G. A subset S of G is said to be an *independent set* if no two members of S are adjacent in G. A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. A complete bipartite graph with vertex partition V_1 and V_2 is called a *star* if either $|V_1| = 1$ or $|V_2| = 1$ [5, Definition 1.1.12].

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Let R be any ring. We denote the set of all zero divisor of ring R by $Z(R)$. A prime ideal P is said to be a *minimal prime ideal over an ideal* I if it is minimal among all prime ideals containing I. A prime ideal is said to be a *minimal prime* ideal if it is a minimal prime ideal over the zero ideal. Recall that an element x of ring R is said to be *nilpotent* if there exist positive integer n such that $x^n = 0$. The set of all nilpotent elements of ring R is said to be *nilradical* and it is denoted by $nil(R)$.

Let R be a ring. In Section 2 of this article, some results regarding diameter of $L(\Gamma(R))$ is proved. It is proved in Theorem 2.1 that if $\Gamma(R)$ is a complete graph then $diam(L(G(R))) \in \{0,1,2\}$. It is shown by means of an examples in Remark 2.2 that $diam(L(G(R)))$ attains all the three values 0,1,2, when $\Gamma(R)$ is a complete graph. In Theorem 2.3 it is proved that When $diam(\Gamma(R) = 2$, then $1 \leq diam(L(\Gamma(R))) \leq 3$. In example 2.4, example of a ring R is given for which $diam(\Gamma(R) = 2$ and $diam(L(\Gamma(R))) = 3$ and in example 2.5, example of a ring is given for which $diam(\Gamma(R) = 2 = diam(L(\Gamma(R)))$.

2. On the diameter of $L(\Gamma(R))$

Theorem 2.1. *Let R be a commutative ring. If $diam(\Gamma(R)) = 1$, then $diam(L(\Gamma(R))) \in \{0,1,2\}$.*

Proof. As $diam(\Gamma(R)) = 1$, it follows from [2, Theorem 2.6 (T.G.Lucas)] that $xy = 0$ for each pair of distinct zero divisors x and y of R and R has atleast two zero divisors. So, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R) = P$ with $P^2 = (0)$.

Case (i): $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Then, $\Gamma(R)$ has only edge $(0,1) - (1,0)$. So, $L(\Gamma(R))$ has only one vertex. So, $diam(L(\Gamma(R))) = 0$.

Case (ii): $Z(R) = P$ with $P^2 = (0)$.

Subcase (i): If $|P| = 3$.

Then, $|Z^*(R)| = |P^*| = 2$. Let $x, y \in Z^*(R), x \neq y$. Then, $\Gamma(R)$ has only one edge $x - y$. So, $L(\Gamma(R))$ has only one vertex $[x, y]$. So, $diam(L(\Gamma(R))) = 0$.

Subcase (ii): If $|P| = 4$.

Then, $|Z^*(R)| = |P^*| = 3$. As, $diam(\Gamma(R)) = 1$, it follows $\Gamma(R)$ is a triangle. So, $L(\Gamma(R))$ is a path on two vertices. So, $diam(L(\Gamma(R))) = 1$.

Subcase (iii): If $|P| \geq 5$. Let $a, b, c, d \setminus P^*$ and $e_1 = [a \quad b]$ and $e_2 = [c \quad d]$ be any two vertices of $L(\Gamma(R))$. Also, note that e_1 and e_2 are not adjacent in $L(\Gamma(R))$. So, $diam(L(\Gamma(R))) \geq 2$. As, $diam(\Gamma(R)) = 1$, it follows that a and c are adjacent in $\Gamma(R)$. So, we have a path $e_1 = [a \quad b] - [a \quad c] - [c \quad d] = e_2$ between e_1 and e_2. So, $diam(L(\Gamma(R))) \leq 2$. Hence, $diam(L(\Gamma(R))) = 2$. □

Remark 2.2. *Here we present examples to show that in above theorem $diam(L(\Gamma(R)))$ attains all the three number $0, 1, 2$. Note that the zero divisor of $\Gamma(R)$ for $R = \mathbb{Z}_6, \frac{F_4[x]}{(x^2)}, \mathbb{Z}_{25}$ is a complete graph. So, for $R \in \{\mathbb{Z}_6, \frac{F_4[x]}{(x^2)}, \mathbb{Z}_{25}\}$, $diam(\Gamma(R)) = 1$. But $diam(L(\Gamma(\mathbb{Z}_6))) = 0$, $diam\left(L\left(\Gamma\left(\frac{F_4[x]}{(x^2)}\right)\right)\right) = 1$ and $diam\left(L\left(\Gamma\left(\mathbb{Z}_{25}\right)\right)\right) = 2$.*

Theorem 2.3. *Let R be a commutative ring. If $diam(\Gamma(R)) = 2$, then $1 \leq diam(L(\Gamma(R))) \leq 3$.*

Proof. Since, $diam(\Gamma(R)) = 2$, it follows from [3, Theorem 2.6] that either R is reduced with exactly two minimal prime ideals and atleast three nonzero zero divisors or $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator.

