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Some Results on Intersection Graphs of Ideals of Commutative Rings

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Abstract: The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. Recall that the intersection graph of ideals of R, denoted by $G(R)$, is an undirected simple graph whose vertex set is the set of all nontrivial ideals of R (an ideal I of R is said to be nontrivial if $I \notin \{(0), R\}$) and distinct vertices I, J are joined by an edge in $G(R)$ if and only if $I \cap J \neq (0)$. Let $r \in \mathbb{N}$. The aim of this article is to characterize rings R such that $G(R)$ is either bipartite or 3-partite.

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1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$. The concept of associating a graph with a ring R and investigating the interplay between the ring theoretic properties of R and the graph theoretic properties of a graph associated with it was initiated by I. Beck in [7]. Subsequently, a lot of research activity has been carried out by several researchers in this area (see, for example [1–4, 8, 9, 12, 15]). The study of intersection graph of ideals of a ring has begun with the work of Chakrabarthy, Ghosh, Mukherjee and Sen [9]. Let R be a ring with identity which is not necessarily commutative and which admits at least one nonzero proper left ideal. Recall from [9] that the intersection graph of ideals of R, denoted by $G(R)$, is an undirected simple graph whose vertex set is the set of all nonzero proper left ideals of R, and two distinct vertices I,J are joined by an edge in this graph if and only if $I \cap J \neq (0)$. The intersection graph of ideals of a ring R was studied by several other researchers (see, for example [2, 9, 12, 14, 15]). The concept of the zero-divisor graph of a commutative ring was introduced and investigated by D.F. Anderson and P.S. Livingston in [4] and subsequently, several mathematicians worked and published research articles in the area of zero-divisor graphs of rings (see for example, [1, 4, 8, 10, 13]). The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $r \geq 2$. Recall from [11] that G is said to be r-partite if the vertex set V can be decomposed into r disjoint nonempty subsets $V_1, V_2, \ldots, V_r$ such that no edge in G joins the vertices in the same subset. A r-partite graph G with vertex partition $\{V_1, V_2, \ldots, V_r\}$ is said to be complete r-partite if for any $i \in \{1, 2, \ldots, r\}$, each $x \in V_i$ is adjacent in G to all the vertices in V_j for each $j \in \{1, 2, \ldots, r\} \backslash \{i\}$. A 2-partite graph (respectively, a complete 2-partite graph) is referred to as a bipartite graph (respectively, a complete bipartite graph). The authors of [1, 8] proved several interesting theorems on bipartite (respectively, complete r-partite)

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zero-divisor graphs of rings. Let $n \in \mathbb{N}$ with $n > 1$. We denote the ring of integers modulo n by $\mathbb{Z}_n$. Let $n \in \mathbb{N}$ be composite. Chakraborthy, Ghosh, Mukherjee, and Sen showed in [9, Theorem 3.3] that the intersection graph of ideals of $\mathbb{Z}_n$ is bipartite if and only if $n = pq$ or $n = p^3$, where p and q are distinct primes.

Let R be a ring which admits at least one nontrivial ideal. Motivated by the work published in the articles [1, 8, 9], in this article, we try to classify the rings R such that $G(R)$, the intersection graph of ideals of R is either bipartite or 3-partite. The main results obtained are presented in Section 2 of this article. Let $r \geq 2$. It is shown in Lemma 2.1 that if $G(R)$ is r-partite, then R can have at most r maximal ideals. With the assumption that R has exactly r maximal ideals, in Lemma 2.2, we classify the rings R such that $G(R)$ is r-partite. In particular, for a ring R with exactly two maximal ideals, we deduce from Lemma 2.2 that $G(R)$ is bipartite if and only if $R \cong K_1 \times K_2$ as rings, where K_1 and K_2 are fields. A ring R which admits a unique maximal ideal is referred to as a quasilocal ring. A Noetherian quasilocal ring is referred to as a local ring. If M is the unique maximal ideal of a quasilocal ring R, then we denote it using the notation that (R, M) is a quasilocal ring. Recall that a principal ideal ring R is said to be a special principal ideal ring (SPIR), if R has a unique prime ideal. If M is the only prime ideal of a SPIR R, then M is necessarily nilpotent. If $n \geq 2$ is least with the property that $M^n = (0)$, then it follows from $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that $\{M^i : i \in \{1, \ldots, n-1\}\}$ is the set of all nontrivial ideals of R. If R is a special principal ideal ring with M as its only prime ideal, then we denote it by saying that (R, M) is a SPIR. Let $r \geq 2$. In Lemmas 2.3 to 2.5, we derive some necessary conditions in order that a quasilocal ring (R, M) to be r-partite. With the assumption that $M^2 = (0)$, in Lemma 2.6, we are able to describe all the ideals of a quasilocal ring (R, M) in order that $G(R)$ to be bipartite. In Theorem 2.7, we classify quasilocal rings (R, M) such that $G(R)$ is bipartite. In Lemma 2.9, for a quasilocal ring (R, M), it is shown that $G(R)$ is 3-partite but not bipartite if and only if (R, M) is a SPIR with $M^3 \neq (0)$ but $M^4 = (0)$. For a ring R with exactly two maximal ideals, it is proved in Lemma 2.8 that $G(R)$ is 3-partite if and only if $R \cong K \times S$ as rings, where K is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$. Let R be a ring. We denote the set of all units of R using the notation $U(R)$. Let A be a set. We use $|A|$ to denote the cardinality of A. We use $\subset$ to denote proper inclusion.

