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Best Proximity Point Theorem for Φ-Weak Contractions

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Abstract: Fixed point theory for generalized ϕ-weak contractions have been extended for Best proximity point using p-property.

Keywords: Best Proximity point, p-property, generalized ϕ-weak contraction, complete metric spaces, lower semi-continuous.

1. Introduction and Preliminaries

Let E be a metric space. A map $T : E \to E$ is a contraction if for each $x, y \in E$, there exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$. A map $T : E \to E$ is a ϕ-weak contraction if for each $x, y \in E$, there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is a positive on $(0, \infty)$ and $\phi(0) = 0$, and $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$. The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. Rhoades [7] showed the results of [1] Albert and Guerre are still true for any Bancah space. Also Rhoades [7] proved the following very interesting fixed point theorem, which is one of the generalizations of the Banach contraction principle because it contains contractions as special case ($\phi(t) = (1 - k)t$).

Theorem 1.1. *Let (E, d) be a complete metric space and let A be a ϕ-weak contraction on E. If $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\phi(\epsilon) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$, then A has a unique fixed point.*

In fact the weak contractions are closely related to maps of Boyd and Wong type ones [3] and Reich types ones [6], namely if ϕ is a lower semi-continuous function from the right then $\psi(t) = 1 = \phi(t)$ is an upper semi-continuous function from the right and moreover Theorem 1.1, turns into

$$d(Tx, Ty) \leq k(d(x, y))d(x, y)$$

Therefore the ϕ-weak contraction becomes a Raich type one. During the last few decades a number of hybrid contraction mapping results have been obtained by many mathematical researchers.

For example, Song [10,11], Thagafi and Shahzad [2], Shahzad [8] and Hussain and Jungch [5] obtained the common fixed point theorems of f-contraction $T(d(Tx, Ty) \leq kd(f_x, f_y))$, and generalized f-contraction.

$$T(d(Tx, Ty) \leq k \max\left\{d(f_x, f_y), d(Tx, f_y), d(Ty, f_y), \frac{1}{2}[d(f_x, f_y) + d(Tx, Ty)]\right\})$$

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and generalized (f,g)-contraction.

$$T(d(Tx,Ty) \leq k \max\left\{d(f_x,g_y), d(Tx,f_y), d(Ty,g_y), \frac{1}{2}[d(f_x,Ty)+d(Tx,g_y)]\right\})$$

respectively. Song [10] extended the above results to g-weak contractions.

$$(d(Tx,Ty) \leq d(f_x,f_y) - \phi(d(f_x,g_y)))$$

Further Zang and Sang proved that there is a unique common fixed point of hybrid generalized ϕ-weak contractions T,S on complete metric space E.

Theorem 1.2 ([12])**.** *Let (E,d) be a complete metric space $T,S: E \to E$ two mappings such that for all $x,y \in E, d(Tx,Ty) \leq M(x,y) - \phi(M(x,y))$, where $\phi: [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$.*

$$M(x,y) = \max\left\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx)+d(x,Sy)]\right\}$$

Then there exists a unique point $u \in E$ such that $u = Tu = Su$.

The aim of this work is to extend the result that there is unoque common best proximity point of hybrid generalized ϕ-weak contraction T,S on complete metric space E.

Definition 1.3 ([4])**.** *Let A and B be nonempty subsets of a metric space (X,d) with $A_0 \neq 0$. The pair (A,B) is said to have the weak p-property if and only if*

$$d(x_1,y_1) = d(A,B)$$
$$d(x_2,y_2) = d(A,B)$$

which implies $d(x_1,x_2) \leq d(y_1,y_2)$, where $x_1,x_2 \in A$ and $y_1,y_2 \in B$.

2. Main Results

Theorem 2.1. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (E,d) such that A_0 is nonempty. Let $T,S: A \to B$ such that $T(A_0) \subseteq B_0, S(A_0) \subseteq B_0$. Suppose*

$$d(Tx,Ty) = d(Sx,Sy) \leq M(x,y) - \phi(M(x,y)) - d(A,B) \tag{1}$$

where $\phi: [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$.

$$M(x,y) = \max\left\{d(x,y), d(Tx,x) - d(A,B), d(Sy,y) - d(A,B), \frac{1}{2}[d(y,Tx)+d(x,Sy)] - d(A,B)\right\}$$

Furthermore the pair (A,B) has p-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*) = d(x^*,Sx^*) = d(A,B)$ that is x^* is the common best proximity point of T and S.

Proof. Choose $x_0 \in A$. Since $Tx_0 \in T(A_0 \subseteq B_0), Sx_0 \in S(A_0 \subseteq B_0)$, there exists $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(x_1, Sx_0) = d(A, B).$$

Analogously, regarding the assumption, $Tx_1 \in T(A_0 \subseteq B_0)$, $Sx_1 \in T(A_0 \subseteq B_0)$. We determine $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(x_2, Sx_1)$$

Recursively, we obtain a sequence (x_n) in A_0 satisfying

$$d(x_{n+1}, x_n) = d(A, B) \quad for\ all \quad n \in N \tag{2}$$

Claim: $d(x_{n+1}, x_n) \to 0$.