Case (i): R is reduced with exactly two minimal prime ideals P_1 and P_2 and at least three nonzero zero divisors.

Then, $Z(R) = P_1 \cup P_2$ and $P_1 \cap P_2 = (0)$.

Subcase (i): $|P_1| = 2$ and $|P_2| \geq 3$.

Then, $\Gamma(R)$ is a star graph $K_{1,n}$, where $|P_2| = n + 1$. So, $L(\Gamma(R))$ is a complete graph on $\frac{n(n+1)}{2}$. So, $diam(L(\Gamma(R))) = 1$.

Subcase(iii) $|P_1| \geq 3$ and $|P_2| \geq 3$. Then, $\Gamma(R)$ is a complete bipartite graph with vertex partition $Z^*(R) = V_1 \cup V_2$, where $V_1 = P_1 \setminus \{0\}$ and $V_2 = P_2 \setminus \{0\}$. Since, $|P_1| \geq 3$ and $|P_2| \geq 3$, it follows that $|V_1| \geq 2$ and $|V_2| \geq 2$. Let $x_1, x \in V_1, x_1 \neq x$ and $y_1, y \in V_2, y_1 \neq y$. Then, $e_1 = [x \quad y]$, $e_2 = [x_1 \quad y_1]$ are vertices of $L(\Gamma(R))$. Note that e_1 and e_2 are not adjacent in $L(\Gamma(R))$. So, $diam(L(\Gamma(R)))\ geq 2$. Let $e_1 = [a \quad b], e_2 = [c \quad d] \in V(L(\Gamma(R)))$ and e_1 and e_2 are not adjacent vertices of $V(L(\Gamma(R)))$. As, $a - b$ is an edge of $\Gamma(R)$, without loss of generality, we can assume that $a \in V_1 = P_1 \setminus \{0\}$ and $b \in V_2 = P_2 \setminus \{0\}$. Similarly, we can assume that $c \in V_1 = P_1 \setminus \{0\}$ and $d \in V_2 = P_2 \setminus \{0\}$. As, $a \in V_1$ and $d \in V_2$, they are adjacent in $\Gamma(R)$. So, we have a path $e_1 = [a \quad b] - [a \quad d] - [c \quad d] = e_2$. Hence, $diam(L(\Gamma(R))) \leq 2$. Therefore, $diam(L(\Gamma(R))) = 2$.

Case (ii): $Z(R)$ is an ideal whose square is not (0) and each pair of zero divisors has a nonzero annihilator.

Let $Z(R) = P, P^2 \neq (0)$. As $diam(\Gamma(R)) = 2$, we can find $x, y \in Z^*(R)$ with $xy \neq 0$ and $x \neq y$. Let $e_1 = [a \quad b]$ and $e_2 = [c \quad d]$ be any two vertices of $L(\Gamma(R))$. Assume that e_1 and e_2 are not adjacent in $L(\Gamma(R))$. If $ac = 0$, then we have a path $e_1 = [a \quad b] - [a \quad c] - [c \quad d] = e_2$ between e_1 and e_2 of length of 2. Similarly, $ac = 0$, then we have a path $e_1 = [a \quad b] - [b \quad c] - [c \quad d] = e_2$ between e_1 and e_2 of length of 2. Similarly, in the case $ad = 0$ and $bd = 0$, there is a path between e_1 and e_2 of length of 2. So, we can assume that $ac \neq 0, ad \neq 0, bc \neq 0, bd \neq 0$. Now, Since a and c are two different zero divisors, by hypothesis there exist $y \in Z^*(R)$ such that $ay = 0 = cy$. Now, $y \neq a$ as $ac \neq 0$, $y \neq b$ as $bc \neq 0$,$y \neq d$ as $ad \neq 0$,$y \neq c$ as $ac \neq 0$. Hence, $y \notin \{a, b, c, d\}$. So, we have a path $e_1 = [a \quad b] - [a \quad y] - [c \quad y] - [c \quad d] = e_2$ between e_1 and e_2 of length 3. So, $diam(L(\Gamma(R))) \leq 3$. □

In the following Example 2.4 we gave an example of a ring R for which $diam(\Gamma(R)) = 2$ and $diam(L(\Gamma(R))) = 3$.