2. Main Results

Let R be a ring with atleast one nontrivial ideal. The aim of this section is to classify rings R such that $G(R)$ is either 2-partite or 3-partite.

Lemma 2.1. *Let R be a ring and let $r \geq 2$. If $G(R)$ is r-partite, then R has at most r maximal ideals.*

Proof. Suppose that R has more than r maximal ideals. Let $\{M_1, M_2 \ldots, M_{r+1}\}$ be a set consisting of $(r+1)$ distinct maximal ideals of R. Let $G(R)$ be r-partite with vertex partition $\{V_1, V_2 \ldots, V_r\}$. Observe that $M_i \cap M_j \neq (0)$ for any distinct $i, j \in \{1, 2, 3, \ldots, r+1\}$. Hence, M_i, M_j cannot be in the same V_k for any $k \in \{1, 2, 3, \ldots, r\}$ and for any distinct $i, j \in \{1, 2, 3, \ldots r+1\}$. We can assume without loss of generality that $M_i \in V_i$ for each $i \in \{1, 2, \ldots, r\}$. Note that, $\bigcap_{i=1}^{r} M_i \neq (0)$ and $\bigcap_{i=1}^{r} M_i \notin V_k$, for any $k \in \{1, 2, 3, \ldots, r\}$. This is a contradiction. Hence, the number of maximal ideals of R is at most r. $\square$

Lemma 2.2. *Let R be a ring such that R has exactly r maximal ideals $(r \geq 2)$. Then the following statements are equivalent:*

(1). $G(R)$ is r-partite.

(2). $r \leq 3$ and $R \cong K_1 \times K_2 \times, \ldots, \times K_r$, where K_i is a field for each $i \in \{1, 2, \ldots, r\}$.

Proof. (1) $\Rightarrow$ (2) Assume that $G(R)$ is r- partite with vertex partition $\{V_1, V_2, ..., V_r\}$. We are assuming that R has exactly r maximal ideals. Let $\{M_1, M_2..., M_r\}$ be the set of all maximal ideals of R. We claim that $M_1 \cap M_2 \cap ... \cap M_r = (0)$. Suppose that $M_1 \cap M_2 \cap ... \cap M_r \neq (0)$. Then $M_i \cap M_j \neq (0)$, for all distinct $i, j \in \{1, 2, 3, ..., r\}$. Hence, for any distinct $i, j \in \{1, 2, 3, ..., r\}$ M_i and M_j cannot be in the same V_k, for any $k \in \{1, 2, 3, ..., r\}$. Without loss of generality, we can assume that $M_i \in V_i$, for each $i \in \{1, 2, ..., r\}$. Then the nontrivial ideal $M_1 \cap M_2 \cap ... \cap M_r \notin V_k$ for any $k \in \{1, 2, ..., r\}$. This is a contradiction. Therefore, $M_1 \cap M_2 \cap ... \cap M_r = (0)$. Since, $M_i + M_j = R$ for any distinct $i, j \in \{1, 2, 3, ..., r\}$, it follows from the Chinese remainder theorem,[5,Proposition 1.10(ii)and(iii)] that $R \cong \frac{R}{M_1} \times \frac{R}{M_2} \times, ..., \times \frac{R}{M_r}$ as rings. Let $\frac{R}{M_i} = K_i$, for each $i \in \{1, 2, ..., r\}$. Then K_i is a field, for each $i \in \{1, 2, ..., r\}$ and $R \cong K_1 \times K_2 \times, ..., \times K_r$ as rings.
We next verify that $r \leq 3$. Suppose that $r \geq 4$. Then $M_i \cap M_j \neq (0)$, for any distinct $i, j \in \{1, 2, 3, ..., r\}$. Note that no V_k ($k \in \{1, 2, ..., r\}$) can contain both M_i and M_j. We can assume without loss of generality that $M_i \in V_i$ for each $i \in \{1, 2, ..., r\}$. Note that $M_1 \cap M_2 \neq (0)$. Observe that $M_1 \cap M_2 \notin V_i$ for any $i \in \{1, 2, ..., r\}$. For if $M_1 \cap M_2 \in V_i$ for some $i \in \{1, 2, ..., r\}$, then $i \geq 3$ and $M_1 \cap M_2, M_i \in V_i$. As $r \geq 4$, $M_1 \cap M_2 \cap M_i \neq (0)$. Hence there is an edge of $G(R)$ joining $M_1 \cap M_2$ and M_i. This is impossible. Therefore, $r \leq 3$.
(2) $\Rightarrow$ (1) Now, $R \cong K_1 \times K_2 \times, ..., \times K_r$ as rings and K_i is a field for each $i \in \{1, 2, ..., r\}$ with $2 \leq r \leq 3$. Then the graph $G(R)$ is isomorphic to $G(K_1 \times K_2 \times, ..., \times K_r)$. Note that,$G(K_1 \times K_2)$ is bipartite with vertex partition $\{V_1, V_2\}$ with $V_1 = \{(0) \times K_2\}$ and $V_2 = \{K_1 \times (0)\}$. Observe that $G(K_1 \times K_2 \times K_3)$ is 3- partite with vertex partition $\{V_1, V_2, V_3\}$ with $V_1 = \{K_1 \times (0) \times (0),\ (0) \times K_2 \times K_3\}, V_2 = \{(0) \times K_2 \times (0),\ K_1 \times (0) \times K_3\}$, and $V_3 = \{(0) \times (0) \times K_3,\ K_1 \times K_2 \times (0)\}$. Therefore, we obtain that $G(R)$ is 3- partite. □

