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Single Variable Generalized Additive - Quadratic and Generalized Cubic- Quartic Functional Equations in Various Banach Spaces

M. Arunkumar[1,*], E. Sathya[2] and C. Pooja[1]

1 Department of Mathematics, Government Arts College, Tiruvannamalai, TamilNadu, India.

2 Department of Mathematics, Shanmuga Industries Arts and Science College, Tiruvannamalai, TamilNadu, India.

Abstract: In this paper, we introduce and examine the generalized Ulam - Hyers stability of single variable generalized additive-quadratic and generalized cubic-quartic functional equations in various Banach spaces with the help of two different methods.

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1. Introduction

In 1940 Ulam [16] proposed the problem regarding the stability of group homomorphisms as Follows "when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation ?" In 1941, Hyers [6]considered the case of approximately additive mappings under the assumption that the groups are Banach spaces. The renowned Hyers stability result that appeared in [1] was generalized by Aoki for the stability of the additive mapping involving a sum of powers of norms. In [5, 8–10] provided a generalization of Hyers' Theorem for the stability of the linearmapping, which allows the Cauchy difference to be unbounded. This result of lead mathematicians working in stability of functional equations to establish what is known today as Generalized Hyers-Ulam-Rassias stability or Cauchy-Rassias stability or Ulam stability. M.Arunkumar et. al., [2] introduced and established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x), \tag{1}$$

and

$$g(2x) = 12g(x) + 4g(-x), \tag{2}$$

* *E-mail: annarun2002@gmail.com*

having solutions

$$f(x) = ax + bx^2 \qquad and \qquad g(x) = cx^3 + dx^4, \tag{3}$$

Also, the generalized Ulam - Hyers stability of the above (1) and (2) functional equations via Quasi-Beta Banach space, Intuitionistic fuzzy Banach space using direct and fixed point methods were discussed in [3]. In this paper, we introduce and examine the generalized Ulam - Hyers stability of single variable generalized additive-quadratic and generalized cubic-quartic functional equations

$$\phi(\lambda w) = \frac{\lambda}{2}(\phi(w) - \phi(-w)) + \frac{\lambda^2}{2}(\phi(w) + \phi(-w)) \tag{4}$$

$$\psi(\mu w) = \frac{\mu^3}{2}(\psi(w) - \psi(-w)) + \frac{\mu^4}{2}(\psi(w) + \psi(-w)) \tag{5}$$

in various Banach spaces with the help of two different methods. Now, we will recall the fundamental results in fixed point theory.

Theorem 1.1 (The alternative of fixed point [7]). *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant* L. *Then, for each given element* $x \in X$, *either*

(F_1) $d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0$,

or

(F_2) *there exists a natural number* n_0 *such that:*

$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;

$(FPC2)$ *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T

$(FPC3)$ y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L} \; d(y, Ty)$ *for all* $y \in Y$.

2. Solution of The Functional Equations

In this section, we judge the solution of the functional equations (4) and (5). For that we assume Λ_1 and Λ_2 are vector spaces. Using oddness and evenness of ϕ and ψ, the following lemmas are trivial.

Lemma 2.1. *An odd function* $\phi : \Lambda_1 \longrightarrow \Lambda_2$ *satisfying (4) then* ϕ *is additive.*

Lemma 2.2. *An even function* $\phi : \Lambda_1 \longrightarrow \Lambda_2$ *satisfying (4) then* ϕ *is quadratic.*

Lemma 2.3. *An odd function* $\psi : \Lambda_1 \longrightarrow \Lambda_2$ *satisfying (5) then* ψ *is additive.*

Lemma 2.4. *An even function* $\psi : \Lambda_1 \longrightarrow \Lambda_2$ *satisfying (5) then* ψ *is quadratic.*

3. Stability In Banach Space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (4) and (5) in Banach space. To prove stability results, let us take Δ_1 be an normed space and Δ_2 be an Banach space.

3.1. Hyers Method of (4)

Theorem 3.1. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\| \phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right) \right\| \leq \Phi(w) \tag{1}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta \to \infty} \frac{\Phi(\lambda^{\eta\tau} w)}{\lambda^{\eta\tau}} = 0 \tag{2}$$

for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_A(w) - \phi(w)\| \leq \frac{1}{\lambda} \sum_{\rho = \frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} \tag{3}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_A(w)$ is defined by

$$\Psi_A(w) = \lim_{\eta \to \infty} \frac{\Phi(\lambda^{\eta\tau} w)}{\lambda^{\eta\tau}} \tag{4}$$

for all $w \in \Delta_1$.

Proof. In order to prove the theorem, we have to prove the following:

(1). The sequence $\left\{ \dfrac{\phi(\lambda^\eta w)}{\lambda^\eta} \right\}$ is a Cauchy sequence for all $w \in \Delta_1$;

(2). The mapping $\Psi_A(w)$ satisfies (4);

(3). $\Psi_A(w)$ is unique.

Using oddness of ϕ in (1), we reach

$$\|\phi(\lambda w) - \lambda \phi(w)\| \leq \Phi(w) \tag{5}$$

for all $w \in \Delta_1$. The above inequality can be rewritten as

$$\left\| \frac{\phi(\lambda w)}{\lambda} - \phi(w) \right\| \leq \frac{\Phi(w)}{\lambda} \tag{6}$$

for all $w \in \Delta_1$. Changing w by λw and multiplying by $\dfrac{1}{\lambda}$ in (6), we arrive

$$\left\| \frac{\phi(\lambda^2 w)}{\lambda^2} - \frac{\phi(\lambda w)}{\lambda} \right\| \leq \frac{\Phi(\lambda w)}{\lambda^2} \tag{7}$$

for all $w \in \Delta_1$. Using triangle inequality on (6) and (7), we have

$$\begin{aligned} \left\| \frac{\phi(\lambda^2 w)}{\lambda^2} - \phi(w) \right\| &= \left\| \frac{\phi(\lambda^2 w)}{\lambda^2} - \frac{\phi(\lambda w)}{\lambda} + \frac{\phi(\lambda w)}{\lambda} - \phi(w) \right\| \\ &\leq \left\| \frac{\phi(\lambda w)}{\lambda} - \phi(w) \right\| + \left\| \frac{\phi(\lambda^2 w)}{\lambda^2} - \frac{\phi(\lambda w)}{\lambda} \right\| \end{aligned}$$

$$\leq \frac{\Phi(w)}{\lambda} + \frac{\Phi(\lambda w)}{\lambda^2} = \frac{\Phi(w)}{\lambda}\left(1 + \frac{\Phi(\lambda w)}{\lambda}\right) \tag{8}$$

for all $w \in \Delta_1$. Generalizing for a positive integer η, we obtain

$$\left\|\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w)\right\| \leq \frac{1}{\lambda}\sum_{\rho=0}^{\eta-1}\frac{\Phi(\lambda^\rho w)}{\lambda^\rho} \tag{9}$$

for all $w \in \Delta_1$. Hence $\left\{\frac{\phi(\lambda^\eta w)}{\lambda^\eta}\right\}$ is a Cauchy sequence and it converges to a point $\Psi_A(w) \in \Delta_2$. This proves the existence of Cauchy sequence. Indeed, replacing w by $\lambda^\kappa w$ and divided by λ^κ in (9), we get

$$\begin{aligned}\left\|\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{\eta+\kappa}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}\right\| &= \frac{1}{\lambda^\kappa}\left\|\frac{\phi(\lambda^\eta \cdot \lambda^\kappa w)}{\lambda^\eta} - \phi(\lambda^\kappa w)\right\| \\ &\leq \frac{1}{\lambda}\sum_{\rho=0}^{\eta-1}\frac{\Phi(\lambda^{\rho+\kappa} w)}{\lambda^{\rho+\kappa}} \\ &\longrightarrow 0 \quad as \quad \kappa \quad \longrightarrow \quad \infty\end{aligned} \tag{10}$$

for all $w \in \Delta_1$. Thus, we define mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ such that

$$\Psi_A(w) = \lim_{\eta\longrightarrow\infty}\frac{\phi(\lambda^\eta w)}{\lambda^\eta}$$

for all $w \in \Delta_1$. Letting limit $\eta \to \infty$ in (9) and using the definition of $\Phi(w)$, we get

$$\left\|\lim_{\eta\longrightarrow\infty}\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w)\right\| \leq \frac{1}{\lambda}\sum_{\rho=0}^{\infty}\frac{\Phi(\lambda^\rho w)}{\lambda^\rho} \Longrightarrow \|\Psi_A(w) - \phi(w)\| \leq \frac{1}{\lambda}\sum_{\rho=0}^{\infty}\frac{\Phi(\lambda^\rho w)}{\lambda^\rho}$$

for all $w \in \Delta_1$. Thus (3) holds for $\tau = 1$. Now, to show that $\Psi_A(w)$ satisfies (4), changing w by $\lambda^\eta w$ and divided by λ^η in (1), we reach

$$\frac{1}{\lambda^\eta}\left\|\phi(\lambda^\eta \cdot \lambda w) - \frac{\lambda}{2}\left(\phi(\lambda^\eta w) - \phi(-\lambda^\eta w)\right) - \frac{\lambda^2}{2}\left(\phi(\lambda^\eta w) + \phi(-\lambda^\eta w)\right)\right\| \leq \frac{1}{\lambda^\eta}\Phi(\lambda^\eta w)$$

for all $w \in \Delta_1$. Approaching $\eta \to \infty$ and using the definition of $\Psi_A(w)$, (2) in the above inequality we can see that

$$\Psi_A(\lambda w) = \frac{\lambda}{2}\left(\Psi_A(w) - \Psi_A(-w)\right) + \frac{\lambda^2}{2}\left(\Psi_A(w) + \Psi_A(-w)\right)$$

for all $w \in \Delta_1$. Hence $\Psi_A(w)$ satisfies the functional equation (4) for all $w \in \Delta_1$. In order to prove the existence of $\Psi_A(w)$ is unique, assume $\Psi_B(w)$ be another additive mapping satisfying (4) and (3). Now,

$$\begin{aligned}\|\Psi_A(w) - \Psi_B(w)\| &= \frac{1}{\lambda^\kappa}\|\Psi_A(\lambda^\kappa w) - \Psi_B(\lambda^\kappa w)\| \\ &= \frac{1}{\lambda^\kappa}\|\Psi_A(\lambda^\kappa w) - \phi(\lambda^\kappa w) + \phi(\lambda^\kappa w - \Psi_B(\lambda^\kappa w)\| \\ &= \frac{1}{\lambda^\kappa}\{\|\Psi_A(\lambda^\kappa w) - \phi(\lambda^\kappa w)\| + \|\Psi_B(\lambda^\kappa w) - \phi(\lambda^\kappa w)\|\} \\ &\leq \frac{2}{\lambda}\sum_{\rho=0}^{\infty}\frac{\Phi(\lambda^{\rho+\kappa} w)}{\lambda^{\rho+\kappa}} \\ &\longrightarrow 0 \quad as \quad \kappa \quad \longrightarrow \quad \infty\end{aligned}$$

for all $w \in \Delta_1$. This proves that $\Psi_A(w) = \Psi_B(w)$ for all $w \in \Delta_1$. Thus $\Psi_A(w)$ is unique. Hence the theorem holds for $\tau = 1$.

Replacing w by $\frac{w}{\lambda}$ in (5), we achieve

$$\left\|\phi(w)-\lambda\phi\left(\frac{w}{\lambda}\right)\right\|\leq\Phi\left(\frac{w}{\lambda}\right) \tag{11}$$

for all $w\in\Delta_1$. Again replacing w by $\frac{w}{\lambda}$ and multiply by λ in (11), we arrive

$$\left\|\lambda\phi\left(\frac{w}{\lambda}\right)-\lambda^2\phi\left(\frac{w}{\lambda^2}\right)\right\|\leq\lambda\Phi\left(\frac{w}{\lambda^2}\right) \tag{12}$$

for all $w\in\Delta_1$. Using triangle inequality on (11) and (12), we have

$$\begin{aligned}\left\|\phi(w)-\lambda^2\phi\left(\frac{w}{\lambda^2}\right)\right\|&\leq\left\|\phi(w)-\lambda\phi\left(\frac{w}{\lambda}\right)\right\|+\left\|\lambda\phi\left(\frac{w}{\lambda}\right)-\lambda^2\phi\left(\frac{w}{\lambda^2}\right)\right\|\\&\leq\Phi\left(\frac{w}{\lambda}\right)+\lambda\Phi\left(\frac{w}{\lambda^2}\right)\end{aligned} \tag{13}$$

for all $w\in\Delta_1$. Generalizing for a positive integer η, we obtain

$$\left\|\phi(w)-\lambda^\eta\phi\left(\frac{w}{\lambda^\eta}\right)\right\|\leq\sum_{\rho=1}^{\eta-1}\lambda^{\rho-1}\Phi\left(\frac{w}{\lambda^\rho}\right)=\frac{1}{\lambda}\sum_{\rho=1}^{\eta-1}\lambda^\rho\Phi\left(\frac{w}{\lambda^\rho}\right) \tag{14}$$

for all $w\in\Delta_1$. The rest of the proof is similar ideas to that of case $\tau=1$. Thus the theorem is true for $\tau=-1$. Hence the proof is complete. □

The following corollary is the immediate consequence of Theorem 3.1 concerning the stabilities of (4).

Corollary 3.2. *Assume σ and α be positive numbers. Let $\phi:\Delta_1\longrightarrow\Delta_2$ be an odd function fulfilling the inequality*

$$\left\|\phi(\lambda w)-\frac{\lambda}{2}\left(\phi(w)-\phi(-w)\right)-\frac{\lambda^2}{2}\left(\phi(w)+\phi(-w)\right)\right\|\leq\begin{cases}\sigma;\\ \sigma||w||^\alpha;\ \ \alpha\neq1\end{cases} \tag{15}$$

for all $w\in\Delta_1$. Then there exists one and only additive mapping $\Psi_A(w):\Delta_1\longrightarrow\Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_A(w)-\phi(w)\|\leq\begin{cases}\dfrac{\sigma}{|\lambda-1|};\\ \dfrac{\sigma||w||^\alpha}{|\lambda-\lambda^\alpha|};\end{cases} \tag{16}$$

for all $w\in\Delta_1$.

The proof of the following theorem and corollary is similar clues that of Theorem 3.1 and Corollary 3.2. Hence the details of the proof are omitted.