Example 2.4. *Consider the ring* $R = \frac{\cup_{n=1}^{\infty} K[[x_1, x_2, \ldots, x_n]]}{I = \langle \{x_i x_j | i \neq j, i, j \in \mathbb{N}\} \rangle}$. *Note that* R *is a reduced Ring. Let* $M = \frac{\{x_i | i \in \mathbb{N}\}}{I}$. *Then* $M = Z(R)$ *is an ideal of* R *with* $M^2 = (0)$. *Let* $X_i = x_i + I$. *Note that* $e_1 = [x_1 + x_3 \quad x_2 + x_4]$ *and* $e_1 = [x_1 + x_2 \quad x_3 + x_4]$ *are vertices of* $L(\Gamma(R))$. *Note that* e_1 *and* e_2 *are not adjacent in* $L(\Gamma(R))$. *Now,*

$$(x_1 + x_3)(x_1 + x_2) = x_1^2 \neq 0$$

$$(x_1 + x_3)(x_3 + x_4) = x_3^2 \neq 0$$

$$(x_2 + x_4)(x_1 + x_2) = x_2^2 \neq 0$$

$$(x_2 + x_4)(x_3 + x_4) = x_4^2 \neq 0.$$

So, $diam(L(\Gamma(R))) \geq 3$. *Now, by Theorem 2.2, we have* $diam(L(\Gamma(R))) \leq 3$. *Hence,* $diam(L(\Gamma(R))) = 3$.

In the following Example 2.5 we gave an example of a ring R for which $diam(\Gamma(R)) = 2$ and $diam(L(\Gamma(R))) = 2$.

Example 2.5. *Consider the ring* $R = \frac{K[x,y]}{(x^3)}$, *where* K *is a field. Then* $M = Z(R) = \frac{(x)}{(x^3)}$ *is an maximal ideal of* R *and* $M^2 \neq (0)$. *Let* $e_1 = [a \quad b]$ *and* $e_2 = [c \quad d]$ *be any two non adjacent vertices of* $L(\Gamma(R))$. *As,* $a, b, c, d \in M$, *we have* $a = \overline{f}x$ *and* $a = \overline{g}x$. *Now, since*$ab = 0$, *we have* $fg \in M$. *Hence, either* $f \in M$ *or* $g \in M$. *Without loss of generality we can assume that* $f \in M$. *Hence,* $f = xs$ *for some* $s \in K$. *So,* $a = x^2 s$. *similarly we can assume that* $c = x^2 r$ *for some* $r \in K$. *So, we have* $ad = 0$. *Hence, we have a path* $e_1 = [a \quad b] - [a \quad d] - [c \quad d] = e_2$ *of length 2 between* e_1 *and* e_2. *So,* $diam(L(\Gamma(R))) \leq 2$. *Now, consider the elements* $e_1 = [x \quad x^2 y]$ *and* $e_2 = [x^2 \quad xy]$ *of* $L(\Gamma(R))$. *Note that* e_1 *and* e_2 *are not adjacent in* $L(\Gamma(R))$. *Therefore,* $diam(L(\Gamma(R))) \geq 2$. *So,* $diam(L(\Gamma(R))) = 2$.

Lemma 2.6. *Let* R *be a ring,* $Z(R)$ *is an ideal of* R *whose square is not* (0) *and each pair of distinct zero divisors has a nonzero annihilator. If there exist* $a, b, c, d \in Z^*(R)$ *such that* $ab = 0, cd = 0, ac \neq 0, ad \neq 0, bc \neq 0, bd \neq 0$. *Then* $d_{L(\Gamma(R))}([a \quad b], [c \quad d]) \geq 3$.