It follows from Lemma 2.2 that for a ring R with exatly two maximal ideals, $G(R)$ is bipartite if and only if $R \cong K_1 \times K_2$ as rings, where K_1 and K_2 are fields.

Lemma 2.3. *Let (R, M) be a quasilocal ring. Let $r \geq 2$. If $G(R)$ is r-partite, then $M^{r+1} = (0)$.*

Proof. Let $G(R)$ be r-partite with vertex partition $\{V_1, V_2, ..., V_r\}$. We claim that $M^{r+1} = (0)$. Suppose that $M^{r+1} \neq (0)$. Then there exist $x_1, x_2, ..., x_{r+1} \in M$ such that $x_1, x_2, ..., x_{r+1} \neq 0$. Observe that if $a, b \in M$ with $a \neq 0$, then $Ra \neq Rab$. For if $Ra = Rab$, then $a = rab$ for some $r \in R$. This implies that $a(1 - rb) = 0$. As $1 - rb \in U(R)$, it follows that $a = 0$. This is a contradiction. Therefore, $Ra \neq Rab$. Hence, we obtain that for each $i \in \{1, 2, ..., r+1\}$. Ra_i is a nontrivial ideal of R, where $a_i = \prod_{k=1}^{i} x_k$. Moreover, note that for all $i, j \in \{1, 2, ..., r+1\}$ with $i < j$, $Ra_j \subset Ra_i$. Thus for distinct $i, j \in \{1, 2, ..., r+1\}$, Ra_i and Ra_j cannot be in the same V_k for any $k \in \{1, 2, ..., r\}$. As for each $i \in \{1, 2, ..., r+1\}$, $Ra_i \in V_k$ for some $k \in \{1, 2, ..., r\}$, it follows from the Pigeon-hole principle that there exists some $i \in \{1, 2, ..., r+1\}$ that $Ra_i \notin V_k$ for any $k \in \{1, 2, ..., r\}$. This is a contradiction. Therefore, $M^{r+1} = (0)$. □

Lemma 2.4. *Let (R, M) be a quasilocal ring with $M \neq (0)$. Let $r \geq 2$ and let $G(R)$ be r-partite. If $I_1 \subset I_2 \subset, ..., \subset I_k = M$ is a chain of nontrivial ideals of R then $k \leq r$. In particular, if $G(R)$ is bipartite, then any nontrivial ideal I of R with $I \neq M$ is minimal.*