Theorem 3.3. *Let $\phi:\Delta_1\longrightarrow\Delta_2$ be an even function fulfilling the inequality*

$$\left\|\phi(\lambda w)-\frac{\lambda}{2}\left(\phi(w)-\phi(-w)\right)-\frac{\lambda^2}{2}\left(\phi(w)+\phi(-w)\right)\right\|\leq\Phi(w) \tag{17}$$

for all $w\in\Delta_1$, where $\Phi:\Delta_1\longrightarrow[0,\infty)$ with the condition

$$\lim_{\eta\to\infty}\frac{\Phi(\lambda^{\eta\tau}w)}{\lambda^{2\eta\tau}}=0 \tag{18}$$

for all $w \in \Delta_1$*. Then there exists one and only quadratic mapping* $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\|\Psi_{Q_2}(w) - \phi(w)\| \leq \frac{1}{\lambda^2} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} \tag{19}$$

for all $w \in \Delta_1$ *with* $\tau = \pm 1$*. The mapping* $\Psi_{Q2}(w)$ *is defined by*

$$\Psi_{Q_2}(w) = \lim_{\eta\to\infty} \frac{\Phi(\lambda^{\eta\tau} w)}{\lambda^{2\eta\tau}} \tag{20}$$

for all $w \in \Delta_1$*.*

Corollary 3.4. *Assume* σ *and* α *be positive numbers. Let* $\phi : \Delta_1 \longrightarrow \Delta_2$ *be an even function fulfilling the inequality*

$$\left\|\phi(\lambda w) - \frac{\lambda}{2}(\phi(w) - \phi(-w)) - \frac{\lambda^2}{2}(\phi(w) + \phi(-w))\right\| \leq \begin{cases} \sigma; \\ \sigma||w||^{\alpha}; \ \ \alpha \neq 2 \end{cases} \tag{21}$$

for all $w \in \Delta_1$*. Then there exists one and only quadratic mapping* $\Psi_{Q_2}(w) : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\|\Psi_{Q_2}(w) - \phi(w)\| \leq \begin{cases} \dfrac{\sigma}{|\lambda^2 - 1|}; \\ \dfrac{\sigma||w||^{\alpha}}{|\lambda^2 - \lambda^{\alpha}|}; \end{cases} \tag{22}$$

for all $w \in \Delta_1$*.*

Theorem 3.5. *Let* $\phi : \Delta_1 \longrightarrow \Delta_2$ *be a function fulfilling the inequality*

$$\left\|\phi(\lambda w) - \frac{\lambda}{2}(\phi(w) - \phi(-w)) - \frac{\lambda^2}{2}(\phi(w) + \phi(-w))\right\| \leq \Phi(w) \tag{23}$$

for all $w \in \Delta_1$*, where* $\Phi : \Delta_1 \longrightarrow [0, \infty)$ *satisfying the conditions (2) and (18) for all* $w \in \Delta_1$*. Then there exists one and only additive mapping* $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ *and one and only quadratic mapping* $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\|\phi(w) - \Psi_A(w) - \Psi_{Q2}(w)\| \leq \frac{1}{2}\left\{\frac{1}{\lambda} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{1}{\lambda^2} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}}\right\} \tag{24}$$

for all $w \in \Delta_1$ *with* $\tau = \pm 1$*. The mappings* $\Psi_A(w)$ *and* $\Psi_{Q_2}(w)$ *are respectively defined in (4) and (20) for all* $w \in \Delta_1$*.*

Proof. Let

$$\phi_a(w) = \frac{\phi(w) - \phi(-w)}{2} \tag{25}$$

for all $w \in \Delta_1$. Then it is easy to verify that

$$\phi_a(0) = 0 \qquad and \qquad \phi_a(-w) = -\phi_a(w)$$

for all $w \in \Delta_1$. By Theorem 3.1 and (25), we arrive

$$\|\Psi_A(w) - \phi_a(w)\| \leq \frac{1}{2\lambda} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} \tag{26}$$

for all $w \in \Delta_1$. Also, let

$$\phi_q(w) = \frac{\phi(w) + \phi(-w)}{2} \tag{27}$$

for all $w \in \Delta_1$. Then it is easy to verify that

$$\phi_q(0) = 0 \qquad \textit{and} \qquad \phi_q(-w) = \phi_q(w)$$

for all $w \in \Delta_1$. By Theorem 3.3 and (27), we arrive

$$\|\Psi_{Q_2}(w) - \phi_q(w)\| \leq \frac{1}{2\lambda^2} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} \tag{28}$$

for all $w \in \Delta_1$. Define

$$\phi(w) = \phi_a(w) + \phi_q(w) \tag{29}$$

for all $w \in \Delta_1$. Now, it follows from (29), (28) and (29), we get

$$\begin{aligned}
\|\phi(w) - \Psi_A(w) - \Psi_{Q_2}(w)\| &= \|\phi_a(w) + \phi_q(w) - \Psi_A(w) - \Psi_{Q_2}(w)\| \\
&\leq \|\phi_a(w) - \Psi_A(w)\| + \|\phi_q(w) - \Psi_{Q_2}(w)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{\lambda} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{1}{\lambda^2} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}}\right\}
\end{aligned}$$

for all $w \in \Delta_1$. □

Corollary 3.6. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\|\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right)\right\| \leq \begin{cases} \sigma; \\ \sigma||w||^{\alpha}; \ \ \alpha \neq 1, 2 \end{cases} \tag{30}$$

for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quadratic mapping $\Psi_{Q_2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w) - \Psi_{Q_2}(w)\| \leq \begin{cases} \sigma\left(\dfrac{1}{|\lambda - 1|} + \dfrac{1}{|\lambda^2 - 1|}\right); \\ \sigma||w||^{\alpha}\left(\dfrac{1}{|\lambda - \lambda^{\alpha}|} + \dfrac{1}{|\lambda^2 - \lambda^{\alpha}|}\right); \end{cases} \tag{31}$$

for all $w \in \Delta_1$.

3.2. Hyers Method of (5)

Theorem 3.7. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\|\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right)\right\| \leq \Phi(w) \tag{32}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta\to\infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{3\eta\tau}} = 0 \tag{33}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_C(w) - \phi(w)\| \leq \frac{1}{\mu^3} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau} w)}{\mu^{3\rho\tau}} \tag{34}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_C(w)$ is defined by

$$\Psi_C(w) = \lim_{\eta\to\infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{3\eta\tau}} \tag{35}$$

for all $w \in \Delta_1$.

Corollary 3.8. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\|\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right)\right\| \leq \begin{cases} \sigma; \\ \sigma||w||^{\alpha}; \;\; \alpha \neq 3 \end{cases} \tag{36}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_C(w) - \phi(w)\| \leq \begin{cases} \dfrac{\sigma}{|\mu^3 - 1|}; \\ \dfrac{\sigma||w||^{\alpha}}{|\mu^3 - \mu^{\alpha}|}; \end{cases} \tag{37}$$

for all $w \in \Delta_1$.

Theorem 3.9. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\|\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right)\right\| \leq \Phi(w) \tag{38}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta\to\infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{4\eta\tau}} = 0 \tag{39}$$

for all $w \in \Delta_1$. Then there exists one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_{Q4}(w) - \phi(w)\| \leq \frac{1}{\mu^4} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau} w)}{\mu^{4\rho\tau}} \tag{40}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_{Q_4}(w)$ is defined by

$$\Psi_{Q4}(w) = \lim_{\eta\to\infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{4\eta\tau}} \tag{41}$$

for all $w \in \Delta_1$.

Corollary 3.10. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\| \leq \begin{cases} \sigma; \\ \sigma ||w||^{\alpha}; \;\; \alpha \neq 4 \end{cases} \tag{42}$$

for all $w \in \Delta_1$. Then there exists one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_{Q4}(w) - \phi(w)\| \leq \begin{cases} \dfrac{\sigma}{|\mu^4 - 1|}; \\ \dfrac{\sigma||w||^{\alpha}}{|\mu^4 - \mu^{\alpha}|}; \end{cases} \tag{43}$$

for all $w \in \Delta_1$.

Theorem 3.11. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\| \leq \Phi(w) \tag{44}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ satisfying the conditions (33) and (18) for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q_4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\phi(w) - \Psi_C(w) - \Psi_{Q_4}(w)\| \leq \frac{1}{2}\left\{ \frac{1}{\mu^3}\sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau}w)}{\mu^{3\rho\tau}} + \frac{\Phi(-\mu^{\rho\tau}w)}{\mu^{3\rho\tau}} + \frac{1}{\mu^4}\sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau}w)}{\mu^{4\rho\tau}} + \frac{\Phi(-\mu^{\rho\tau}w)}{\mu^{4\rho\tau}} \right\} \tag{45}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mappings $\Psi_C(w)$ and $\Psi_{Q_4}(w)$ are respectively defined in (35) and (20) for all $w \in \Delta_1$.

Corollary 3.12. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\| \leq \begin{cases} \sigma; \\ \sigma ||w||^{\alpha}; \;\; \alpha \neq 3, 4 \end{cases} \tag{46}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q_4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\phi(w) - \Psi_C(w) - \Psi_{Q_4}(w)\| \leq \begin{cases} \sigma\left(\dfrac{1}{|\mu^3 - 1|} + \dfrac{1}{|\mu^4 - 1|}\right); \\ \sigma||w||^{\alpha}\left(\dfrac{1}{|\mu^3 - \mu^{\alpha}|} + \dfrac{1}{|\mu^4 - \mu^{\alpha}|}\right); \end{cases} \tag{47}$$

for all $w \in \Delta_1$.

3.3. Radus Method of (4)

Theorem 3.13. *If $\phi : \Delta_1 \to \Delta_2$ be an odd function satisfying the inequality (1) and $\Phi : \Delta_1^2 \to [0, \infty)$ be function with the condition*

$$\lim_{\eta \to \infty} \frac{1}{\theta_j^{\eta}} \Phi\left(\theta_j^{\eta} w\right) = 0 \tag{48}$$

for all $w \in \Delta_1$, where

$$\theta_j = \begin{cases} \lambda & j = 0; \\ \frac{1}{\lambda} & j = 1. \end{cases} \tag{49}$$

If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right),$$

has the property

$$\frac{1}{\theta_j}\ \Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{50}$$

Then there exists a unique additive mapping $\Psi_A : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w)\| \le \left(\frac{L^{1-j}}{1-L}\right)\Phi(w) \tag{51}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r, s) = \inf\{K \in (0, \infty) : \|r(w) - s(w)\| \le K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \le Ld(r, s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L. With the help of (50), it follows from (6) for the case $j = 0$ it reduces to

$$\left\|\frac{\phi(\lambda w)}{\lambda} - \phi(w)\right\| \le \frac{\Phi\left(w\right)}{\lambda} \Rightarrow \left\|\frac{\phi(\lambda w)}{\lambda} - \phi(w)\right\| \le \frac{\Phi(w)}{\lambda} \Rightarrow d(T\phi, \phi) \le L < \infty. \tag{52}$$

for all $w \in \Delta_1$.

With the help of (50), it follows from (5) for the case $j = 1$ it reduces to

$$\left\|\phi\left(w\right) - \lambda\ \phi\left(\frac{w}{\lambda}\right)\right\| \le \Phi\left(\frac{w}{\lambda}\right) \Rightarrow \left\|\phi(w) - \lambda\phi\left(\frac{w}{\lambda}\right)\right\| \le\ \Phi(w) \Rightarrow d(\phi, T\phi) \le 1 < \infty. \tag{53}$$

for all $w \in \Delta_1$. Combining the above two cases, we arrive

$$d(\phi, T\phi) \le L^{1-j}.$$

Therefore (FPC1) of Theorem 1.1 holds. By (FPC2) of Theorem 1.1, it follows that there exists a fixed point Ψ_A of T in $\mathcal{X}$ such that

$$\Psi_A(w) = \lim_{\eta\to\infty} \frac{\phi(\theta_j^\eta w)}{\theta_j^\eta}, \qquad \forall\ \ w \in \Delta_1. \tag{54}$$

To order to prove $\Psi_A : \Delta_1 \to \Delta_2$ is additive the proof is similar ideas to that of Theorem 3.1. Again by (FPC3) of Theorem 1.1, Ψ_A is the unique fixed point of T in the set $\mathcal{Y} = \{\Psi_A \in \mathcal{X} : d(\phi, \Psi_A) < \infty\}, \Psi_A$ is the unique function such that $\|\phi(w) - \Psi_A(w)\| \le K\ \Phi(w)$ for all $w \in \Delta_1$ and $K > 0$. Finally by (FPC4) of Theorem 1.1, we obtain $d(\phi, \Psi_A) \le \frac{1}{1-L}\ d(\phi, T\phi)$ this implies $d(\phi, \Psi_A) \le \frac{L^{1-j}}{1-L}$ which yields our desired result. □

The following corollary is an immediate consequence of Theorem 3.13 concerning the some stabilities of (4).

Corollary 3.14. *Let $\phi : \Delta_1 \to \Delta_2$ be an odd mapping satisfying the inequality (15) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique additive function $\Psi_A : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and*

$$\|\phi(w) - \Psi_A(w)\| \le \begin{cases} \dfrac{\sigma}{|\lambda - 1|}; \\ \dfrac{\sigma||w||^\alpha}{|\lambda - \lambda^\alpha|}; \end{cases} \tag{55}$$

for all $w \in \Delta_1$.

Proof. If we take

$$\Phi(w)=\begin{cases}\sigma;\\ \sigma||w||^{\alpha};\ \ \alpha\neq 1\end{cases}\tag{56}$$

for all $w\in\Delta_1$ in Theorem 3.13. Replacing w by $\theta_j^{\eta}\ w$ and dividing by θ_j^{η} in (56) one can see that (48) holds. Now, by definition of $\Phi(w)$ and its property, we have

$$\Phi(w)=\Phi\left(\frac{w}{\lambda}\right)=\begin{cases}\sigma\\ \dfrac{\sigma}{\lambda^{\alpha}}\ ||w||^{\alpha},\end{cases}$$

and

$$\frac{1}{\theta_j}\Phi(\theta_j w)=\begin{cases}\dfrac{\sigma}{\theta_j}\\ \dfrac{\sigma}{\lambda^{\alpha}\theta_j}||\theta_j\ w||^{\alpha},\end{cases}=\begin{cases}\theta_j^{-1}\ \Phi(w),\\ \\ \theta_j^{\alpha-1}\ \Phi(w),\end{cases}$$

for all $w\in\Delta_1$. Hence the property (50) and the inequality (51) holds for the following:

$$L=\theta_j^{-1}=\lambda^{-1}\quad for\quad j=0$$
$$\|\phi(w)-\Psi_A(w)\|\leq\left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w)=\left(\frac{\lambda^{-1}}{1-\lambda^{-1}}\right)\ \sigma=\left(\frac{\sigma}{\lambda-1}\right)$$
$$L=\theta_j^{-1}=\frac{1}{\lambda^{-1}}=\lambda\quad for\quad j=1$$
$$\|\phi(w)-\Psi_A(w)\|\leq\left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w)=\left(\frac{1}{1-\lambda}\right)\ \sigma=\left(\frac{\sigma}{1-\lambda}\right)$$
$$L=\theta_j^{\alpha-1}=\lambda^{\alpha-1}\quad for\quad j=0$$
$$\|\phi(w)-\Psi_A(w)\|\leq\left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w)=\left(\frac{\lambda^{\alpha-1}}{1-\lambda^{\alpha-1}}\right)\ \frac{\sigma}{\lambda^{\alpha}}\ ||w||^{\alpha}=\frac{\sigma}{\lambda-\lambda^{\alpha}}\ ||w||^{\alpha}$$
$$L=\theta_j^{\alpha-1}=\frac{1}{\lambda^{\alpha-1}}=\lambda^{1-\alpha}\quad for\quad j=1$$
$$\|\phi(w)-\Psi_A(w)\|\leq\left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w)=\left(\frac{1}{1-\lambda^{1-d}}\right)\ \frac{\sigma}{\lambda^{\alpha}}\ ||w||^{\alpha}=\frac{\sigma}{\lambda^{\alpha}-\lambda}\ ||w||^{\alpha}$$

Hence the proof is complete. □

Theorem 3.15. *If $\phi:\Delta_1\to\Delta_2$ be an even function satisfying the inequality (17) and $\Phi:\Delta_1^2\to[0,\infty)$ be function with the condition*

$$\lim_{\eta\to\infty}\frac{1}{\theta_j^{2\eta}}\Phi\left(\theta_j^{\eta}w\right)=0\tag{57}$$

for all $w\in\Delta_1$, where θ_j is defined in (49). If there exists $L=L(j)$ such that the function

$$\Phi(w)=\Phi\left(\frac{w}{\lambda}\right),$$

has the property

$$\frac{1}{\theta_j^2}\ \Phi\left(\theta_j w\right)=L\ \Phi(w).\tag{58}$$

Then there exists a unique quadratic mapping $\Psi_{Q2}:\Delta_1\to\Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w)-\Psi_{Q2}(w)\|\leq\left(\frac{L^{1-j}}{1-L}\right)\Phi(w)\tag{59}$$

for all $w\in\Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\| \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^2} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.15 concerning the some stabilities of (4).

Corollary 3.16. *Let $\phi : \Delta_1 \to \Delta_2$ be an even mapping satisfying (21) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique quadratic function $\Psi_{Q2} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and*

$$\|\phi(w) - \Psi_{Q2}(w)\| \leq \begin{cases} \dfrac{\sigma}{|\lambda^2 - 1|}; \\ \dfrac{\sigma ||w||^\alpha}{|\lambda^2 - \lambda^\alpha|}; \end{cases} \tag{60}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 3.17. *If $\phi : \Delta_1 \to \Delta_2$ be a function satisfying the inequality (23) and $\Phi : \Delta_1^2 \to [0,\infty)$ be function with the conditions (48) and (57) for all $w \in \Delta_1$, where θ_j is defined in (49). If there exists $L = L(j)$ such that the function*

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right),$$

has the properties (50) and (58). Then there exists a unique additive mapping $\Psi_A : \Delta_1 \to \Delta_2$ and a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right)[\Phi(w) + \Phi(-w)] \tag{61}$$

for all $w \in \Delta_1$.

Proof. The proof is similar lines to that of Theorem 3.5. □

Corollary 3.18. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality (30) for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quadratic mapping $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and the inequality (31) for all $w \in \Delta_1$.*

3.4. Radus Method of (5)

Theorem 3.19. *If $\psi\Delta_1 \to \Delta_2$ be a function satisfying the inequality (32) and $\Phi : \Delta_1^2 \to [0,\infty)$ be a function with the condition*

$$\lim_{\eta\to\infty} \frac{1}{\theta_j^{3\eta}} \Phi\left(\theta_j^\eta w\right) = 0 \tag{62}$$

for all $w \in \Delta_1$, where

$$\theta_j = \begin{cases} \mu & j = 0; \\ \frac{1}{\mu} & j = 1. \end{cases} \tag{63}$$

If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the property

$$\frac{1}{\theta_j^3}\,\Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{64}$$

Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \to \Delta_2$ *satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_C(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right)\Phi(w) \tag{65}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\| \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^3} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.19 concerning the some stabilities of (5).