Proof. Since, $[a \quad b]$ and $[c \quad d]$ are not adjacent in $L(\Gamma(R))$. So, $d_{L(\Gamma(R))}([a \quad b],[c \quad d]) \geq 2$. Suppose that there exist a path of length 2 between $[a \quad b]$ and $[c \quad d]$ in $L(\Gamma(R))$. Let $e_1 = [a \quad b]-[x \quad y]-[c \quad d]$ is a path of length 2 between $[a \quad b]$ and $[c \quad d]$ in $L(\Gamma(R))$. Note that $\{x,y\} \cap \{a,b\}$ is a singleton set. Without loss of generality we can assume that $\{x,y\} \cap \{a,b\} = \{x\}$ with $x = a$. Then $\{a,y\} \cap \{c,d\} = \{y\}$. So, $y \in \{c,d\}$. without loss of generality we can assume that $y = c$. Then $ac = 0$. This is in contradiction to the hypothesis. So, there is no path of length 2 between $[a \quad b]$ and $[c \quad d]$ in $L(\Gamma(R))$. $d_{L(\Gamma(R))}([a \quad b],[c \quad d]) \geq 3$. □

Lemma 2.7. *Let R be a reduced ring, $Z(R) = P$ is an ideal of R whose square is not (0). Let $\{a,b,c,d\} \subseteq P^*$ and the subgraph of $\Gamma(R)$ induced on $\{a,b,c,d\}$ is a clique, then $diam(L(\Gamma(R))) = 3$.*

Proof. Note that $e_1 = [a+c \quad b+d]$ and $e_2 = [a+b \quad c+d]$ are vertices of $L(\Gamma(R))$. Also we have

$$(a+c)(a+b) = a^2 \neq 0$$
$$(a+c)(c+d) = c^2 \neq 0$$
$$(b+d)(a+b) = b^2 \neq 0$$
$$(b+d)(c+d) = d^2 \neq 0$$

If $a+c = b+d$, then $d(a+c) = d(b+d)$. Hence, $d^2 = 0$. This is not possible as R is reduced. so, $a+c \neq b+d$. Similarly, $a+c \neq c+d$, $b+d \neq c+d$, $a+b \neq c+d$. So, from Lemma 2.6, we obtain that $d_{L(\Gamma(R))}([a \quad b],[c \quad d]) \geq 3$. Therefore, $diam(L(\Gamma(R))) \geq 3$. As, $diam(\Gamma(R)) = 2$, we have $diam(L(\Gamma(R))) \leq 3$. Hence, $diam(L(\Gamma(R))) = 3$. □

Corollary 2.8. *Let R be a reduced ring, $Z(R) = P$ is an ideal of R whose square is not (0) and each pair of distinct zero divisors has a non zero annhilator. If $\omega(\Gamma(R)) \geq 4$, then $diam(L(\Gamma(R))) = 3$.*

Lemma 2.9. *Let R be a reduced ring with exactly three minimal prime ideals then $diam(L(\Gamma(R))) = 2$.*

Proof. Let P_1, P_2, P_3 are three minimal prime ideals of R. Let $e_1 = [a \quad b]$ and $e_1 = [a \quad b]$ be any two non adjacent vertices of $L(\Gamma(R))$. If $ac = 0$, then we have a path $e_1 = [a \quad b] - [a \quad c] - [c \quad d]$ of length 2 between e_1 and e_2 in $L(\Gamma(R))$. Similarly, if $ad = 0, bc = 0$ or $bd = 0$ then we have a path of length 2 between e_1 and e_2 in $L(\Gamma(R))$. So, we assume that $ac \neq 0$. Without loss of generality we can assume that $ac \notin P_1$. Hence, $a \notin P_1$ and $c \notin P_1$. Now from $ab = 0 \in P_1$ and $a \notin P_1$, we have $b \in P_1$. Similarly from $cd = 0$ and $c \notin P_1$, we have $d \in P_1$. Now, $bd \neq 0$. Hence, $bd \notin P_2$. So, $b \notin P_2$ and $d \notin P_2$. As, $ab = 0$ and $cd = 0$, we have $a \in P_2$ and $c \in P_2$. Now from $ad \neq 0$, we obtain that $ad \notin P_3$. Hence, $d \notin P_3$ and $a \notin P_3$. Therefore, $c \in P_3$ and $b \in P_3$. So, $c \in P_2 \cap P_3$ and $b \in P_1 \cap P_3$. Hence, $bc \in P_1 \cap P_2 \cap P_3 = (0)$. Hence, $bc = 0$. So, we have a path $e_1 = [a \quad b] - [b \quad c] - [c \quad d]$ of length 2 between e_1 and e_2 in $L(\Gamma(R))$. So, $diam(L(\Gamma(R))) = 2$. □

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