Proof. Let $G(R)$ be r-partite with vertex partition $\{V_1, V_2 ..., V_r\}$. Suppose that $k \geq r+1$. Then $I_1 \subset I_2 \subset, ..., \subset I_{r+1} = M$ is a chain of $(r+1)$ nontrivial ideals of R. Observe that, for any distinct $i, j \in \{1, 2, ..., r+1\}$, I_i and I_j cannot be in the same V_k for any $k \in \{1, 2, ..., r\}$. Without loss of generality, we can assume that $I_i \in V_i$ for each $i \in \{1, 2, ..., r\}$. Now, $I_{r+1} \notin V_k$ for any $k \in \{1, 2, ..., r\}$. This is a contradiction. Therefore, $k \leq r$. Assume that $r = 2$. Let I be any nontrivial ideal of R with $I \neq M$. Since, $G(R)$ is bipartite, it follows from the previous paragraph that there is no nontrivial ideal J of R with $J \subset I$. Hence, I is a minimal ideal of R. □

Lemma 2.5. *Let (R, M) be a quasilocal ring. Let $r \geq 2$. Suppose that $G(R)$ is r-partite. If $M^r \neq (0)$, then M is principal.*

Proof. Let $G(R)$ be r-partite with vertex partition $\{V_1, V_2, ..., V_r\}$. As $G(R)$ is r-partite, we know from Lemma 2.3 that, $M^{r+1} = (0)$. By hypothesis, $M^r \neq (0)$. Hence, it follows that $M^i \neq M^j$ for all distinct $i, j \in \{1, 2, ..., r\}$. Observe that given any nontrivial ideal I of R, then I must be in V_k for some $k \in \{1, 2, ..., r\}$. Note that for distinct $i, j \in \{1, 2, ..., r\}$, M^i and M^j cannot be in the same V_k for any $k \in \{1, 2, ..., r\}$. Without loss of generality, we can assume that $M^i \in V_i$ for each $i \in \{1, 2, ..., r\}$. Let $x \in M \backslash M^2$. As $M^i \subseteq M^2 + Rx$ for each $i \in \{2, ..., r\}$, it follows that $M^2 + Rx \notin V_k$ for each $k \in \{2, ..., r\}$. Hence, $M^2 + Rx \in V_1$ and so, $M^2 + Rx = M$. Thus, $M = M^2 + Rx = \left(M^2 + Rx\right)^2 + Rx = M^4 + Rx = \left(M^2 + Rx\right)^4 + Rx = M^8 + Rx$. Continuing in this way, we get that $M = M^{2^k} + Rx$ for each $k \geq 1$. As $M^{r+1} = (0)$, it follows that $M^{2^{r+1}} = (0)$ and so , $M = Rx$. This proves that M is principal. □

Lemma 2.6. *Let (R, M) be a quasilocal ring with $M \neq (0)$. If $G(R)$ is bipartite and if $M^2 = (0)$, then M is not principal but two generated and moreover, $G(R)$ is bipartite with vertex partition $\{V_1, V_2\}$ with $V_1 = \{M = Rx + Ry\}$ and $V_2 = \{Rx, Ry, R(x - u_\alpha y) : \alpha \in \Lambda\}$, where $\{u_\alpha\}_{\alpha\in\Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$.*

Proof. Let $G(R)$ be bipartite with vertex partition $\{V_1, V_2\}$. It is given that $M^2 = (0)$. We claim that M is generated by at most two elements. Suppose not. Then there exist $\{x, y, z\} \subseteq M \setminus \{0\}$ such that $\{x, y, z\}$ is linearly independent over $\frac{R}{M}$. Note that $M, Rx, Rx + Ry$ are distinct nontrivial ideals of R which are pairwise comparable under inclusion. Hence, $Rx - M - Rx + Ry - Rx$ is a cycle of length 3 in $G(R)$. This is impossible, since $G(R)$ is bipartite. Thus, M can be generated by at most two elements. We assert that M is not principal. If M is principal, then it follows from $M^2 = (0)$ that M is the only vertex of $G(R)$. This is impossible, since, |the vertex set of G(R)| $= |V_1 \cup V_2| \geq 2$. Hence, M is generated by exactly two elements. Thus, there exist $x, y \in M$ such that $M = Rx + Ry$. Let A be any nontrivial ideal of R with $A \neq M$. We know from Lemma 2.4, that A is minimal. Let $a \in A$ be such that $A = Ra$. We claim that either $\{a, x\}$ is linearly independent over $\frac{R}{M}$ or $\{a, y\}$ is linearly independent over $\frac{R}{M}$. Suppose that a, x are linearly dependent over $\frac{R}{M}$. Now, there exist λ and μ in R with at least one of which belongs to $R\backslash M$ such that