Corollary 3.20. *Let* $\psi : \Delta_1 \to \Delta_2$ *be an odd mapping satisfying (36) and there exists real numbers* σ *and* α *such that for all* $w \in \Delta_1$. *Then there exists a unique cubic function* $\Psi_C : \Delta_1 \to \Delta_2$ *satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_C(w)\| \leq \begin{cases} \dfrac{\sigma}{|\mu^3 - 1|}; \\ \dfrac{\sigma ||w||^\alpha}{|\mu^3 - \mu^\alpha|}; \end{cases} \tag{66}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 3.21. *If* $\psi : \Delta_1 \to \Delta_2$ *be an even function satisfying the inequality (38) and* $\Phi : \Delta_1^2 \to [0,\infty)$ *be a function with the condition*

$$\lim_{\eta\to\infty} \frac{1}{\theta_j^{4\eta}} \Phi\left(\theta_j^\eta w\right) = 0 \tag{67}$$

for all $w \in \Delta_1$, *where* θ_j *is defined in (63). If there exists* $L = L(j)$ *such that the function*

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the property

$$\frac{1}{\theta_j^4}\,\Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{68}$$

Then there exists a unique quartic mapping $\Psi_{Q4} : \Delta_1 \to \Delta_2$ *satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_{Q4}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right)\Phi(w) \tag{69}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\| \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^4} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.21 concerning the some stabilities of (5).

Corollary 3.22. *Let $\psi : \Delta_1 \to \Delta_2$ be an even mapping satisfying (42) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique quartic function $\Psi_{Q4} : \Delta_1 \to \Delta_2$ satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_{Q4}(w)\| \leq \begin{cases} \dfrac{\sigma}{|\mu^3 - 1|}; \\ \dfrac{\sigma||w||^\alpha}{|\mu^3 - \mu^\alpha|}; \end{cases} \tag{70}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 3.23. *If $\psi : \Delta_1 \to \Delta_2$ and $\Phi : \Delta_1^2 \to [0, \infty)$ are functions satisfying the inequality (44) and the conditions (62) and (67) for all $w \in \Delta_1$, where θ_j is defined in (63). If there exists $L = L(j)$ such that the function*

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the properties (64) and (68). Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \to \Delta_2$ and a unique quartic mapping $\Psi_{Q4} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\psi(w) - \Psi_C(w) - \Psi_{Q4}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right)[\Phi(w) + \Phi(-w)] \tag{71}$$

for all $w \in \Delta_1$.

Proof. The proof is similar lines to that of Theorem 3.5. □

Corollary 3.24. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality (46) for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and the inequality (47) for all $w \in \Delta_1$.*

4. Stability In $(\beta; p)$ Banach Space

In this section, we study the generalized Ulam - Hyers stability of the functional equations (4) and (5) in $(\beta; p)$ Banach Space. To prove stability results, let us take Δ_1 be an Quasi normed space and Δ_2 be an $(\beta; p)$ Banach space. In this section, we present some basic facts concerning quasi-β-Normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

4.1. Definitions and Notations On Quasi Beta Banach space

Definition 4.1. *Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:*

(Q1) $\| x \| \geq 0$ *for all* $x \in X$ *and* $\| x \| = 0$ *if and only if* $x = 0$.

(Q2) $\| \lambda x \| = | \lambda |^\beta . \| x \|$ *for all* $\lambda \in \mathbb{K}$ *and all* $x \in X$.

(Q3) *There is a constant* $K \geq 1$ *such that* $\| x + y \| \leq K (\| x \| + \| y \|)$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 4.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 4.3. *A qusi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if $\| x + y \|^p \leq \| x \|^p + \| y \|^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.*

More details, one can refer [4, 11, 15] for the concepts of quasi-normed spaces and p-Banach space. Given a $p-$norm, the formula $d(x, y) := ||x-y||^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [11], each quasi-norm is equivalent to some $p-$norm. Since it is much easier to work with $p-$norms than quasi-norms, henceforth we restrict our attention mainly to $p-$norms.

4.2. Hyers Method of (4)

Theorem 4.4. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\|\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right)\right\|_{\Delta_2} \leq \Phi(w) \tag{72}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta\to\infty} \frac{\Phi(\lambda^{\eta\tau} w)}{\lambda^{\eta\tau}} = 0 \tag{73}$$

for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_A(w) - \phi(w)\|_{\Delta_2}^p \leq \left(\frac{K^{\eta-1}}{\lambda^\beta} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}}\right)^p \tag{74}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_A(w)$ is defined by

$$\Psi_A(w) = \lim_{\eta\to\infty} \frac{\Phi(\lambda^{\eta\tau} w)}{\lambda^{\eta\tau}} \tag{75}$$

for all $w \in \Delta_1$.

Proof. Using oddness of ϕ in (72), we reach

$$\|\phi(\lambda w) - \lambda\phi(w)\|_{\Delta_2} \leq \Phi(w) \tag{76}$$

for all $w \in \Delta_1$. The above inequality can be rewritten as

$$\left\|\frac{\phi(\lambda w)}{\lambda} - \phi(w)\right\|_{\Delta_2} \leq \frac{\Phi(w)}{\lambda^\beta} \tag{77}$$

for all $w \in \Delta_1$. Changing w by λw and multiplying by $\dfrac{1}{\lambda}$ in (77), we arrive

$$\left\|\frac{\phi(\lambda^2 w)}{\lambda^2} - \frac{\phi(\lambda w)}{\lambda}\right\|_{\Delta_2} \leq \frac{\Phi(\lambda w)}{\lambda^\beta \cdot \lambda} \tag{78}$$

for all $w \in \Delta_1$. Using triangle inequality on (77) and (78), we have

$$\begin{aligned}\left\|\frac{\phi(\lambda^2 w)}{\lambda^2}-\phi(w)\right\|_{\Delta_2} &= \left\|\frac{\phi(\lambda^2 w)}{\lambda^2}-\frac{\phi(\lambda w)}{\lambda}+\frac{\phi(\lambda w)}{\lambda}-\phi(w)\right\|_{\Delta_2}\\ &\leq K\left\{\left\|\frac{\phi(\lambda w)}{\lambda}-\phi(w)\right\|_{\Delta_2}+\left\|\frac{\phi(\lambda^2 w)}{\lambda^2}-\frac{\phi(\lambda w)}{\lambda}\right\|_{\Delta_2}\right\}\\ &\leq K\left\{\frac{\Phi(w)}{\lambda^\beta}+\frac{\Phi(\lambda w)}{\lambda^\beta\cdot\lambda}\right\}=\frac{K}{\lambda^\beta}\left(\Phi(w)+\frac{\Phi(\lambda w)}{\lambda}\right)\end{aligned} \tag{79}$$

for all $w \in \Delta_1$. Generalizing for a positive integer η, we obtain

$$\left\|\frac{\phi(\lambda^\eta w)}{\lambda^\eta}-\phi(w)\right\|_{\Delta_2} \leq \frac{K^{\eta-1}}{\lambda^\beta}\sum_{\rho=0}^{\eta-1}\frac{\Phi(\lambda^\rho w)}{\lambda^\rho} \tag{80}$$

for all $w \in \Delta_1$. The rest of the proof is similar ideas to that of Theorem 3.1. Hence the proof is complete. □

The following corollary is the immediate consequence of Theorem 4.4 concerning the stabilities of (4).

Corollary 4.5. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\|\phi(\lambda w)-\frac{\lambda}{2}\left(\phi(w)-\phi(-w)\right)-\frac{\lambda^2}{2}\left(\phi(w)+\phi(-w)\right)\right\|_{\Delta_2} \leq \begin{cases}\sigma;\\ \sigma||w||^\alpha;\ \ \alpha\neq 1\end{cases} \tag{81}$$

for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_A(w)-\phi(w)\|^p_{\Delta_2} \leq \begin{cases}\left(\dfrac{K^{\eta-1}\ \sigma}{\lambda^{\beta-1}|\lambda-1|}\right)^p;\\ \left(\dfrac{K^{\eta-1}\ \sigma||w||^\alpha}{\lambda^{\beta-1}|\lambda-\lambda^{\alpha\beta}|}\right)^p;\end{cases} \tag{82}$$

for all $w \in \Delta_1$.

The proof of the following theorem and corollary is similar clues that of Theorem 4.4 and Corollary 4.5. Hence the details of the proof are omitted.

Theorem 4.6. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\|\phi(\lambda w)-\frac{\lambda}{2}\left(\phi(w)-\phi(-w)\right)-\frac{\lambda^2}{2}\left(\phi(w)+\phi(-w)\right)\right\|_{\Delta_2} \leq \Phi(w) \tag{83}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0,\infty)$ with the condition

$$\lim_{\eta\to\infty}\frac{\Phi(\lambda^{\eta\tau}w)}{\lambda^{2\eta\tau}}=0 \tag{84}$$

for all $w \in \Delta_1$. Then there exists one and only quadratic mapping $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_{Q_2}(w)-\phi(w)\|^p_{\Delta_2} \leq \left(\frac{K^{\eta-1}}{\lambda^{2\beta}}\sum_{\rho=\frac{1-\tau}{2}}^{\infty}\frac{\Phi(\lambda^{\rho\tau}w)}{\lambda^{2\rho\tau}}\right)^p \tag{85}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_{Q2}(w)$ is defined by

$$\Psi_{Q_2}(w)=\lim_{\eta\to\infty}\frac{\Phi(\lambda^{\eta\tau}w)}{\lambda^{2\eta\tau}} \tag{86}$$

for all $w \in \Delta_1$.

Corollary 4.7. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\| \phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right) \right\|_{\Delta_2} \leq \begin{cases} \sigma; \\ \sigma ||w||^\alpha; \ \ \alpha \neq 2 \end{cases} \tag{87}$$

for all $w \in \Delta_1$. Then there exists one and only quadratic mapping $\Psi_{Q_2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\Psi_{Q_2}(w) - \phi(w)\|^p_{\Delta_2} \leq \begin{cases} \left(\frac{K^{\eta-1} \ \sigma}{\lambda^{2\beta-1}|\lambda^2 - 1|} \right)^p; \\ \left(\frac{K^{\eta-1} \ \sigma ||w||^\alpha}{\lambda^{2\beta-1}|\lambda^2 - \lambda^{\alpha\beta}|} \right)^p; \end{cases} \tag{88}$$

for all $w \in \Delta_1$.

Theorem 4.8. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right) \right\|_{\Delta_2} \leq \Phi(w) \tag{89}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ satisfying the conditions (73) and (84) for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quadratic mapping $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w) - \Psi_{Q2}(w)\|^p_{\Delta_2} \leq \left(\frac{K^\eta}{2^\beta} \left\{ \frac{1}{\lambda^\beta} \sum_{\rho = \frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{\rho\tau}} + \frac{1}{\lambda^{2\beta}} \sum_{\rho = \frac{1-\tau}{2}}^{\infty} \frac{\Phi(\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} + \frac{\Phi(-\lambda^{\rho\tau} w)}{\lambda^{2\rho\tau}} \right\} \right)^p \tag{90}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mappings $\Psi_A(w)$ and $\Psi_{Q_2}(w)$ are respectively defined in (75) and (86) for all $w \in \Delta_1$.

Corollary 4.9. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right) \right\|_{\Delta_2} \leq \begin{cases} \sigma; \\ \sigma ||w||^\alpha; \ \ \alpha \neq 1, 2 \end{cases} \tag{91}$$

for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quadratic mapping $\Psi_{Q_2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w) - \Psi_{Q_2}(w)\|^p_{\Delta_2} \leq \begin{cases} \left(\frac{K^\eta \ \sigma}{2^\beta} \left(\frac{1}{\lambda^{\beta-1}|\lambda - 1|} + \frac{1}{\lambda^{2\beta-1}|\lambda^2 - 1|} \right) \right)^p; \\ \left(\frac{K^\eta \ \sigma ||w||^\alpha}{2^\beta} \left(\frac{1}{\lambda^{\beta-1}|\lambda - \lambda^{\alpha\beta}|} + \frac{1}{\lambda^{2\beta-1}|\lambda^2 - \lambda^{\alpha\beta}|} \right) \right)^p; \end{cases} \tag{92}$$

for all $w \in \Delta_1$.

4.3. Hyers Method of (5)

Theorem 4.10. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\|_{\Delta_2} \leq \Phi(w) \tag{93}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta \to \infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{3\eta\tau}} = 0 \tag{94}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_C(w) - \phi(w)\|_{\Delta_2}^p \leq \left(\frac{K^{\eta-1}}{\mu^{3\beta}} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau} w)}{\mu^{3\rho\tau}} \right)^p \tag{95}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_C(w)$ is defined by

$$\Psi_C(w) = \lim_{\eta \to \infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{3\eta\tau}} \tag{96}$$

for all $w \in \Delta_1$.

Corollary 4.11. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an odd function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2} \left(\psi(w) - \psi(-w) \right) - \frac{\mu^4}{2} \left(\psi(w) + \psi(-w) \right) \right\|_{\Delta_2} \leq \begin{cases} \sigma; \\ \sigma ||w||^{\alpha}; \;\; \alpha \neq 3 \end{cases} \tag{97}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_C(w) - \phi(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\frac{K^{\eta-1} \; \sigma}{\mu^{3\beta-1}|\mu^3 - 1|} \right)^p; \\ \left(\frac{K^{\eta-1} \; \sigma ||w||^{\alpha}}{\mu^{3\beta-1}|\mu^3 - \mu^{\alpha\beta}|} \right)^p; \end{cases} \tag{98}$$

for all $w \in \Delta_1$.

Theorem 4.12. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2} \left(\psi(w) - \psi(-w) \right) - \frac{\mu^4}{2} \left(\psi(w) + \psi(-w) \right) \right\|_{\Delta_2} \leq \Phi(w) \tag{99}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ with the condition

$$\lim_{\eta \to \infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{4\eta\tau}} = 0 \tag{100}$$

for all $w \in \Delta_1$. Then there exists one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_{Q4}(w) - \phi(w)\|_{\Delta_2}^p \leq \left(\frac{K^{\eta-1}}{\mu^{4\beta}} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau} w)}{\mu^{4\rho\tau}} \right)^p \tag{101}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mapping $\Psi_{Q_4}(w)$ is defined by

$$\Psi_{Q4}(w) = \lim_{\eta \to \infty} \frac{\Phi(\mu^{\eta\tau} w)}{\mu^{4\eta\tau}} \tag{102}$$

for all $w \in \Delta_1$.

Corollary 4.13. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an even function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\|_{\Delta_2} \leq \begin{cases} \sigma; \\ \sigma ||w||^{\alpha}; \ \ \alpha \neq 4 \end{cases} \tag{103}$$

for all $w \in \Delta_1$. Then there exists one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\Psi_{Q4}(w) - \phi(w)\|^p_{\Delta_2} \leq \begin{cases} \left(\dfrac{K^{\eta-1}\ \sigma}{\mu^{4\beta-1}|\mu^4 - 1|} \right)^p; \\ \left(\dfrac{K^{\eta-1}\ \sigma ||w||^{\alpha}}{\mu^{4\beta-1}|\mu^4 - \mu^{\alpha\beta}|} \right)^p; \end{cases} \tag{104}$$

for all $w \in \Delta_1$.

Theorem 4.14. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\|_{\Delta_2} \leq \Phi(w) \tag{105}$$

for all $w \in \Delta_1$, where $\Phi : \Delta_1 \longrightarrow [0, \infty)$ satisfying the conditions (94) and (84) for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q_4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\begin{aligned} &\|\phi(w) - \Psi_C(w) - \Psi_{Q_4}(w)\|_{\Delta_2} \\ &\leq \left(\frac{K^{\eta}}{2^{\beta}} \left\{ \frac{1}{\mu^{3\beta}} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau}w)}{\mu^{3\rho\tau}} + \frac{\Phi(-\mu^{\rho\tau}w)}{\mu^{3\rho\tau}} + \frac{1}{\mu^{4\beta}} \sum_{\rho=\frac{1-\tau}{2}}^{\infty} \frac{\Phi(\mu^{\rho\tau}w)}{\mu^{4\rho\tau}} + \frac{\Phi(-\mu^{\rho\tau}w)}{\mu^{4\rho\tau}} \right\} \right)^p \end{aligned} \tag{106}$$

for all $w \in \Delta_1$ with $\tau = \pm 1$. The mappings $\Psi_C(w)$ and $\Psi_{Q_4}(w)$ are respectively defined in (96) and (86) for all $w \in \Delta_1$.

Corollary 4.15. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality*

$$\left\| \psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \right\|_{\Delta_2} \leq \begin{cases} \sigma; \\ \sigma ||w||^{\alpha}; \ \ \alpha \neq 3, 4 \end{cases} \tag{107}$$

for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q_4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\|\phi(w) - \Psi_C(w) - \Psi_{Q_4}(w)\|^p_{\Delta_2} \leq \begin{cases} \left(\dfrac{K^{\eta}\ \sigma}{2^{\beta}} \left(\dfrac{1}{\mu^{3\beta-1}|\mu^3 - 1|} + \dfrac{1}{\mu^{4\beta-1}|\mu^4 - 1|} \right) \right)^p; \\ \left(\dfrac{K^{\eta}\ \sigma ||w||^{\alpha}}{2^{\beta}} \left(\dfrac{1}{\mu^{3\beta-1}|\mu^3 - \mu^{\alpha\beta}|} + \dfrac{1}{\mu^{4\beta-1}|\mu^4 - \mu^{\alpha\beta}|} \right) \right)^p; \end{cases} \tag{108}$$

for all $w \in \Delta_1$.

4.4. Radus Method of (4)

Theorem 4.16. *If $\phi : \Delta_1 \rightarrow \Delta_2$ be an odd function satisfying the inequality (72) and $\Phi : \Delta_1^2 \rightarrow [0, \infty)$ be function with the condition*

$$\lim_{\eta \rightarrow \infty} \frac{1}{\theta_j^{\eta}} \Phi\left(\theta_j^{\eta} w\right) = 0 \tag{109}$$

for all $w \in \Delta_1$, where θ_j is defined in (49). If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right),$$

has the property

$$\frac{1}{\theta_j}\ \Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{110}$$

Then there exists a unique additive mapping $\Psi_A : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w)\|^p_{\Delta_2} \leq \left(\left(\frac{L^{1-j}}{1-L}\right)\Phi(w)\right)^p \tag{111}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\|_{\Delta_2} \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr,Ts) \leq Ld(r,s)$, for all $r,s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L. With the help of (110), it follows from (77) for the case $j = 0$ it reduces to

$$\left\|\frac{\phi(\lambda w)}{\lambda} - \phi(w)\right\|_{\Delta_2} \leq \frac{\Phi\left(w\right)}{\lambda^{\beta}} \Rightarrow \left\|\frac{\phi(\lambda w)}{\lambda} - \phi(w)\right\|_{\Delta_2} \leq \frac{\Phi(w)}{\lambda^{\beta}} \Rightarrow d(T\phi,\phi) \leq L < \infty. \tag{112}$$

for all $w \in \Delta_1$.

With the help of (110), it follows from (76) for the case $j = 1$ it reduces to

$$\left\|\phi\left(w\right) - \lambda\ \phi\left(\frac{w}{\lambda}\right)\right\|_{\Delta_2} \leq \Phi\left(\frac{w}{\lambda}\right) \Rightarrow \left\|\phi(w) - \lambda\phi\left(\frac{w}{\lambda}\right)\right\|_{\Delta_2} \leq\ \Phi(w) \Rightarrow d(\phi, T\phi) \leq 1 < \infty. \tag{113}$$

for all $w \in \Delta_1$. Combining the above two cases, we arrive

$$d(\phi, T\phi) \leq L^{1-j}.$$

The rest of the proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 4.16 concerning the some stabilities of (4).

Corollary 4.17. *Let $\phi : \Delta_1 \to \Delta_2$ be an odd mapping satisfying the inequality (81) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique additive function $\Psi_A : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and*

$$\|\phi(w) - \Psi_A(w)\|^p_{\Delta_2} \leq \begin{cases} \left(\dfrac{\sigma}{|\lambda - 1|}\right)^p; \\ \left(\dfrac{\sigma||w||^{\alpha}}{|\lambda - \lambda^{\alpha\beta}|}\right)^p; \end{cases} \tag{114}$$

for all $w \in \Delta_1$.

Proof. If we take

$$\Phi(w) = \begin{cases} \sigma; \\ \sigma||w||^{\alpha};\ \ \alpha \neq 1 \end{cases} \tag{115}$$

for all $w \in \Delta_1$ in Theorem 3.13. Replacing w by $\theta_j^{\eta}\ w$ and dividing by θ_j^{η} in (115) one can see that (109) holds. Now, by definition of $\Phi(w)$ and its property, we have

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right) = \begin{cases} \sigma \\ \dfrac{\sigma}{\lambda^{\alpha\beta}}\ ||w||^{\alpha}, \end{cases}$$

and

$$\frac{1}{\theta_j}\Phi(\theta_j w) = \begin{cases} \dfrac{\sigma}{\theta_j} \\ \dfrac{\sigma}{\lambda^{\alpha\beta}\theta_j}||\theta_j\ w||^{\alpha}, \end{cases} = \begin{cases} \theta_j^{-1}\ \Phi(w), \\ \\ \theta_j^{\alpha\beta-1}\ \Phi(w), \end{cases}$$

for all $w \in \Delta_1$. Hence the property (110) and the inequality (111) holds for the following:

$$\begin{aligned}
L = \theta_j^{-1} &= \lambda^{-1} \quad for \quad j=0\\
\|\phi(w) - \Psi_A(w)\|_{\Delta_2} &\leq \left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w) = \left(\frac{\lambda^{-1}}{1-\lambda^{-1}}\ \right)\ \sigma = \left(\frac{\sigma}{\lambda-1}\right)\\
L = \theta_j^{-1} &= \frac{1}{\lambda^{-1}} = \lambda \quad for \quad j=1\\
\|\phi(w) - \Psi_A(w)\|_{\Delta_2} &\leq \left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w) = \left(\frac{1}{1-\lambda}\ \right)\ \sigma = \left(\frac{\sigma}{1-\lambda}\ \right)\\
L = \theta_j^{\alpha\beta-1} &= \lambda^{\alpha\beta-1} \quad for \quad j=0\\
\|\phi(w) - \Psi_A(w)\|_{\Delta_2} &\leq \left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w) = \left(\frac{\lambda^{\alpha\beta-1}}{1-\lambda^{\alpha\beta-1}}\ \right)\ \frac{\sigma}{\lambda^{\alpha\beta}}\ ||w||^{\alpha} = \frac{\sigma}{\lambda-\lambda^{\alpha\beta}}\ ||w||^{\alpha}\\
L = \theta_j^{\alpha\beta-1} &= \frac{1}{\lambda^{\alpha\beta-1}} = \lambda^{1-\alpha\beta} \quad for \quad j=1\\
\|\phi(w) - \Psi_A(w)\|_{\Delta_2} &\leq \left(\frac{L^{1-j}}{1-L}\right)\ \Phi(w) = \left(\frac{1}{1-\lambda^{1-\alpha\beta}}\ \right)\ \frac{\sigma}{\lambda^{\alpha\beta}}\ ||w||^{\alpha} = \frac{\sigma}{\lambda^{\alpha\beta}-\lambda}\ ||w||^{\alpha}
\end{aligned}$$

Hence the proof is complete. □

Theorem 4.18. *If $\phi : \Delta_1 \to \Delta_2$ be an even function satisfying the inequality (83) and $\Phi : \Delta_1^2 \to [0, \infty)$ be function with the condition*

$$\lim_{\eta\to\infty} \frac{1}{\theta_j^{2\eta}}\Phi\left(\theta_j^{\eta} w\right) = 0 \tag{116}$$

for all $w \in \Delta_1$, where θ_j is defined in (49). If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right),$$

has the property

$$\frac{1}{\theta_j^2}\ \Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{117}$$

Then there exists a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_{Q2}(w)\|_{\Delta_2}^{p} \leq \left(\left(\frac{L^{1-j}}{1-L}\right)\Phi(w)\right)^{p} \tag{118}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\|_{\Delta_2} \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^2} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.15 concerning the some stabilities of (4).

Corollary 4.19. *Let $\phi : \Delta_1 \to \Delta_2$ be an even mapping satisfying (87) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique quadratic function $\Psi_{Q2} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and*

$$\|\phi(w) - \Psi_A(w) - \Psi_{Q2}(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\dfrac{\sigma}{|\lambda^2 - 1|}\right)^p; \\ \left(\dfrac{\sigma||w||^\alpha}{|\lambda^2 - \lambda^{\alpha\beta}|}\right)^p; \end{cases} \tag{119}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 4.20. *If $\phi : \Delta_1 \to \Delta_2$ be a function satisfying the inequality (89) and $\Phi : \Delta_1^2 \to [0,\infty)$ be function with the conditions (109) and (116) for all $w \in \Delta_1$, where θ_j is defined in (49). If there exists $L = L(j)$ such that the function*

$$\Phi(w) = \Phi\left(\frac{w}{\lambda}\right),$$

has the properties (110) and (117). Then there exists a unique additive mapping $\Psi_A : \Delta_1 \to \Delta_2$ and a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\phi(w) - \Psi_A(w)\|_{\Delta_2}^p \leq \left(\frac{K}{2^\beta}\left(\frac{L^{1-j}}{1-L}\right)[\Phi(w) + \Phi(-w)]\right)^p \tag{120}$$

for all $w \in \Delta_1$.

Proof. The proof is similar lines to that of Theorem 4.8. □

Corollary 4.21. *Assume σ and α be positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality (91) for all $w \in \Delta_1$. Then there exists one and only additive mapping $\Psi_A(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quadratic mapping $\Psi_{Q2}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and the inequality*

$$\|\phi(w) - \Psi_A(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\dfrac{K}{2^\beta}\left(\dfrac{\sigma}{|\lambda - 1|} + \dfrac{\sigma}{|\lambda^2 - 1|}\right)\right)^p; \\ \left(\dfrac{K}{2^\beta}\left(\dfrac{\sigma||w||^\alpha}{|\lambda^2 - \lambda^{\alpha\beta}|} + \dfrac{\sigma||w||^\alpha}{|\lambda - \lambda^{\alpha\beta}|}\right)\right)^p; \end{cases} \tag{121}$$

for all $w \in \Delta_1$.

4.5. Radus Method of (5)

Theorem 4.22. *If $\psi : \Delta_1 \to \Delta_2$ be a function satisfying the inequality (93) and $\Phi : \Delta_1^2 \to [0,\infty)$ be a function with the condition*

$$\lim_{\eta\to\infty} \frac{1}{\theta_j^{3\eta}} \Phi\left(\theta_j^\eta w\right) = 0 \tag{122}$$

for all $w \in \Delta_1$, where θ_j is defined in (63). If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the property

$$\frac{1}{\theta_j^3}\ \Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{123}$$

Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \to \Delta_2$ satisfying the functional equation (5) and

$$\|\psi(w) - \Psi_C(w)\|_{\Delta_2}^p \leq \left(\left(\frac{L^{1-j}}{1-L}\right)\Phi(w)\right)^p \tag{124}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\|_{\Delta_2} \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^3} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr,Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.19 concerning the some stabilities of (5).

Corollary 4.23. *Let $\psi : \Delta_1 \to \Delta_2$ be an odd mapping satisfying (97) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique cubic function $\Psi_C : \Delta_1 \to \Delta_2$ satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_C(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\dfrac{\sigma}{|\mu^3 - 1|}\right)^p; \\ \left(\dfrac{\sigma||w||^\alpha}{|\mu^3 - \mu^{\alpha\beta}|}\right)^p; \end{cases} \tag{125}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 4.24. *If $\psi : \Delta_1 \to \Delta_2$ be an even function satisfying the inequality (99) and $\Phi : \Delta_1^2 \to [0,\infty)$ be a function with the condition*

$$\lim_{\eta\to\infty} \frac{1}{\theta_j^{4\eta}}\Phi\left(\theta_j^\eta w\right) = 0 \tag{126}$$

for all $w \in \Delta_1$, where θ_j is defined in (63). If there exists $L = L(j)$ such that the function

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the property

$$\frac{1}{\theta_j^4}\ \Phi\left(\theta_j w\right) = L\ \Phi(w). \tag{127}$$

Then there exists a unique quartic mapping $\Psi_{Q4} : \Delta_1 \to \Delta_2$ satisfying the functional equation (5) and

$$\|\psi(w) - \Psi_{Q4}(w)\|_{\Delta_2}^p \leq \left(\left(\frac{L^{1-j}}{1-L}\right)\Phi(w)\right)^p \tag{128}$$

for all $w \in \Delta_1$.

Proof. Define a set $\mathcal{X} = \{r/r : \Delta_1 \to \Delta_2,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathcal{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\|_{\Delta_2} \leq K\Phi(w), w \in \Delta_1\}$. It is easy to see that $(\mathcal{X}, d)$ is complete. Also, define a function $T : \mathcal{X} \to \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^4} r(\theta_j w)$, for all $w \in \Delta_1$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r, s \in \mathcal{X}$. i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L The rest of The proof is similar lines to that of Theorem 3.13. □

The following corollary is an immediate consequence of Theorem 3.21 concerning the some stabilities of (5).

Corollary 4.25. *Let $\psi : \Delta_1 \to \Delta_2$ be an even mapping satisfying (103) and there exists real numbers σ and α such that for all $w \in \Delta_1$. Then there exists a unique quartic function $\Psi_{Q4} : \Delta_1 \to \Delta_2$ satisfying the functional equation (5) and*

$$\|\psi(w) - \Psi_{Q4}(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\dfrac{\sigma}{|\mu^4 - 1|}\right)^p; \\ \left(\dfrac{\sigma ||w||^\alpha}{|\mu^4 - \mu^{\alpha\beta}|}\right)^p; \end{cases} \tag{129}$$

for all $w \in \Delta_1$.

Proof. The proof of the corollary is similar to that of Corollary 3.14 □

Theorem 4.26. *If $\psi : \Delta_1 \to \Delta_2$ and $\Phi : \Delta_1^2 \to [0,\infty)$ are functions satisfying the inequality (105) and the conditions (122) and (126) for all $w \in \Delta_1$, where θ_j is defined in (63). If there exists $L = L(j)$ such that the function*

$$\Phi(w) = \Phi\left(\frac{w}{\mu}\right),$$

has the properties (123) and (127). Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \to \Delta_2$ and a unique quartic mapping $\Psi_{Q4} : \Delta_1 \to \Delta_2$ satisfying the functional equation (4) and

$$\|\psi(w) - \Psi_C(w) - \Psi_{Q4}(w)\|_{\Delta_2} \leq \left(\frac{K}{2^\beta}\left(\frac{L^{1-j}}{1-L}\right)[\Phi(w) + \Phi(-w)]\right)^p \tag{130}$$

for all $w \in \Delta_1$.

Proof. The proof is similar lines to that of Theorem 4.8. □

Corollary 4.27. *Assume σ and α be positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be a function fulfilling the inequality (107) for all $w \in \Delta_1$. Then there exists one and only cubic mapping $\Psi_C(w) : \Delta_1 \longrightarrow \Delta_2$ and one and only quartic mapping $\Psi_{Q4}(w) : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and the inequality*

$$\|\phi(w) - \Psi_C(w) - \Psi_{Q4}(w)\|_{\Delta_2}^p \leq \begin{cases} \left(\dfrac{K}{2^\beta}\left(\dfrac{\sigma}{|\mu^3 - 1|} + \dfrac{\sigma}{|\mu^4 - 1|}\right)\right)^p; \\ \left(\dfrac{K}{2^\beta}\left(\dfrac{\sigma||w||^\alpha}{|\mu^3 - \mu^{\alpha\beta}|} + \dfrac{\sigma||w||^\alpha}{|\mu^4 - \mu^{\alpha\beta}|}\right)\right)^p; \end{cases} \tag{131}$$

for all $w \in \Delta_1$.

5. Stability In Intuitionistic Fuzzy Banach Space

In this section, we scrutinize the generalized Ulam - Hyers stability of the functional equations (4) and (5) in Intuitionistic Fuzzy Banach Space. To prove stability results, let us take Δ_1 be an Intuitionistic Fuzzy normed space and Δ_2 be an Intuitionistic Fuzzy Banach space.

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space given in [12].