$$\lambda a + \mu x = 0 \tag{1}$$

We assert that both λ and μ are units in R. Suppose that $\lambda \in U(R)$ and $\mu \in M$. Then $\mu x = 0$. Hence, we obtain from (1) that $\lambda a = 0$. which implies that $a = 0$. This is a contradiction. Therefore, $\mu \in U(R)$. Similarly, if $\mu \in U(R)$, then $\lambda \in U(R)$. Thus both λ and μ are units in R. Now, from (1), $a = -\lambda^{-1}\mu x$. This implies that $Ra = Rx$. Thus, $A = Rx$. Similarly, if a, y are linearly dependent over $\frac{R}{M}$, then we get that $A = Ry$. Hence,$A = Rx = Ry$. This is impossible. Therefore, either $\{a, x\}$ is linearly independent over $\frac{R}{M}$ or $\{a, y\}$ is linearly independent over $\frac{R}{M}$. Suppose that $\{a, x\}$ is linearly independent over $\frac{R}{M}$. As $\dim_{\frac{R}{M}} M = 2$, it follows that $M = Ra + Rx$. Now, $y \in M$ and so, $y = r_1a + r_2x$ for some $r_1, r_2 \in R$.

We claim that $r_1 \in U(R)$. If $r_1 \notin U(R)$, then from $M^2 = (0)$, we obtain that $r_1a = 0$ and so, $y = r_2x$. This is impossible. since $\{x, y\}$ is linearly independent over $\frac{R}{M}$. Therefore, r_1 is a unit in R. Hence, $r_1a = y - r_2x$ and so, $Ra = R(y - r_2x)$. Similarly, if $\{a, y\}$ is linearly independent over $\frac{R}{M}$, then we obtain that $A = Ra = R(x - sy)$ for some $s \in R$. It is clear that for any unit u of R, $Rx, Ry, R(x - uy)$ are distinct. Let $u_1, u_2 \in U(R)$ be such that $u_1 - u_2 \in M$. Then $x - u_1y = x - (u_1 - u_2 + u_2)y = x - u_2y$, since $(u_1 - u_2)y \in M^2 = (0)$. Hence, $R(x - u_1y) = R(x - u_2y)$.

Conversely, if $v_1, v_2 \in U(R)$ be such that $R(x - v_1y) = R(x - v_2y)$. Now, $(x - v_2y) - (x - v_1y) \in R(x - v_1y)$ and so, $(v_1 - v_2)y \in R(x - v_1y)$. As $y \notin R(x - v_1y)$, it follows that $v_1 - v_2 \in M$. This shows that for units u_1, u_2 of R, $R(x - u_1y) = R(x - u_2y)$ if and only if $u_1 - u_2 \in M$. Let $\{u_\alpha\}_{\alpha\in\Lambda} \subseteq U(R)$ be such that $u_\alpha + M \neq u_\beta + M$ for all distinct $\alpha, \beta \in \Lambda$. From the above discussion, we obtain that the set of all nontrivial ideals of R equals $\{Rx, Ry, R(x - u_\alpha y), M = Rx + Ry : \alpha \in \Lambda\}$. Note that $Rx, Ry, R(x - u_\alpha y)$ are distinct minimal ideals of R, where $\{u_\alpha\}_{\alpha\in\Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$. Hence, we obtain that $G(R)$ is bipartite with vertex partition $V_1 = \{M = Rx+Ry\}$ and $V_2 = \{Rx, Ry, R(x - u_\alpha y) : \alpha \in \Lambda\}$. □

Theorem 2.7. *Let (R, M) be quasilocal. Then the following statements are equivalent:*

(1). $G(R)$ is bipartite.

(2). $M^3 = (0)$ and if $M^2 \neq (0)$ then M must be principal and so (R, M) is a SPIR. If $M^2 = (0)$, then M is not principal but there exist $x, y \in M$ such that $M = Rx+Ry$ and the set of all nontrivial ideals of $R = \{M, Rx, Ry, R(x-u_\alpha y) : \alpha \in \Lambda\}$, where, $\{u_\alpha\}_{\alpha\in\Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$.

Proof. (1) ⇒ (2) It follows from Lemma 2.3 that $M^3 = (0)$. If $M^2 \neq (0)$, then it follows from Lemma 2.4 that M is principal. Now, it follows from the proof of $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that $\{M, M^2\}$ are the only nontrivial ideals of R. Hence, (R, M) is SPIR. If $M^2 = (0)$, then it follows from Lemma 2.6 that M is not principal but there exist $x, y \in M$ such that $M = Rx + Ry$ and moreover, the set of all nontrivial ideals of $R = \{M, Rx, Ry, R(x - u_\alpha y) : \alpha \in \Lambda\}$, where, $\{u_\alpha\}_{\alpha\in\Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$.
(2) ⇒ (1)
Case 1: $M^3 = (0)$ but $M^2 \neq (0)$. In this case, (R, M) is a SPIR with the set of nontrivial ideals of R equals $\{M, M^2\}$. It is then clear that $G(R)$ is bipartite with vertex partition $V_1 = \{M\}$ and $V_2 = \{M^2\}$.
Case 2: $M^2 = (0)$. In this case, M is not principal but M is two generated and the set of all nontrivial ideals of $R = \{M = Rx + Ry, Rx, Ry, R(x - u_\alpha y) : \alpha \in \Lambda\}$, where, $\{u_\alpha\}_{\alpha\in\Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$. It is already verified in the proof of Lemma 2.6 that $G(R)$ is bipartite. □

Lemma 2.8. *Let R be a ring with exactly two maximal ideals. Then the following statements are equivalent:*

(1). $G(R)$ is 3- partite.