5.1. Definitions and Notations of Intuitionistic Fuzzy Banach Space

Definition 5.1. *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is said to be continuous* t*-norm if* $*$ *satisfies the following conditions:*

(1). $*$ *is commutative and associative;*

(2). $*$ *is continuous;*

(3). $a * 1 = a$ *for all* $a \in [0,1]$*;*

(4). $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 5.2. *A binary operation* $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ *is said to be continuous* t*-conorm if* $\diamond$ *satisfies the following conditions:*

(1'). $\diamond$ *is commutative and associative;*

(2'). $\diamond$ *is continuous;*

(3'). $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*

(4'). $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [12] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 5.3. *The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamond$ *is a continuous* $t-$ *conorm, and* μ, ν *are fuzzy sets on* $X \times (0,\infty)$ *satisfying the following conditions. For every* $x, y \in X$ *and* $s, t > 0$

(IFN1) $\mu(x,t) + \nu(x,t) \leq 1,$

(IFN2) $\mu(x,t) > 0,$

(IFN3) $\mu(x,t) = 1,$ *if and only if* $x = 0.$

(IFN4) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{\alpha}\right)$ *for each* $\alpha \neq 0,$

(IFN5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s),$

(IFN6) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ *is continuous,*

(IFN7) $\lim\limits_{t\to\infty} \mu(x,t) = 1$ *and* $\lim\limits_{t\to 0} \mu(x,t) = 0,$

(IFN8) $\nu(x,t) < 1,$

(IFN9) $\nu(x,t) = 0,$ *if and only if* $x = 0.$

(IFN10) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{\alpha}\right)$ *for each* $\alpha \neq 0,$

(IFN11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s),$

(IFN12) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ *is continuous,*

(IFN13) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case, (μ,ν) is called an intuitionistic fuzzy norm.

Example 5.4. *Let $(X, \|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider*

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0; \end{cases} \qquad and \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFN-space.

Definition 5.5. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L \in X$ if*

$$\lim \quad \mu(x_k - L, t) = 1 \quad and \quad \lim \quad \nu(x_k - L, t) = 0$$

for all $t > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty$$

Definition 5.6. *Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\left(x_{k+p} - x_k, t\right) = 1 \qquad and \qquad \nu\left(x_{k+p} - x_k, t\right) = 0$$

for all $t > 0$, and $p = 1, 2 \cdots$.

Definition 5.7. *Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.*

5.2. Hyers Method of (4)

Theorem 5.8. *Let $\phi : \Delta_1 \to \Delta_2$ be an odd function satisfying the inequality*

$$\left.\begin{array}{l} \mu\left(\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \ge \mu'\left(\Phi\left(w\right), t\right) \\ \nu\left(\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \le \nu'\left(\Phi\left(w\right), t\right) \end{array}\right\} \tag{132}$$

for all $w \in \Delta_1$ and all $t > 0$ where $\Phi : \Delta_1 \longrightarrow (0,1]$ with the conditions

$$\left.\begin{array}{l} \mu'\left(\Phi\left(\lambda^{\eta\tau} w\right), t\right) \ge \mu'\left(\omega^{\eta\tau} \Phi\left(w\right), t\right) \\ \nu'\left(\Phi\left(\lambda^{\eta\tau} w\right), t\right) \le \nu'\left(\omega^{\eta\tau} \Phi\left(w\right), t\right) \end{array}\right\} \tag{133}$$

and

$$\left.\begin{array}{l} \lim_{n\to\infty} \mu'\left(\Phi\left(\lambda^{\eta\tau} w\right), \lambda^{\eta\tau} t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Phi\left(\lambda^{\eta\tau} w\right), \lambda^{\eta\tau} t\right) = 0 \end{array}\right\} \tag{134}$$

for all $w \in \Delta_1$ and all $t > 0$ with

$$0 < \left(\frac{p}{\lambda}\right)^{\eta} < 1.$$

Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_A(w), t\right) \ge \mu'\left(\Phi\left(w\right), |\lambda - \omega| t\right) \\ \nu\left(\phi(w) - \Psi_A(w), t\right) \le \nu'\left(\Phi\left(w\right), |\lambda - \omega| t\right) \end{array}\right\} \tag{135}$$

for all $w \in \Delta_1$ and all $t > 0$ with $\tau \ne 1$.

Proof. **Case (i):** Let $\tau = 1$. Using oddness of ϕ in (132), we reach

$$\left.\begin{aligned} \mu\left(\phi(\lambda w) - \lambda\phi\left(w\right), t\right) \geq \mu'\left(\Phi\left(w\right), t\right) \\ \nu\left(\phi(\lambda w) - \lambda\phi\left(w\right), t\right) \leq \nu'\left(\Phi\left(w\right), t\right) \end{aligned}\right\} \tag{136}$$

for all $w \in \Delta_1$ and all $t > 0$. Using (IFN4) and (IFN10) in (136), can be rewritten as

$$\left.\begin{aligned} \mu\Big(\frac{\phi(\lambda w)}{\lambda} - \phi(w), \frac{t}{\lambda}\Big) \geq \mu'\left(\Phi(w), t\right) \\ \nu\Big(\frac{\phi(\lambda w)}{\lambda} - \phi(w), \frac{t}{\lambda}\Big) \leq \nu'\left(\Phi(w), t\right) \end{aligned}\right\} \tag{137}$$

for all $w \in \Delta_1$ and all $t > 0$. Substituting w by $\lambda^\eta w$ in (137), we arrive

$$\left.\begin{aligned} \mu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda} - \phi(\lambda^\eta w), \frac{t}{\lambda}\Big) \geq \mu'\left(\Phi(\lambda^\eta w), t\right) \\ \nu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda} - \phi(\lambda^\eta w), \frac{t}{\lambda}\Big) \leq \nu'\left(\Phi(\lambda^\eta w), t\right) \end{aligned}\right\} \tag{138}$$

for all $w \in \Delta_1$ and all $t > 0$. Using (IFN4), (IFN10) and (133) in (138), we have

$$\left.\begin{aligned} \mu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda^{(\eta+1)}} - \frac{\phi(\lambda^\eta w)}{\lambda^\eta}, \frac{t}{\lambda\cdot\lambda^\eta}\Big) \geq \mu'\left(\Phi(w), \frac{t}{\omega^\eta}\right) \\ \nu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda^{(\eta+1)}} - \frac{\phi(\lambda^\eta w)}{\lambda^\eta}, \frac{t}{\lambda\cdot\lambda^\eta}\Big) \leq \nu'\left(\Phi(w), \frac{t}{\omega^\eta}\right) \end{aligned}\right\} \tag{139}$$

for all $w \in \Delta_1$ and all $t > 0$. Interchanging t into $\omega^\eta t$ in (139), we obtain

$$\left.\begin{aligned} \mu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda^{(\eta+1)}} - \frac{\phi(\lambda^\eta w)}{\lambda^\eta}, \frac{t\cdot\omega^\eta}{\lambda\cdot\lambda^\eta}\Big) \geq \mu'\left(\Phi(w), t\right) \\ \nu\Big(\frac{\phi(\lambda^{\eta+1} w)}{\lambda^{(\eta+1)}} - \frac{\phi(\lambda^\eta w)}{\lambda^\eta}, \frac{t\cdot\omega^\eta}{\lambda\cdot\lambda^\eta}\Big) \leq \nu'\left(\Phi(w), t\right) \end{aligned}\right\} \tag{140}$$

for all $w \in \Delta_1$ and all $t > 0$. It is easy to see that

$$\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w) = \sum_{\rho=0}^{\eta-1} \frac{\phi(\lambda^{\rho+1} w)}{\lambda^{(\rho+1)}} - \frac{\phi(\lambda^\rho w)}{\lambda^\rho} \tag{141}$$

for all $w \in \Delta_1$. It follows from (140) and (141), we get

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) = \mu\left(\sum_{\rho=0}^{\eta-1}\frac{\phi(\lambda^{\rho+1} w)}{\lambda^{(\rho+1)}} - \frac{\phi(\lambda^\rho w)}{\lambda^\rho}, \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) \\ \nu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) = \nu\left(\sum_{\rho=0}^{\eta-1}\frac{\phi(\lambda^{\rho+1} w)}{\lambda^{(\rho+1)}} - \frac{\phi(\lambda^\rho w)}{\lambda^\rho}, \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) \end{aligned}\right\} \tag{142}$$

for all $w \in \Delta_1$ and all $t > 0$. Using (IFNS5) and (IFNA11) in (142), we reach

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho t}{\lambda\cdot\lambda^\rho}\right) \geq \prod_{\rho=0}^{\eta-1}\mu\left(\frac{\phi(\lambda^{\rho+1} w)}{\lambda^{(\rho+1)}} - \frac{\phi(\lambda^\rho w)}{\lambda^\rho}, \frac{\omega^\rho\ tr}{\lambda\cdot\lambda^\rho}\right) \\ \nu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1}\frac{\omega^\rho t}{\lambda\cdot\lambda^\rho}\right) \leq \coprod_{\rho=0}^{\eta-1}\nu\left(\frac{\phi(\lambda^{\rho+1} w)}{\lambda^{(\rho+1)}} - \frac{\phi(\lambda^\rho w)}{\lambda^\rho}, \frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) \end{aligned}\right\} \tag{143}$$

where

$$\prod_{\rho=0}^{\eta-1} c_j = c_1 * c_2 * \cdots * c_n \qquad \textit{and} \qquad \coprod_{\rho=0}^{\eta-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_n$$

for all $w \in \Delta_1$ and all $t > 0$. Hence

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1} \frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) \geq \prod_{\rho=0}^{\eta-1} \mu'\left(\Phi(w), t\right) = \mu'\left(\Phi(w), t\right) \\ \nu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), \sum_{\rho=0}^{\eta-1} \frac{\omega^\rho\ t}{\lambda\cdot\lambda^\rho}\right) \leq \prod_{\rho=0}^{\eta-1} \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi(w), t\right) \end{aligned}\right\} \tag{144}$$

for all $w \in \Delta_1$ and all $t > 0$. Replacing w by $\lambda^\kappa w$ in (144) and using (134), (IFN4), (IFN10), we achieve

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, \sum_{\rho=0}^{\eta-1} \frac{\omega^\rho\ t}{\lambda\cdot\lambda^{(\rho+\kappa)}}\right) \geq \mu'\left(\Phi(\lambda^\kappa w), t\right) = \mu'\left(\Phi(w), \frac{t}{\omega^\kappa}\right) \\ \nu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, \sum_{\rho=0}^{\eta-1} \frac{\omega^\rho\ t}{\lambda\cdot\lambda^{(\rho+\kappa)}}\right) \leq \nu'\left(\Phi(\lambda^\kappa w), t\right) = \nu'\left(\Phi(w), \frac{t}{\omega^\kappa}\right) \end{aligned}\right\} \tag{145}$$

for all $w \in \Delta_1$ and all $t > 0$ and all $\kappa, \eta \geq 0$. Replacing t by $\omega^\kappa t$ in (145), we get

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, \sum_{\rho=0}^{\eta-1} \frac{\omega^{\rho+\kappa}\ t}{\lambda\cdot\lambda^{(\rho+\kappa)}}\right) \geq \mu'\left(\Phi(w), t\right) \\ \nu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, \sum_{\rho=0}^{\eta-1} \frac{\omega^{\rho+\kappa}\ t}{\lambda\cdot\lambda^{(\rho+\kappa)}}\right) \leq \nu'\left(\Phi(w), t\right) \end{aligned}\right\} \tag{146}$$

for all $w \in \Delta_1$ and all $t > 0$ and all $\kappa, \eta \geq 0$. The relation (146) implies that

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, t\right) \geq \mu'\left(\Phi(w), \frac{t}{\sum_{\rho=\kappa}^{\eta-1} \frac{\omega^\rho}{\lambda\cdot\lambda^\rho}}\right) \\ \nu\left(\frac{\phi(\lambda^{\eta+\kappa} w)}{\lambda^{(\eta+\kappa)}} - \frac{\phi(\lambda^\kappa w)}{\lambda^\kappa}, t\right) \leq \nu'\left(\Phi(w), \frac{t}{\sum_{\rho=\kappa}^{\eta-1} \frac{\omega^\rho}{\lambda\cdot\lambda^\rho}}\right) \end{aligned}\right\} \tag{147}$$

holds for all $w \in \Delta_1$ and all $t > 0$ and all $\kappa, \eta \geq 0$. Since $0 < \omega < 1$ and $\sum_{\rho=0}^{\eta} \left(\frac{\omega}{\lambda}\right)^\rho < \infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{\phi(\lambda^\eta w)}{\lambda^\eta}\right\}$ is Cauchy in (Δ_2, μ, ν). Since (Δ_2, μ, ν) is a complete IFN-space this sequence converges to some point $\Psi_A(w) \in \Delta_2$. So, one can define the mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ by

$$\lim_{n\to\infty} \mu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \Psi_A(w), t\right) = 1, \lim_{n\to\infty} \nu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \Psi_A(w), t\right) = 0$$

for all $w \in \Delta_1$ and all $t > 0$. Hence

$$\frac{\phi(\lambda^\eta w)}{\lambda^\eta} \xrightarrow{IF} \Psi_A(w), \quad as \quad n \to \infty.$$

Letting $\kappa = 0$ in (147), we arrive

$$\left.\begin{aligned} \mu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), t\right) \geq \mu'\left(\Phi(w), \frac{t}{\sum_{\rho=0}^{\eta-1} \frac{\omega^\rho}{\lambda\cdot\lambda^\rho}}\right) \\ \nu\left(\frac{\phi(\lambda^\eta w)}{\lambda^\eta} - \phi(w), t\right) \leq \nu'\left(\Phi(w), \frac{t}{\sum_{\rho=0}^{\eta-1} \frac{\omega^\rho}{\lambda\cdot\lambda^\rho}}\right) \end{aligned}\right\} \tag{148}$$

for all $w \in \Delta_1$ and all $t > 0$. Letting $n \to \infty$ in (148), we arrive

$$\left.\begin{aligned} \mu\left(\Psi_A(w) - \phi(w), t\right) \geq \mu'\left(\Phi(w), t(\lambda - \omega)\right) \\ \nu\left(\Psi_A(w) - \phi(w), t\right) \leq \nu'\left(\Phi(w), t(\lambda - \omega)\right) \end{aligned}\right\} \tag{149}$$

for all $w \in \Delta_1$ and all $t > 0$. To prove Ψ_A satisfies (4), replacing w by $\lambda^\eta w$ in (132), we obtain

$$\left.\begin{array}{l} \mu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},t\right)\geq\mu'\left(\Phi(\lambda^\eta w),\lambda^\eta t\right)\\ \nu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},t\right)\leq\nu'\left(\Phi(\lambda^\eta w),\lambda^\eta t\right)\end{array}\right\} \tag{150}$$

for all $w \in \Delta_1$ and all $t > 0$. Also,

$$\left.\begin{array}{l} \lim\limits_{\eta\to\infty}\mu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},t\right)=1\\ \lim\limits_{\eta\to\infty}\nu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},t\right)=0\end{array}\right\} \tag{151}$$

for all $w \in \Delta_1$ and all $t > 0$. Now,

$$\begin{aligned} &\mu\left(\Psi_A(\lambda w)-\frac{\lambda}{2}\left(\Psi_A(w)-\Psi_A(-w)\right)-\frac{\lambda^2}{2}\left(\Psi_A(w)+\Psi_A(-w)\right),t\right)\\ &\geq\mu\Big(\Psi_A(\lambda w)\frac{1}{\lambda^\eta}\phi(\lambda w),\frac{t}{4}\Big)*\\ &\quad\mu\Big(-\frac{\lambda}{2}\left(\Psi_A(w)-\Psi_A(-w)\right)+\frac{\lambda}{2}\frac{1}{\lambda^\eta}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right),\frac{t}{4}\Big)*\\ &\quad\mu\Big(-\frac{\lambda^2}{2}\left(\Psi_A(w)+\Psi_A(-w)\right)+\frac{\lambda^2}{2}\frac{1}{\lambda^\eta}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right),\frac{t}{4}\Big)*\\ &\quad\mu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},\frac{t}{4}\right) \end{aligned} \tag{152}$$

and

$$\begin{aligned} &\nu\left(\Psi_A(\lambda w)-\frac{\lambda}{2}\left(\Psi_A(w)-\Psi_A(-w)\right)-\frac{\lambda^2}{2}\left(\Psi_A(w)+\Psi_A(-w)\right),t\right)\\ &\geq\nu\Big(\Psi_A(\lambda w)\frac{1}{\lambda^\eta}\phi(\lambda w),\frac{t}{4}\Big)*\\ &\quad\nu\Big(-\frac{\lambda}{2}\left(\Psi_A(w)-\Psi_A(-w)\right)+\frac{\lambda}{2}\frac{1}{\lambda^\eta}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right),\frac{t}{4}\Big)*\\ &\quad\nu\Big(-\frac{\lambda^2}{2}\left(\Psi_A(w)+\Psi_A(-w)\right)+\frac{\lambda^2}{2}\frac{1}{\lambda^\eta}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right),\frac{t}{4}\Big)*\\ &\quad\nu\left(\frac{1}{\lambda^\eta}\left\{\phi(\lambda^\eta\cdot\lambda w)-\frac{\lambda}{2}\left(\phi(\lambda^\eta w)-\phi(-\lambda^\eta w)\right)-\frac{\lambda^2}{2}\left(\phi(\lambda^\eta w)+\phi(-\lambda^\eta w)\right)\right\},\frac{t}{4}\right) \end{aligned} \tag{153}$$

for all $w \in \Delta_1$ and all $t > 0$. Letting $n \to \infty$ in (152), (153) and using (151), we can see that Ψ_A satisfies the functional equation (4). In order to prove the existence of $\Psi_A(w)$ is unique, let $\Psi_B(w)$ be another additive functional equation satisfying (4) and (135). Hence,

$$\begin{aligned} \mu(\Psi_A(w)-\Psi_B(w),2t)&\geq\mu\left(\Psi_A(\lambda^\eta w)-\phi(\lambda^\eta w),\frac{t.\lambda^\eta}{\lambda}\right)*\mu\left(\phi(\lambda^\eta w)-\Psi_B(\lambda^\eta w),\frac{t.\lambda^\eta}{\lambda}\right)\\ &\geq\mu'\left(\Phi(\lambda^\eta w),\frac{t\ \lambda^\eta|\lambda-\omega|}{\lambda}\right)\geq\mu'\left(\Phi(w),\frac{t\ \lambda^\eta|\lambda-\omega|}{\omega^\eta}\right)\\ \nu(\Psi_A(w)-\Psi_B(w),2t)&\leq\nu\left(\Psi_A(\lambda^\eta w)-\phi(\lambda^\eta w),\frac{t.\lambda^\eta}{\lambda}\right)\diamond\nu\left(\phi(\lambda^\eta w)-\Psi_B(\lambda^\eta w),\frac{t.\lambda^\eta}{\lambda}\right)\\ &\leq\nu'\left(\Phi(\lambda^\eta w),\frac{t\ \lambda^\eta|\lambda-\omega|}{\lambda}\right)\leq\nu'\left(\Phi(w),\frac{t\ \lambda^\eta|\omega-\omega|}{\omega^\eta}\right) \end{aligned}$$

for all $w \in \Delta_1$ and all $t > 0$. Since $\lim\limits_{\eta\to\infty}\dfrac{t\ \lambda^\eta|\lambda-\omega|}{\omega^\eta}=\infty$, we obtain

$$\left.\begin{array}{l} \lim\limits_{\eta\to\infty}\mu'\left(\Phi(w),\frac{t\ \lambda^\eta|\lambda-\omega|}{\omega^\eta}\right)=1\\ \lim\limits_{\eta\to\infty}\nu'\left(\Phi(w),\frac{t\ \lambda^\eta|\lambda-\omega|}{\omega^\eta}\right)=0\end{array}\right\}$$

for all $w \in \Delta_1$ and all $t > 0$. Thus

$$\left.\begin{aligned} \mu(\Psi_A(w) - \Psi_B(w), t) = 1 \\ \nu(\Psi_A(w) - \Psi_B(w), t) = 0 \end{aligned}\right\}$$

for all $w \in \Delta_1$ and all $t > 0$. Hence, $\Psi_A(w) = \Psi_B(w)$. Therefore, $\Psi_A(w)$ is unique.

Case 2: For $\tau = -1$. Putting w by $\frac{w}{\lambda}$ in (136), we get

$$\left.\begin{aligned} \mu\left(\phi(w) - \lambda f\left(\tfrac{w}{\lambda}\right), t\right) \geq \mu'\left(\Phi\left(\tfrac{w}{\lambda}\right), t\right) \\ \nu\left(\phi(w) - \lambda f\left(\tfrac{w}{\lambda}\right), t\right) \leq \nu'\left(\Phi\left(\tfrac{w}{\lambda}\right), t\right) \end{aligned}\right\} \tag{154}$$

for all $w \in \Delta_1$ and all $t > 0$. The rest of the proof is similar to that of Case 1. This completes the proof. □

The following corollary is an immediate consequence of Theorem 5.8, regarding the stability of (4)

Corollary 5.9. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ an odd function fulfilling the inequality*

$$\left.\begin{aligned} \mu\left(\phi(\lambda w) - \tfrac{\lambda}{2}(\phi(w) - \phi(-w)) - \tfrac{\lambda^2}{2}(\phi(w) + \phi(-w)), t\right) \geq \begin{cases} \mu'(\sigma, t), \\ \mu'(\sigma|w|^\alpha, t), \end{cases} \\ \nu\left(\phi(\lambda w) - \tfrac{\lambda}{2}(\phi(w) - \phi(-w)) - \tfrac{\lambda^2}{2}(\phi(w) + \phi(-w)), t\right) \leq \begin{cases} \nu'(\sigma, t), \\ \nu'(\sigma|w|^\alpha, t), \end{cases} \end{aligned}\right\} \tag{155}$$

for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 1$. Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{aligned} \mu(\phi(w) - \Psi_A(w), t) \geq \begin{cases} \mu'(\sigma, |\lambda - 1|t), \\ \mu'(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t), \end{cases} \\ \nu(\phi(w) - \Psi_A(w), t) \leq \begin{cases} \nu'(\sigma, |\lambda - 1|t), \\ \nu'(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t), \end{cases} \end{aligned}\right\} \tag{156}$$

for all $w \in \Delta_1$ and all $t > 0$.

The proof of the following theorem and corollary is similar clues that of Theorem 5.8 and Corollary 5.9. Hence the details of the proof are omitted.

Theorem 5.10. *Let $\phi : \Delta_1 \to \Delta_2$ be an even function satisfying the inequality*

$$\left.\begin{aligned} \mu\left(\phi(\lambda w) - \tfrac{\lambda}{2}(\phi(w) - \phi(-w)) - \tfrac{\lambda^2}{2}(\phi(w) + \phi(-w)), t\right) \geq \mu'(\Phi(w), t) \\ \nu\left(\phi(\lambda w) - \tfrac{\lambda}{2}(\phi(w) - \phi(-w)) - \tfrac{\lambda^2}{2}(\phi(w) + \phi(-w)), t\right) \leq \nu'(\Phi(w), t) \end{aligned}\right\} \tag{157}$$

for all $w \in \Delta_1$ and all $t > 0$ where $\Phi : \Delta_1 \longrightarrow (0, 1]$ with the conditions

$$\left.\begin{aligned} \mu'\left(\Phi\left(\lambda^{\eta\tau} w\right), t\right) \geq \mu'\left(\omega^{\eta\tau}\Phi(w), t\right) \\ \nu'\left(\Phi\left(\lambda^{\eta\tau} w\right), t\right) \leq \nu'\left(\omega^{\eta\tau}\Phi(w), t\right) \end{aligned}\right\} \tag{158}$$

and

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'\left(\Phi\left(\lambda^{\eta\tau}w\right), \lambda^{2\eta\tau}t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Phi\left(\lambda^{\eta\tau}w\right), \lambda^{2\eta\tau}t\right) = 0 \end{aligned}\right\} \tag{159}$$

for all $w \in \Delta_1$ *and all* $t > 0$ *with*

$$0 < \left(\frac{p}{\lambda}\right)^{2\eta} < 1.$$

Then there exists a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\left.\begin{aligned} \mu\left(\phi(w) - \Psi_{Q2}(w), t\right) \geq \mu'\left(\Phi\left(w\right), |\lambda^2 - \omega|t\right) \\ \nu\left(\phi(w) - \Psi_{Q2}(w), t\right) \leq \nu'\left(\Phi\left(w\right), |\lambda^2 - \omega|t\right) \end{aligned}\right\} \tag{160}$$

for all $w \in \Delta_1$ *and all* $t > 0$ *with* $\tau \neq 1$.

Corollary 5.11. *Assume* σ *and* α *are positive numbers. Let* $\phi : \Delta_1 \longrightarrow \Delta_2$ *an even function fulfilling the inequality*

$$\left.\begin{aligned} \mu\left(\phi(\lambda w) - \tfrac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \tfrac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \geq \begin{cases} \mu'\left(\sigma, t\right), \\ \mu'\left(\sigma|w|^\alpha, t\right), \end{cases} \\ \nu\left(\phi(\lambda w) - \tfrac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \tfrac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \leq \begin{cases} \nu'\left(\sigma, t\right), \\ \nu'\left(\sigma|w|^\alpha, t\right), \end{cases} \end{aligned}\right\} \tag{161}$$

for all $w \in \Delta_1$ *and all* $t > 0$*, with* $\sigma > 0$ *and* $\alpha \neq 2$*. Then there exists a unique quadratic mapping* $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\left.\begin{aligned} \mu\left(\phi(w) - \Psi_{Q2}(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\lambda^2 - \lambda^\alpha|t\right), \end{cases} \\ \nu\left(\phi(w) - \Psi_{Q2}(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\lambda^2 - \lambda^\alpha|t\right), \end{cases} \end{aligned}\right\} \tag{162}$$

for all $w \in \Delta_1$ *and all* $t > 0$.

Theorem 5.12. *Let* $\phi : \Delta_1 \to \Delta_2$ *be a function satisfying the inequality*

$$\left.\begin{aligned} \mu\left(\phi(\lambda w) - \tfrac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \tfrac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \geq \mu'\left(\Phi\left(w\right), t\right) \\ \nu\left(\phi(\lambda w) - \tfrac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \tfrac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \leq \nu'\left(\Phi\left(w\right), t\right) \end{aligned}\right\} \tag{163}$$

for all $w \in \Delta_1$ *and all* $t > 0$ *where* $\Phi : \Delta_1 \longrightarrow (0, 1]$ *with the conditions (133), (134), (158) and (159) for all* $w \in \Delta_1$ *and all* $t > 0$ *with*

$$0 < \left(\frac{p}{\lambda}\right)^{\eta} < 1; 0 < \left(\frac{p}{\lambda}\right)^{2\eta} < 1$$

Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ *and a unique quadratic mapping* $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ *satisfying the functional equation (4) and*

$$\left.\begin{aligned} & \mu\left(\phi(w) - \Psi_A(w) - \Psi_{Q2}(w), t\right) \\ & \geq \mu'\left(\Phi\left(w\right), |\lambda - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\lambda - \omega|t\right) * \mu'\left(\Phi\left(w\right), |\lambda^2 - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\lambda^2 - \omega|t\right) \\ & \nu\left(\phi(w) - \Psi_A(w) - \Psi_{Q2}(w), t\right) \\ & \leq \nu'\left(\Phi\left(w\right), |\lambda - \omega|t\right) \diamond \nu'\left(\Phi\left(-w\right), |\lambda - \omega|t\right) \diamond \nu'\left(\Phi\left(w\right), |\lambda^2 - \omega|t\right) \diamond \nu'\left(\Phi\left(-w\right), |\lambda^2 - \omega|t\right) \end{aligned}\right\} \tag{164}$$

for all $w \in \Delta_1$ *and all* $t > 0$ *with* $\tau \neq 1$.

Proof. Assume that

$$\phi_{add}(w) = \frac{\phi(w) - \phi(-w)}{2} \quad for \;\; all \quad w \in \Delta_1.$$

Then

$$\phi_{add}(0) = 0 \quad and \quad \phi_{add}(-w) = -\phi_{add}(w) \quad for \;\; all \quad w \in \Delta_1.$$

By Theorem 5.8, we arrive that

$$\left.\begin{array}{l} \mu\left(\phi_{add}(w) - \Psi_A(w), t\right) \geq \mu'\left(\Phi\left(w\right), |\lambda - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\lambda - \omega|t\right) \\ \nu\left(\phi_{add}(w) - \Psi_A(w), t\right) \leq \nu'\left(\Phi\left(w\right), |\lambda - \omega|t\right) \diamond \nu'\left(\Phi\left(-w\right), |\lambda - \omega|t\right) \end{array}\right\} \tag{165}$$

for all $w \in \Delta_1$ and all $t > 0$. Assume that

$$\phi_{quad}(w) = \frac{\phi(w) + \phi(-w)}{2} \quad for \;\; all \quad w \in \Delta_1.$$

Then

$$\phi_{quad}(0) = 0 \quad and \quad \phi_{quad}(-w) = \phi_{quad}(w) \quad for \;\; all \quad w \in \Delta_1.$$

By Theorem 5.10, we arrive that

$$\left.\begin{array}{l} \mu\left(\phi_{quad}(w) - \Psi_{Q2}(w), t\right) \geq \mu'\left(\Phi\left(x\right), |\lambda^2 - \omega|t\right) * \mu'\left(\Phi\left(-x\right), |\lambda^2 - \omega|t\right) \\ \nu\left(\phi_{quad}(w) - \Psi_{Q2}(w), t\right) \leq \nu'\left(\Phi\left(x\right), |\lambda^2 - \omega|t\right) \diamond \nu'\left(\Phi\left(-x\right), |\lambda^2 - \omega|t\right) \end{array}\right\} \tag{166}$$

for all $w \in \Delta_1$ and all $t > 0$. If we define

$$\phi(w) = \phi_{add}(w) + \phi_{quad}(w) \tag{167}$$

for all $w \in \Delta_1$. From (165),(166) and (167), we arrive

$$\begin{aligned} \mu\left(\phi(w) - \Psi_A(w) - \Psi_{Q2}(w), 2t\right) &= \mu\left(\phi_{add}(w) + \phi_{quad}(w) - \Psi_A(w) - \Psi_{Q2}(w), 2t\right) \\ &\geq \mu\left(\phi_{add}(w) - \Psi_A(w), t\right) * \mu\left(\phi_{quad}(w) - \Psi_{Q2}(w), t\right) \\ &\geq \mu'\left(\Phi\left(w\right), |\lambda - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\lambda - \omega|t\right) * \mu'\left(\Phi\left(w\right), |\lambda^2 - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\lambda^2 - \omega|t\right) \end{aligned}$$

and

$$\begin{aligned} \nu\left(f(w) - \Psi_A(w) - \Psi_{Q2}(w), 2t\right) &= \nu\left(\phi_o(w) + \phi_{quad}(w) - \Psi_A(w) - \Psi_{Q2}(w), 2t\right) \\ &\leq \nu\left(\phi_o(w) - \Psi_A(w), t\right) * \nu\left(\phi_{quad}(w) - \Psi_{Q2}(w), t\right) \\ &\leq \nu'\left(\Phi\left(x\right), |\lambda - \omega|t\right) \diamond \nu'\left(\Phi\left(-x\right), |\lambda - \omega|t\right) \diamond \nu'\left(\Phi\left(x\right), |\lambda^2 - \omega|t\right) \diamond \nu'\left(\Phi\left(-x\right), |\lambda^2 - \omega|t\right) \end{aligned}$$

for all $w \in \Delta_1$ and $t > 0$. □

Corollary 5.13. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ an even function fulfilling the inequality*

$$\left.\begin{array}{l} \mu\left(\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \geq \left\{\begin{array}{l} \mu'\left(\sigma, t\right), \\ \mu'\left(\sigma|w|^\alpha, t\right), \end{array}\right. \\ \nu\left(\phi(\lambda w) - \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right), t\right) \leq \left\{\begin{array}{l} \nu'\left(\sigma, t\right), \\ \nu'\left(\sigma|w|^\alpha, t\right), \end{array}\right. \end{array}\right\} \tag{168}$$

for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 1, 2$. Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ and a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_A(w) - \Psi_{Q2}(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\lambda - 1|t\right) * \mu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t\right) * \mu'\left(\sigma|w|^\alpha, |\lambda^2 - \lambda^\alpha|t\right), \end{cases} \\ \nu\left(\phi(w) - \Psi_A(w) - \Psi_{Q2}(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\lambda - 1|t\right) \diamond \nu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \nu'\left(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t\right) \diamond \nu'\left(\sigma|w|^\alpha, |\lambda^2 - \lambda^\alpha|t\right), \end{cases} \end{array}\right\} \tag{169}$$

for all $w \in \Delta_1$ and all $t > 0$.