(2). $R \cong K \times S$ as rings, where K is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$.

Proof. (1) ⇒ (2): Let $\{M_1, M_2\}$ denote the set of all maximal ideals of R and let $G(R)$ be 3- partite with vertex partition $\{V_1, V_2, V_3\}$. We claim that $M_1 \cap M_2 \neq (0)$. For if $M_1 \cap M_2 = (0)$, then $R \cong \frac{R_1}{M_1} \times \frac{R_2}{M_2}$ as rings and in such a case, R has exactly two nontrivial ideals. However, $G(R)$ is 3- partite implies that R has at least three nontrivial ideals. Therefore, $M_1 \cap M_2 \neq (0)$. As $M_1 \cap M_2 \neq (0)$, M_1, M_2 cannot be in the same V_k, for any $k \in \{1, 2, 3\}$. Without loss of generality we can assume that $M_1 \in V_1$ and $M_2 \in V_2$. Then $M_1 \cap M_2 \in V_3$. Let $x \in M_1 \cap M_2, x \neq (0)$. As $Rx \notin V_1 \cup V_2$, it follows that $Rx \in V_3$ and hence, $Rx = M_1 \cap M_2$. As $M_1 + M_2 = R$, it follows that $M_1 \cap M_2 = M_1M_2$. Thus, $Rx = M_1M_2$. We assert that $x^2 = (0)$. If $x^2 \neq (0)$, then $Rx^2 \in V_3$ and so, $Rx^2 = Rx$. This implies that $x = rx^2$ for some $r \in R$. Hence,

$$x(1 - rx) = 0 \tag{2}$$

As $x \in M_1 \cap M_2 =$ Jacobson radical of R, $1-rx$ is a unit in R. Therefore, from (2), we obtain that $x = (0)$. This is impossible. Therefore, $Rx^2 = (0)$. Thus, ${M_1}^2{M_2}^2 = (0)$. From ${M_1}^2{M_2}^2 = (0)$ but $M_1M_2 \neq (0)$, it follows that either $M_1 \neq {M_1}^2$ or $M_2 \neq {M_2}^2$. Without loss of generality, we can assume that $M_2 \neq {M_2}^2$. We assert that $M_1{M_2}^2 = (0)$. Suppose that $M_1{M_2}^2 \neq (0)$. As $M_1 \in V_1, M_1{M_2}^2 \notin V_1$. Since,$M_2 \in V_2, M_1{M_2}^2 \notin V_2$. Hence, $M_1{M_2}^2 \in V_3$ and so $M_1{M_2}^2 = M_1M_2$. Now, $M_2 \neq {M_2}^2$ and so ${M_2}^2 \notin V_2$. As $M_1 \in V_1$, $M_1{M_2}^2 \neq (0)$, ${M_2}^2 \notin V_1$. As $M_1{M_2}^2 \cap {M_2}^2 = M_1{M_2}^2 \neq (0)$, ${M_2}^2 \notin V_3$. This is a contradiction. Therefore, $M_1{M_2}^2 = (0)$. Note that the mapping $f : R \to \frac{R}{M_1} \times \frac{R}{{M_2}^2}$ defined by $f(r) = (r + M_1, r + {M_2}^2)$ is an isomorphism of rings by [5, Proposition 1.10 (ii) and (iii)].