5.3. Hyers Method of (5)

Theorem 5.14. *Let $\psi : \Delta_1 \to \Delta_2$ be an odd function satisfying the inequality*

$$\left.\begin{array}{l} \mu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \geq \mu'\left(\Phi\left(w\right), t\right) \\ \\ \nu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \leq \nu'\left(\Phi\left(w\right), t\right) \end{array}\right\} \tag{170}$$

for all $w \in \Delta_1$ and all $t > 0$ where $\Phi : \Delta_1 \longrightarrow (0, 1]$ with the conditions

$$\left.\begin{array}{l} \mu'\left(\Phi\left(\mu^{\eta\tau} w\right), t\right) \geq \mu'\left(\omega^{\eta\tau}\Phi\left(w\right), t\right) \\ \nu'\left(\Phi\left(\mu^{\eta\tau} w\right), t\right) \leq \nu'\left(\omega^{\eta\tau}\Phi\left(w\right), t\right) \end{array}\right\} \tag{171}$$

and

$$\left.\begin{array}{l} \lim_{n\to\infty} \mu'\left(\Phi\left(\mu^{\eta\tau} w\right), \mu^{3\eta\tau} t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Phi\left(\mu^{\eta\tau} w\right), \mu^{3\eta\tau} t\right) = 0 \end{array}\right\} \tag{172}$$

for all $w \in \Delta_1$ and all $t > 0$ with

$$0 < \left(\frac{p}{\mu}\right)^{3\eta} < 1.$$

Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{array}{l} \mu\left(\psi(w) - \Psi_C(w), t\right) \geq \mu'\left(\Phi\left(w\right), |\mu^3 - \omega|t\right) \\ \nu\left(\psi(w) - \Psi_C(w), t\right) \leq \nu'\left(\Phi\left(w\right), |\mu^3 - \omega|t\right) \end{array}\right\} \tag{173}$$

for all $w \in \Delta_1$ and all $t > 0$ with $\tau \neq 1$.

Corollary 5.15. *Assume σ and α are positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ an odd function fulfilling the inequality*

$$\left.\begin{array}{l} \mu\left(\psi(\mu w) - \frac{\mu}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \geq \begin{cases} \mu'\left(\sigma, t\right), \\ \mu'\left(\sigma|w|^\alpha, t\right), \end{cases} \\ \nu\left(\psi(\mu w) - \frac{\mu}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \leq \begin{cases} \nu'\left(\sigma, t\right), \\ \nu'\left(\sigma|w|^\alpha, t\right), \end{cases} \end{array}\right\} \tag{174}$$

for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 3$. Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{array}{l} \mu\left(\psi(w) - \Psi_C(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\mu^3 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right), \end{cases} \\ \nu\left(\psi(w) - \Psi_C(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\mu^3 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right), \end{cases} \end{array}\right\} \tag{175}$$

for all $w \in \Delta_1$ and all $t > 0$.

Theorem 5.16. *Let $\psi : \Delta_1 \to \Delta_2$ be an even function satisfying the inequality*

$$\left.\begin{array}{l} \mu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \geq \mu'\left(\Phi\left(w\right), t\right) \\ \nu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \leq \nu'\left(\Phi\left(w\right), t\right) \end{array}\right\} \tag{176}$$

for all $w \in \Delta_1$ and all $t > 0$ where $\Phi : \Delta_1 \longrightarrow (0, 1]$ with the conditions

$$\left.\begin{array}{l} \mu'\left(\Phi\left(\mu^{\eta\tau} w\right), t\right) \geq \mu'\left(\omega^{\eta\tau}\Phi\left(w\right), t\right) \\ \nu'\left(\Phi\left(\mu^{\eta\tau} w\right), t\right) \leq \nu'\left(\omega^{\eta\tau}\Phi\left(w\right), t\right) \end{array}\right\} \tag{177}$$

and

$$\left.\begin{array}{l} \lim_{n\to\infty} \mu'\left(\Phi\left(\mu^{\eta\tau} w\right), \mu^{4\eta\tau} t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Phi\left(\mu^{\eta\tau} w\right), \mu^{4\eta\tau} t\right) = 0 \end{array}\right\} \tag{178}$$

for all $w \in \Delta_1$ and all $t > 0$ with

$$0 < \left(\frac{p}{\mu}\right)^{4\eta} < 1.$$

Then there exists a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{array}{l} \mu\left(\psi(w) - \Psi_{Q4}(w), t\right) \geq \mu'\left(\Phi\left(w\right), |\mu^4 - \omega|t\right) \\ \nu\left(\psi(w) - \Psi_{Q4}(w), t\right) \leq \nu'\left(\Phi\left(w\right), |\mu^4 - \omega|t\right) \end{array}\right\} \tag{179}$$

for all $w \in \Delta_1$ and all $t > 0$ with $\tau \neq 1$.

Corollary 5.17. *Assume σ and α are positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ an even function fulfilling the inequality*

$$\left.\begin{array}{l} \mu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \geq \begin{cases} \mu'\left(\sigma, t\right), \\ \mu'\left(\sigma|w|^\alpha, t\right), \end{cases} \\ \nu\left(\psi(\mu w) - \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) - \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right), t\right) \leq \begin{cases} \nu'\left(\sigma, t\right), \\ \nu'\left(\sigma|w|^\alpha, t\right), \end{cases} \end{array}\right\} \tag{180}$$

for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 4$. Then there exists a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{array}{l} \mu\left(\psi(w) - \Psi_{Q4}(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\mu^4 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^4 - \mu^\alpha|t\right), \end{cases} \\ \nu\left(\psi(w) - \Psi_{Q4}(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\mu^4 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^4 - \mu^\alpha|t\right), \end{cases} \end{array}\right\} \tag{181}$$

for all $w \in \Delta_1$ and all $t > 0$.

Theorem 5.18. *Let $\phi : \Delta_1 \to \Delta_2$ be a function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(\phi(\mu w) - \frac{\mu^3}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\mu^4}{2}\left(\phi(w) + \phi(-w)\right), t\right) \geq \mu'\left(\Phi\left(w\right), t\right) \\ \nu\left(\phi(\mu w) - \frac{\mu^3}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\mu^4}{2}\left(\phi(w) + \phi(-w)\right), t\right) \leq \nu'\left(\Phi\left(w\right), t\right)\end{array}\right\} \tag{182}$$

for all $w \in \Delta_1$ and all $t > 0$ where $\Phi : \Delta_1 \longrightarrow (0,1]$ with the conditions (171), (172), (177) and (178) for all $w \in \Delta_1$ and all $t > 0$ with

$$0 < \left(\frac{p}{\mu}\right)^{3\eta} < 1; 0 < \left(\frac{p}{\mu}\right)^{4\eta} < 1$$

Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ and a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l}\mu\left(\phi(w) - \Psi_C(w) - \Psi_{Q4}(w), t\right) \\ \geq \mu'\left(\Phi\left(w\right), |\mu^3 - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\mu^3 - \omega|t\right) * \mu'\left(\Phi\left(w\right), |\mu^4 - \omega|t\right) * \mu'\left(\Phi\left(-w\right), |\mu^4 - \omega|t\right) \\ \nu\left(\phi(w) - \Psi_C(w) - \Psi_{Q4}(w), t\right) \\ \leq \nu'\left(\Phi\left(w\right), |\mu^3 - \omega|t\right) \diamond \nu'\left(\Phi\left(-w\right), |\mu^3 - \omega|t\right) \diamond \nu'\left(\Phi\left(w\right), |\mu^4 - \omega|t\right) \diamond \nu'\left(\Phi\left(-w\right), |\mu^4 - \omega|t\right)\end{array}\right\} \tag{183}$$

for all $w \in \Delta_1$ and all $t > 0$ with $\tau \neq 1$.

Corollary 5.19. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ a function fulfilling the inequality*

$$\left.\begin{array}{l}\mu\left(\phi(\mu w) - \frac{\mu^3}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\mu^4}{2}\left(\phi(w) + \phi(-w)\right), t\right) \geq \begin{cases}\mu'\left(\sigma, t\right), \\ \mu'\left(\sigma|w|^\alpha, t\right),\end{cases} \\ \nu\left(\phi(\mu w) - \frac{\mu^3}{2}\left(\phi(w) - \phi(-w)\right) - \frac{\mu^4}{2}\left(\phi(w) + \phi(-w)\right), t\right) \leq \begin{cases}\nu'\left(\sigma, t\right), \\ \nu'\left(\sigma|w|^\alpha, t\right),\end{cases}\end{array}\right\} \tag{184}$$

for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 3, 4$. Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ and a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l}\mu\left(\phi(w) - \Psi_C(w) - \Psi_{Q4}(w), t\right) \geq \begin{cases}\mu'\left(\sigma, |\mu^3 - 1|t\right) * \mu'\left(\sigma, |\mu^4 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right) * \mu'\left(\sigma|w|^\alpha, |\mu^4 - \mu^\alpha|t\right),\end{cases} \\ \nu\left(\phi(w) - \Psi_{Q4}(w), t\right) \leq \begin{cases}\nu'\left(\sigma, |\mu^3 - 1|t\right) \diamond \nu'\left(\sigma, |\mu^4 - 1|t\right), \\ \nu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right) \diamond \nu'\left(\sigma|w|^\alpha, |\mu^4 2 - \mu^\alpha|t\right),\end{cases}\end{array}\right\} \tag{185}$$

for all $w \in \Delta_1$ and all $t > 0$.

5.4. Radus Method of (4)

Theorem 5.20. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an odd function satisfying the inequality (132) and $\Phi : \Delta_1 \longrightarrow (0,1]$ be function with the condition*

$$\left.\begin{array}{l}\lim_{\eta\to\infty} \mu'\left(\Phi\left(\theta_j^\eta w\right), \theta_j^\eta t\right) = 1 \\ \lim_{\eta\to\infty} \nu'\left(\Phi\left(\theta_j^\eta w\right), \theta_j^\eta t\right) = 0\end{array}\right\} \tag{186}$$

for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (49). If there exists $L = L(j)$ such that the functions

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right)$$

has the property

$$\left.\begin{array}{l}\mu'\left(\frac{1}{\theta_j}\Phi(\theta_j w),t\right)=\mu'\left(L\ \Phi(w),t\right)\\ \nu'\left(\frac{1}{\theta_j}\Phi(\theta_j w),t\right)=\nu'\left(L\ \Phi(w),t\right)\end{array}\right\}\tag{187}$$

for all $w\in\Delta_1$ and all $t>0$. Then there exists a unique additive function $\Psi_A:\Delta_1\longrightarrow\Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l}\mu\left(\phi(w)-\Psi_A(w),t\right)\geq\mu'\left(\frac{L^{1-i}}{1-L}\Phi(w),t\right)\\ \nu\left(\phi(w)-\Psi_A(w),t\right)\leq\nu'\left(\frac{L^{1-i}}{1-L}\Phi(w),t\right)\end{array}\right\}\tag{188}$$

for all $w\in\Delta_1$ and all $t>0$.

Proof. Define a set

$$\mathcal{X}=\{r|\,r:\Delta_1\longrightarrow\Delta_2,\ r(0)=0\}$$

and introduce the generalized metric on $\mathcal{X}$,

$$d(r,s)=\inf\left\{L\in(0,\infty):\left\{\begin{array}{l}\mu(r(w)-s(w),t)\geq\mu'(L\ \Phi(w),t),w\in\Delta_1,t>0\\ \nu(r(w)-s(w),t)\leq\nu'(L\ \Phi(w),t),w\in\Delta_1,t>0\end{array}\right\}\right\}\tag{189}$$

It is easy to see that (189) is complete with respect to the defined metric. Define $T:\mathcal{X}\longrightarrow\mathcal{X}$ by $Tr(w)=\frac{1}{\theta_j}r(\theta_j w)$, for all $w\in\mathcal{X}$. This implies $d(Tr,Ts)\leq Ld(r,s)$, for all $r,s\in\mathcal{X}$ i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant L.

With the help of (187) it follows from (136) for the case $j=0$ it reduces to

$$\left.\begin{array}{l}\mu\Big(\frac{f(\lambda w)}{\lambda}-f(w),t\Big)\geq\mu'\left(\frac{1}{\lambda}\Phi(w),t\right)\Longrightarrow\mu\Big(Tf(w)-f(w),t\Big)\geq\mu'\left(L^1\Phi(w),t\right)\\ \nu\Big(\frac{f(\lambda w)}{\lambda}-f(w),t\Big)\geq\nu'\left(\frac{1}{\lambda}\Phi(w),t\right)\Longrightarrow\nu\Big(Tf(w)-f(w),t\Big)\geq\nu'\left(L^1\Phi(w),t\right)\end{array}\right\}\tag{190}$$

for all $w\in\Delta_1$ and all $t>0$.

With the help of (187) it follows from (137) for the case $j=1$ it reduces to

$$\left.\begin{array}{l}\mu\Big(f(w)-\lambda f\left(\frac{w}{\lambda}\right),t\Big)\geq\mu'\left(\Phi\left(\frac{w}{\lambda}\right),t\right)\Longrightarrow\mu\Big(f(w)-Tf(w),t\Big)\geq\mu'\left(L^0\Phi(w),t\right)\\ \nu\Big(f(w)-\lambda f\left(\frac{w}{\lambda}\right),t\Big)\geq\nu'\left(\Phi\left(\frac{w}{\lambda}\right),t\right)\Longrightarrow\nu\Big(f(w)-Tf(w),t\Big)\geq\nu'\left(L^0\Phi(w),t\right)\end{array}\right\}\tag{191}$$

for all $w\in\Delta_1$ and all $t>0$. Combining the above two cases, we arrive

$$\inf\left\{L^{1-i}\in(0,\infty):\left\{\begin{array}{l}\mu(T\ \phi(w)-\phi(w),t)\geq\mu'(L^{1-i}\Phi(w),t),w\in\Delta_1\\ \nu(T\ \phi(w)-\phi(w),t)\leq\nu'(L^{1-i}\Phi(w),t),w\in\Delta_1\end{array}\right\}\right\}\tag{192}$$

Hence property (FPC1) of Theorem 1.1 holds. By (FPC2) of Theorem 1.1, it follows that there exists a fixed point Ψ_A of T in $\mathcal{X}$ such that

$$\lim_{\eta\to\infty}\mu\left(\frac{\phi(\theta_j^\eta w)}{\theta_j^\eta}-\Psi_A(w),t\right)=1,\lim_{\eta\to\infty}\nu\left(\frac{\phi(\theta_j^\eta w)}{\theta_j^\eta}-\Psi_A(w),t\right)=0$$

for all $w\in\Delta_1$ and all $t>0$. To order to prove $A:\Delta_1\longrightarrow\Delta_2$ is additive, the proof is similar to that of Theorem 5.8. By (FPC3) of Theorem 1.1, Ψ_A is the unique fixed point of T in the set $\mathcal{Y}=\{\Psi_A\in\mathcal{X}:d(\phi,\Psi_A)<\infty\},\Psi_A$ is the unique function such that

$$\left.\begin{array}{l}\mu(\phi(w)-\Psi_A(w),t)\geq\mu'(L^{1-i}\Phi(w),t),w\in\Delta_1\\ \nu(\phi(w)-\Psi_A(w),t)\leq\nu'(L^{1-i}\Phi(w),t),w\in\Delta_1\end{array}\right\}$$

for all $w \in \Delta_1$ and and all $t > 0$. Finally by (FPC4) of Theorem 1.1, we obtain

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_A(w), t\right) \geq \mu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) \\ \nu\left(\phi(w) - \Psi_A(w), t\right) \leq \nu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) \end{array}\right\}$$

for all $w \in \Delta_1$ and all $t > 0$. So, the proof is complete. □

The following corollary is an immediate consequence of Theorem 5.20 concerning the some stabilities of (4).

Corollary 5.21. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ an odd function fulfilling the inequality (155) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 1$. Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and*

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_A(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\lambda - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t\right), \end{cases} \\ \nu\left(\phi(w) - \Psi_A(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\lambda - 1|t\right), \\ \nu'\left(\sigma|w|^\alpha, |\lambda - \lambda^\alpha|t\right), \end{cases} \end{array}\right\} \tag{193}$$

for all $w \in \Delta_1$ and all $t > 0$.

Proof. If we take

$$\mu'\left(\Phi(w), t\right) = \begin{cases} \mu'\left(\sigma, t\right), \\ \mu'\left(\sigma||w||^\alpha, t\right), \end{cases} \quad \nu'\left(\Phi(w), t\right) = \begin{cases} \nu'\left(\sigma, t\right), \\ \nu'\left(\sigma||w||^\alpha, t\right), \end{cases}$$

in Theorem 5.8. Replacing w by $\theta_j^\eta w$ in (132) one can see that (186) holds.