We claim that there exist $a \in M_1 \backslash M_2$ and $b \in M_2 \backslash M_1$ such that $ab \neq 0$. Since $M_1 + M_2 = R$, there exist $x \in M_1$ and $y \in M_2$ such that $x + y = 1$. Clearly, $x \notin M_2$ and $y \notin M_1$. Let $w \in M_1 \cap M_2, w \neq 0$. Then $w = xw + yw$. Either $xw \neq 0$ or $yw \neq 0$. Without loss of generality, we can assume that $xw \neq 0$. If $xy \neq 0$, then with $a = x$ and $b = y$, we get that $ab \neq 0$. Suppose that $xy = 0$. Then with $a = x$ and $b = y + w$, we obtain that $a \in M_1 \backslash M_2$ and $b \in M_2 \backslash M_1$ and $ab = xw \neq 0$. Now, $ab \in M_1 \cap M_2$ and as $M_1 \cap M_2$ is a minimal ideal of R, it follows that $M_1 \cap M_2 = Rab$. Note that, $Ra \cap M_2 \neq (0)$, $Rb \cap M_1 \neq (0)$. Moreover, $Ra \neq Rab$ and $Rb \neq Rab$. Now, $M_1 \cap M_2 = Rab \in V_3$. Hence, $Ra \notin V_3$ and $Rb \notin V_3$. As $Ra \neq M_2$, $Ra \notin V_2$. Therefore, $Ra \in V_1$ Hence, $M_1 = Ra$. Similarly, $Rb \neq M_1$ and $Rb \cap M_1 \neq (0)$. Therefore, $Rb \notin V_1$ and so, $Rb \in V_2$. Thus, $M_2, Rb \in V_2$. This implies that $M_2 = Rb$. Hence, $\frac{R}{{M_2}^2}$ is a quasilocal ring with $M = \frac{M_2}{{M_2}^2}$ as its unique maximal ideal with $M \neq (0 + {M_2}^2)$ but $M^2 = (0 + {M_2}^2)$. Moreover, as M is principal, it follows from $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that $\left(\frac{R}{{M_2}^2}, \frac{M_2}{{M_2}^2}\right)$ is a SPIR. We have already verified that $R \cong \frac{R}{M_1} \times \frac{R}{{M_2}^2}$ as rings. Let $K = \frac{R}{M_1}$ and $S = \frac{R}{{M_2}^2}$. Note that K is a field and $\left(S,\, M = \frac{M_2}{{M_2}^2}\right)$ is SPIR with $M \neq \left(0 + {M_2}^2\right)$ but $M^2 = \left(0 + {M_2}^2\right)$. This proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Assume that $R \cong K \times S$ as rings, where K is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = 0$. Let $T = K \times S$. Note that $G(T)$ is a graph on the vertex set $\{(0) \times S, (0) \times M, K \times (0), K \times M\}$. Let $W_1 = \{(0) \times S, K \times (0)\}$, $W_2 = \{K \times M\}$, $W_3 = \{(0) \times M\}$. Then it is clear that $G(T)$ is 3-partite with vertex partition $\{W_1, W_2, W_3\}$. □

Lemma 2.9. *Let (R, M) be a quasilocal ring. Then the following statements are equivalent:*

(1). $G(R)$ is 3- partite but not 2-partite.

(2). (R, M) is a SPIR with $M^3 \neq (0)$ but $M^4 = (0)$.

Proof. (1) $\Rightarrow$ (2): Assume that $G(R)$ is 3-partite but not 2-partite. We know from Lemma 2.3 that $M^4 = (0)$. If $M^3 \neq (0)$, then we know from Lemma 2.5 and $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that (R, M) is a SPIR. Suppose that $M^3 = 0$. We claim that M can be generated by at most two elements. Otherwise, we can find $\{x, y, z\} \subseteq M$ such that $\left\{x + M^2, y + M^2, z + M^2\right\}$ is linearly independent over $\frac{R}{M}$. The ideals $Rx, Rx + Ry, Rx + Ry + Rz$ are distinct nontrivial ideals of R. Let $G(R)$ be 3-partite with vertex partition $\{V_1, V_2, V_3\}$. Let $I_1 = Rx + Ry + Rz$, $I_2 = Rx + Ry, I_3 = Rx$. Observe that $I_i \cap I_j \neq (0)$, for all distinct $i, j \in \{1, 2, 3\}$. Hence, no two distinct I_i, I_j $(i, j \in \{1, 2, 3\})$ can belong to the same V_k for any $k \in \{1, 2, 3\}$.