Now, by definition of $\Phi(w)$ and its property, we have

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right) = \begin{cases} \mu'\left(\sigma, t\right) \\ \mu'\left(\frac{\sigma||w||^\alpha}{\lambda^\alpha}, t\right) \end{cases}$$

$$\nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right) = \begin{cases} \nu'\left(\sigma, t\right) \\ \nu'\left(\frac{\sigma||w||^\alpha}{\lambda^\alpha}, t\right) \end{cases}$$

and

$$\mu'\left(\frac{1}{\theta_j}\Phi(\theta_j w), t\right) = \begin{cases} \mu'\left({\theta_j}^{-1}\ \sigma, t\right) \\ \mu'\left({\theta_j}^{\alpha-1}\sigma||w||^\alpha, t\right) \end{cases}$$

$$\nu'\left(\frac{1}{\theta_j}\Phi(\theta_j w), t\right) = \begin{cases} \nu'\left({\theta_j}^{-1}\ \sigma, t\right) \\ \nu'\left({\theta_j}^{\alpha-1}\ \sigma||w||^\alpha, t\right) \end{cases}$$

for all $w \in \Delta_1$ and all $t > 0$. Hence the property (187) and the inequality (188) holds for the following:

$$L = \theta_j^{-1} = \lambda^{-1} \quad for \quad j = 0$$

$$\left.\begin{array}{l} \mu\left(f(w) - \Psi_A(w), t\right) \geq \mu'\left(\frac{(\lambda^{-1})^{1-0}}{1-\lambda^{-1}}\ \Phi(w), t\right) = \mu'\left(\sigma, (\lambda - 1)t\right) \\ \nu\left(f(w) - \Psi_A(w), t\right) \leq \nu'\left(\frac{(\lambda^{-1})^{1-0}}{1-\lambda^{-1}}\ \Phi(w), t\right) = \nu'\left(\sigma, (\lambda - 1)t\right) \end{array}\right\}$$

$$L = \theta_j^{-1} = \frac{1}{\lambda^{-1}} = \lambda \quad for \quad j = 1$$

$$\left.\begin{array}{l} \mu\left(f(w) - \Psi_A(w), t\right) \geq \mu'\left(\frac{(\lambda)^{1-1}}{1-\lambda}\ \Phi(w), t\right) = \mu'\left(\sigma, (1-\lambda)t\right) \\ \nu\left(f(w) - \Psi_A(w), t\right) \leq \nu'\left(\frac{(\lambda)^{1-1}}{1-\lambda}\ \Phi(w), t\right) = \nu'\left(\sigma, (1-\lambda)t\right) \end{array}\right\}$$

$$L = \theta_j^{\alpha-1} = \lambda^{\alpha-1} \quad for \quad j = 0$$

$$\left.\begin{array}{l} \mu\left(f(w) - \Psi_A(w), t\right) \geq \mu'\left(\frac{(\lambda^{\alpha-1})^{1-0}}{1-\lambda^{\alpha-1}}\ \Phi(w), t\right) = \mu'\left(\sigma||w||^{\alpha}, (\lambda-\lambda^{\alpha})t\right) \\ \nu\left(f(w) - \Psi_A(w), t\right) \leq \nu'\left(\frac{(\lambda^{\alpha-1})^{1-0}}{1-\lambda^{\alpha-1}}\ \Phi(w), t\right) = \nu'\left(\sigma||w||^{\alpha}, (\lambda-\lambda^{\alpha})t\right) \end{array}\right\}$$

$$L = \theta_j^{\alpha-1} = \frac{1}{\lambda^{\alpha-1}} = \lambda^{1-\alpha} \quad for \quad j = 1$$

$$\left.\begin{array}{l} \mu\left(f(w) - \Psi_A(w), t\right) \geq \mu'\left(\frac{(\lambda^{1-\alpha})^{1-1}}{1-\lambda^{1-\alpha}}\ \Phi(w), t\right) = \mu'\left(\sigma||w||^{\alpha}, (\lambda^{\alpha}-\lambda)t\right) \\ \nu\left(f(w) - \Psi_A(w), t\right) \leq \nu'\left(\frac{(\lambda^{1-\alpha})^{1-1}}{1-\lambda^{1-\alpha}}\ \Phi(w), t\right) = \nu'\left(\sigma||w||^{\alpha}, (\lambda^{\alpha}-\lambda)t\right) \end{array}\right\}$$

Hence the proof is complete. □

Theorem 5.22. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an even function satisfying the inequality (157) and $\Phi : \Delta_1 \longrightarrow (0,1]$ be function with the condition*

$$\left.\begin{array}{l} \lim_{\eta\to\infty} \mu'\left(\Phi\left(\theta_j^{\eta} w\right), \theta_j^{2\eta} t\right) = 1 \\ \lim_{\eta\to\infty} \nu'\left(\Phi\left(\theta_j^{\eta} w\right), \theta_j^{2\eta} t\right) = 0 \end{array}\right\} \tag{194}$$

for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (49). If there exists $L = L(j)$ such that the functions

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right)$$

has the property

$$\left.\begin{array}{l} \mu'\left(\frac{1}{\theta_j^2}\Phi(\theta_j w), t\right) = \mu'\left(L\ \Phi(w), t\right) \\ \nu'\left(\frac{1}{\theta_j^2}\Phi(\theta_j w), t\right) = \nu'\left(L\ \Phi(w), t\right) \end{array}\right\} \tag{195}$$

for all $w \in \Delta_1$ and all $t > 0$. Then there exists a unique quadratic function $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_{Q2}(w), t\right) \geq \mu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) \\ \nu\left(\phi(w) - \Psi_{Q2}(w), t\right) \leq \nu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) \end{array}\right\} \tag{196}$$

for all $w \in \Delta_1$ and all $t > 0$.

Proof. The proof of the theorem is similar lines to that of Theorem 5.20, by defining $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^2} r(\theta_j w)$, for all $w \in \mathcal{X}$. □

The following corollary is an immediate consequence of Theorem 5.22 concerning the some stabilities of (4).

Corollary 5.23. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ an even function fulfilling the inequality (161) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 1$. Then there exists a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and*

$$\left.\begin{aligned} \mu\left(\phi(w) - \Psi_{Q2}(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \mu'\left(\sigma|w|^{\alpha}, |\lambda^2 - \lambda^{\alpha}|t\right), \end{cases} \\ \nu\left(\phi(w) - \Psi_{Q2}(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\lambda^2 - 1|t\right), \\ \nu'\left(\sigma|w|^{\alpha}, |\lambda^2 - \lambda^{\alpha}|t\right), \end{cases} \end{aligned}\right\} \tag{197}$$

for all $w \in \Delta_1$ and all $t > 0$.

Theorem 5.24. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be an odd function satisfying the inequality (163) and $\Phi : \Delta_1 \longrightarrow (0, 1]$ be function with the conditions (186), (194) for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (49). If there exists $L = L(j)$ such that the functions*

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right)$$

has the properties (187) and (195)

for all $w \in \Delta_1$ and all $t > 0$. Then there exists a unique additive function $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ and a unique quadratic function $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and

$$\left.\begin{aligned} \mu\left(\phi(w) - \Psi_A(w), t\right) \geq \mu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) * \mu'\left(\tfrac{L^{1-i}}{1-L}\Phi(-w), t\right) \\ \nu\left(\phi(w) - \Psi_A(w), t\right) \leq \nu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) \diamond \nu'\left(\tfrac{L^{1-i}}{1-L}\Phi(-w), t\right) \end{aligned}\right\} \tag{198}$$

for all $w \in \Delta_1$ and all $t > 0$.

Corollary 5.25. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ an even function fulfilling the inequality (168) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 1, 2$. Then there exists a unique additive mapping $\Psi_A : \Delta_1 \longrightarrow \Delta_2$ and a unique quadratic mapping $\Psi_{Q2} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and (169) for all $w \in \Delta_1$ and all $t > 0$.*

5.5. Radus Method of (5)

Theorem 5.26. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an odd function satisfying the inequality (170) and $\Phi : \Delta_1 \longrightarrow (0, 1]$ be function with the condition*

$$\left.\begin{aligned} \lim_{\eta\to\infty} \mu'\left(\Phi\left(\theta_j^{\eta} w\right), \theta_j^{3\eta} t\right) = 1 \\ \lim_{\eta\to\infty} \nu'\left(\Phi\left(\theta_j^{\eta} w\right), \theta_j^{3\eta} t\right) = 0 \end{aligned}\right\} \tag{199}$$

for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (63). If there exists $L = L(j)$ such that the functions

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\mu}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\mu}\right), t\right)$$

has the property

$$\left.\begin{aligned} \mu'\left(\tfrac{1}{\theta_j^3}\Phi(\theta_j w), t\right) = \mu'\left(L\ \Phi(w), t\right) \\ \nu'\left(\tfrac{1}{\theta_j^3}\Phi(\theta_j w), t\right) = \nu'\left(L\ \Phi(w), t\right) \end{aligned}\right\} \tag{200}$$

for all $w \in \Delta_1$ and all $t > 0$. Then there exists a unique cubic function $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{aligned} \mu\left(\psi(w) - \Psi_C(w), t\right) \geq \mu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) \\ \nu\left(\psi(w) - \Psi_C(w), t\right) \leq \nu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) \end{aligned}\right\} \tag{201}$$

for all $w \in \Delta_1$ and all $t > 0$.

Proof. The proof of the theorem is similar lines to that of Theorem 5.20, by defining $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^3} r(\theta_j w)$, for all $w \in \mathcal{X}$. □

The following corollary is an immediate consequence of Theorem 5.20 concerning the some stabilities of (5).

Corollary 5.27. *Assume σ and α are positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ an odd function fulfilling the inequality (174) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 3$. Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and*

$$\left.\begin{aligned} \mu\left(\psi(w) - \Psi_C(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\mu^3 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right), \end{cases} \\ \nu\left(\psi(w) - \Psi_C(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\mu^3 - 1|t\right), \\ \nu'\left(\sigma|w|^\alpha, |\mu^3 - \mu^\alpha|t\right), \end{cases} \end{aligned}\right\} \tag{202}$$

for all $w \in \Delta_1$ and all $t > 0$.

Theorem 5.28. *Let $\psi : \Delta_1 \longrightarrow \Delta_2$ be an even function satisfying the inequality (176) and $\Phi : \Delta_1 \longrightarrow (0, 1]$ be function with the condition*

$$\left.\begin{aligned} \lim_{\eta\to\infty} \mu'\left(\Phi\left(\theta_j^\eta w\right), \theta_j^{4\eta} t\right) = 1 \\ \lim_{\eta\to\infty} \nu'\left(\Phi\left(\theta_j^\eta w\right), \theta_j^{4\eta} t\right) = 0 \end{aligned}\right\} \tag{203}$$

for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (63). If there exists $L = L(j)$ such that the functions

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\mu}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\mu}\right), t\right)$$

has the property

$$\left.\begin{aligned} \mu'\left(\tfrac{1}{\theta_j^4}\Phi(\theta_j w), t\right) = \mu'\left(L\ \Phi(w), t\right) \\ \nu'\left(\tfrac{1}{\theta_j^4}\Phi(\theta_j w), t\right) = \nu'\left(L\ \Phi(w), t\right) \end{aligned}\right\} \tag{204}$$

for all $w \in \Delta_1$ and all $t > 0$. Then there exists a unique quartic function $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{aligned} \mu\left(\psi(w) - \Psi_{Q4}(w), t\right) \geq \mu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) \\ \nu\left(\psi(w) - \Psi_{Q4}(w), t\right) \leq \nu'\left(\tfrac{L^{1-i}}{1-L}\Phi(w), t\right) \end{aligned}\right\} \tag{205}$$

for all $w \in \Delta_1$ and all $t > 0$.

Proof. The proof of the theorem is similar lines to that of Theorem 5.20, by defining $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $Tr(w) = \frac{1}{\theta_j^4} r(\theta_j w)$, for all $w \in \mathcal{X}$. □

The following corollary is an immediate consequence of Theorem 5.22 concerning the some stabilities of (5).

Corollary 5.29. *Assume σ and α are positive numbers. Let $\psi : \Delta_1 \longrightarrow \Delta_2$ an even function fulfilling the inequality (180) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 4$. Then there exists a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and*

$$\left.\begin{array}{l} \mu\left(\psi(w) - \Psi_{Q4}(w), t\right) \geq \begin{cases} \mu'\left(\sigma, |\mu^4 - 1|t\right), \\ \mu'\left(\sigma|w|^\alpha, |\mu^4 - \mu^\alpha|t\right), \end{cases} \\ \nu\left(\psi(w) - \Psi_{Q4}(w), t\right) \leq \begin{cases} \nu'\left(\sigma, |\mu^4 - 1|t\right), \\ \nu'\left(\sigma|w|^\alpha, |\mu^4 - \mu^\alpha|t\right), \end{cases} \end{array}\right\} \tag{206}$$

for all $w \in \Delta_1$ and all $t > 0$.

Theorem 5.30. *Let $\phi : \Delta_1 \longrightarrow \Delta_2$ be a function satisfying the inequality (182) and $\Phi : \Delta_1 \longrightarrow (0, 1]$ be function with the conditions (199), (203) for all $w \in \Delta_1$ and all $t > 0$ where θ_j is defined in (63). If there exists $L = L(j)$ such that the functions*

$$\mu'\left(\Phi(w), t\right) = \mu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right); \nu'\left(\Phi(w), t\right) = \nu'\left(\Phi\left(\frac{w}{\lambda}\right), t\right)$$

has the properties (200) and (204) for all $w \in \Delta_1$ and all $t > 0$. Then there exists a unique cubic function $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ and a unique quartic function $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (5) and

$$\left.\begin{array}{l} \mu\left(\phi(w) - \Psi_C(w) - \Psi_{Q4}(w), t\right) \geq \mu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) * \mu'\left(\frac{L^{1-i}}{1-L}\Phi(-w), t\right) \\ \nu\left(\phi(w) - \Psi_C(w) - \Psi_{Q4}(w), t\right) \leq \nu'\left(\frac{L^{1-i}}{1-L}\Phi(w), t\right) \diamond \nu'\left(\frac{L^{1-i}}{1-L}\Phi(-w), t\right) \end{array}\right\} \tag{207}$$

for all $w \in \Delta_1$ and all $t > 0$.

Corollary 5.31. *Assume σ and α are positive numbers. Let $\phi : \Delta_1 \longrightarrow \Delta_2$ a function fulfilling the inequality (184) for all $w \in \Delta_1$ and all $t > 0$, with $\sigma > 0$ and $\alpha \neq 3, 4$. Then there exists a unique cubic mapping $\Psi_C : \Delta_1 \longrightarrow \Delta_2$ and a unique quartic mapping $\Psi_{Q4} : \Delta_1 \longrightarrow \Delta_2$ satisfying the functional equation (4) and (185) for all $w \in \Delta_1$ and all $t > 0$.*

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[2] M. Arunkumar, C. Devi Shyamala Mary and G. Shobana, *Simple AQ And Simple CQ Functional Equations*, Journal Of Concrete And Applicable Mathematics, 13(1/2)(2015), 120-151.

[3] M. Arunkumar, E. Sathya, C. Devi Shyamala Mary and S. Hema Latha, *AQ and CQ Functional Equations*, Malaya Journal of Matematik, 6(1)(2018), 182-205.

[4] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, vol. 1, Colloq. Publ., vol. 48, Amer. Math. Soc., Providence, RI, (2000).

[5] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184(3)(1994), 431-436.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., 27(1941), 222-224.

[7] B. Margolis and J. B. Diaz, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., 74(1968), 305-309.

[8] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal., 46(1)(1982), 126-130.

[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(2)(1978), 297-300.

[10] K. Ravi, M. Arunkumar and J. M. Rassias, *Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, Int. J. Math. Stat., 3(A08)(2008), 36-46.

[11] S. Rolewicz, *Metric Linear Spaces*, Reidel, Dordrecht, (1984).

[12] R. Saadati and J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons and Fractals, 27(2006), 331-344.

[13] R. Saadati and J. H. Park, *Intuitionstic fuzzy Euclidean normed spaces*, Commun. Math. Anal., 1(2006), 85-90.

[14] R. Saadati, S. Sedghi and N. Shobe, *Modified intuitionistic fuzzy metric spaces and some fixed point theorems*, Chaos Solitons and Fractals, 38(2008), 36-47.

[15] J. Tabor, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math., 83(2004), 243-255.

[16] S. M. Ulam, *Problems in modern mathematics*, Science Editions, John Wiley & Sons Inc., New York, (1964).