Without loss of generality, we can assume that $Rx + Ry + Rz \in V_1$, $Rx + Ry \in V_2$ and $Rx \in V_3$. Observe that $Rx + Rz \notin \{I_1, I_2, I_3\}$. It is clear that $Rx + Rz \notin V_1 \cup V_2 \cup V_3$. This is a contradiction. Hence, M can be generated by at most two elements. We are assuming that $M^3 = (0)$. Then either $M^2 = (0)$ or $M^2 \neq (0)$. If M is principal, then M is the only nontrivial ideal of R in the case $M^2 = (0)$ and $\{M, M^2\}$ is the set of all nontrivial ideals of R in the case $M^2 \neq (0)$. However, as $G(R)$ is 3-partite but not 2-partite , R has at least three nontrivial ideals. Therefore, M cannot be principal. Thus, M is two generated but not principal. Let $\{a, b\} \subseteq M$ be such that $M = Ra + Rb$. If $M^2 = (0)$, then we know that the set of nontrivial ideals of R equals $\{M = Ra + Rb, Ra, Rb, R(a - u_\alpha) : \alpha \in \Lambda\}$, where, $\{u_\alpha\}_{\alpha \in \Lambda}$ is the set of distinct representatives of nonzero elements of $\frac{R}{M}$ and in this case, $G(R)$ is a 2-partite with vertex partition $W_1 = \{M\}$ and $W_2 = \{Ra, Rb, R(a - u_\alpha) : \alpha \in \Lambda\}$. As we are assuming that $G(R)$ is not 2-partite, we obtain that $M^2 \neq (0)$. We claim that $M^2 = Rx$ for any $x \in M^2, x \neq 0$. As $M^2 \neq (0)$ there exist $x_1, x_2 \in M$ such that $x_1 x_2 \neq 0$. Observe that the ideals $J_1 = Rx_1$, $J_2 = Rx_2$ and $J_3 = Rx_1 x_2$ are nontrivial ideals of R. As $x_1 \in M$ $x_1 \neq 0$, it follows that $Rx_1 \neq Rx_1 x_2$. As $Rx_1 \cap Rx_1 x_2 = Rx_1 x_2 \neq 0$, Rx_1 and $Rx_1 x_2$ cannot be in the same V_k for any $k \in \{1, 2, 3\}$. Without loss of generality we can assume that $M \in V_1, Rx_1 \in V_2, Rx_1 x_2 \in V_3$. As $M^2 \notin V_1 \cup V_2$, we must have $M^2 \in V_3$. Hence, $M^2 = Rx_1 x_2$. Let $x \in M^2$, $x \neq 0$. As $M^2 = Rx_1 x_2$, $Rx \notin V_1 \cup V_2$. Therefore, $Rx \in V_3$. Hence, $M^2 = Rx$. We next assert that $z^2 = (0)$ for any $z \in M$. Suppose that $z^2 \neq 0$ for some $z \in M$. Consider the mapping $f : M \to M^2$ given by $f(m) = zm$. It is clear that f is R- linear. As $M^2 = Rz^2$, it follows that f is onto. We claim that $kerf = M^2$. If $m \in M^2$, then $zm \in M^3 = (0)$. Hence, $M^2 \subseteq \ker f$. Observe that as $z^2 \neq 0$, $z \in M \setminus \ker f$. Hence, $M^2 \subseteq \ker f \subset M$. Now, $M \in V_1$ and $M^2 \in V_3$. Observe that $Rz \notin V_1 \cup V_3$. Hence, $Rz \in V_2$. Observe that $Rz \neq kerf$ and $Rz \cap \ker f \supseteq Rz^2$. Therefore, $Rz \cap \ker f \neq (0)$. Hence, there is an edge of $G(R)$ joining Rz and ker f. Therefore,ker $f \notin V_2$. Thus, ker $f \in V_3$. As $M^2 \cap \ker f = M^2 \neq (0)$, it follows that ker $f = M^2$. Now, $f : M \to M^2$ is a surjective R- linear map with ker $f = M^2$. Therefore, by the Fundamental theorem of homomorphism of modules, we obtain that $\frac{M}{M^2} \cong M^2$ as R - modules. As M^2 is a minimal ideal of R, it follows that $\frac{M}{M^2}$ is generated by any nonzero element of $\frac{M}{M^2}$. This is impossible. since $\dim_{\frac{R}{M}} \left(\frac{M}{M^2}\right) = 2$. Thus, $z^2 = 0$ for any $z \in M$. Now, $M = Ra + Rb$. Hence, $M^2 = Ra^2 + Rab + Rb^2 = Rab$. Observe that $Ra \neq Rb$. Moreover, $Ra \cap Rb \neq (0)$. It is clear that $Ra, Rb \notin V_1 \cup V_3$. Hence, $Ra, Rb \in V_2$. This is impossible as there is an edge of $G(R)$ joining Ra and Rb. This proves that$M^3 \neq (0)$. In such a case (R, M) is a SPIR with $\{M, M^2, M^3\}$ as its set of nontrivial ideals.

(2) $\Rightarrow$ (1): Note that $\{M, M^2, M^3\}$ is the set of all nontrivial ideals of R. It is clear that $G(R)$ is 3-partite with vertex partition $\{V_1, V_2, V_3\}$ with $V_1 = \{M\}$, $V_2 = \{M^2\}$, and $V_3 = \{M^3\}$. Clearly, $G(R)$ is not 2-partite. □

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