If $x_N = x_{N+1}$, then x_N is best proximity point. By the p-property, we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) = d(Sx_n, Sx_{n+1})$$

Hence we assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since $d(x_{n+1}, Tx_n) = d(x_{n+1}, Sx_n) = d(A, B)$, from (1), we have for all $n \in N$.

$$\begin{aligned}
& d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) = d(Sx_n, Sx_{n+1}) \\
& \leq \left\{ d(x_n, y_n), d(x_n, Tx_n) - d(A, B), d(x_n, Sx_n) - d(A, B), \frac{1}{2}[d(x_n, Tx_n) + d(x_n, Sx_n)] - d(A, B) \right\} \\
& - \phi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B), d(x_{n+1}, Sx_{n+1}) - d(A, B)\right\}\right) \\
& \leq \psi\left(\max\left\{ d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), d(x_{n+1}, Tx_{n+1}) - d(A, B), \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] - d(A, B) \right\}\right) \\
& - \phi(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\right\}) \\
& \leq \psi\left(\max\left\{ d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), d(x_{n+1}, Tx_{n+1}) - d(A, B), \frac{1}{2}(d(x_n, Tx_{n+1}) - d(A, B)) \right\}\right) \\
& - \phi(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\right\})
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2}(d(x_n, x_{n+1}) - d(A, B) &\leq \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})] - d(A, B), \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, Sx_{n+1})] - d(A, B) \\
&\leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1})) - d(A, B), d(x_{n+1}, Sx_{n+1})) - d(A, B)\right\} \\
d(x_n, Tx_n) - d(A, B) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\
&= d(x_n, x_{n+1}) \\
d(x_n, Sx_n) - d(A, B) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Sx_n) - d(A, B) \\
&= d(x_n, x_{n+1})
\end{aligned}$$

It follows that

$$d(Tx_n, Tx_{n+1}) = d(Sx_n, Sx_{n+1}) \leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B), d(x_{n+1}, Sx_{n+1}) - d(A, B)\right\}$$

$$- \phi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A,B), d(x_{n+1}, Sx_{n+1}) - d(A,B)\right\}\right)$$

$$d(x_{n+1}, x_{n+2}) \leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} - \phi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \tag{3}$$

Then from (3), we have

$$d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x_{n+2})) - \phi(d(x_{n+1}, x_{n+2}))$$

that is $\psi(d(x_{n+1}, x_{n+2})) \leq 0$ which implies that $d(x_{n+1}, x_{n+2}) = 0$ contracting our assumption. Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for any $n \in N$ and hence $\{d(x_n, x_{n+1})\}$ is monotone decreasing sequence of nonnegative real numbers, hence there exists $r \geq 0$ such that $lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

In the view of the fact from (3), for any $n \in N$, we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \phi(d(x_n, x_{n+1}))$. Taking the limit as $n \to \infty$ in the above inequality, and using the conditions of ψ and ϕ we have $d(r) \leq d(r) - \phi(r)$ which implies $\phi(r) = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{4}$$

Now we show that (x_n) is a cauchy sequence. If otherwise there exists $\epsilon > 0$, for which we can find two sequences of positive integers (m_k) and (n_k) such that for all positive integers $n_k > m_k > k, d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_{k-1}}) < \epsilon$. Now

$$\epsilon \geq d(x_{m_k}, x_{n_k}) \geq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}),$$

that is $\epsilon \geq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{n_{k-1}}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequalities and using (4) we have

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon \tag{5}$$

Again $d(x_{m_k}, x_{n_k}) \geq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon \tag{6}$$

Again $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$ and $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$. Letting $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) = \epsilon \tag{7}$$

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon \tag{8}$$

For $x = x_{m_k}, y = y_{m_k}$ we have

$$\begin{aligned} d(x_{m_k}, Tx_{m_k}) - d(A,B) &= d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, Tx_{m_k}) - d(A,B) \\ &= d(x_{m_k}, x_{m_{k+1}}) \end{aligned}$$

Similarly $d(x_{n_k}, Sx_{n_k}) - d(A,B) = d(x_{n_k}, x_{n_{k+1}})$. Also

$$d(x_{m_k}, Tx_{n_k}) - d(A,B) = d(x_{m_k}, x_{n_{k+1}})$$

$$d(x_{n_k}, Sx_{m_k}) - d(A,B) = d(x_{n_k}, x_{m_{k+1}})$$

From (1) we have

$$\begin{aligned}
d(x_{m_{k+1}}, x_{n_{k+1}}) &= d(Tx_{m_k}, Sx_{n_k}) \\
&\leq \max\left\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}) - d(A,B), d(x_{n_k}, Sx_{n_k}) - d(A,B), \frac{1}{2}[d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Sx_{m_k})] - d(A,B)\right\} \\
&- \phi\left(\max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, Tx_{n_k}) - d(A,B), d(x_{m_k}, Sx_{m_k}) - d(A,B)\right\}\right) \\
&\leq \max\left\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \frac{1}{2}[d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})]\right\} \\
&- \phi(\max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_{k+1}})\right\})
\end{aligned}$$

It follows that

$$\begin{aligned}
d(Tx_{m_k}, Tx_{n_k}) &\leq \psi\left(\max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, Tx_{n_{k+1}}), \frac{1}{2}[d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})]\right\}\right) \\
&- \phi(\max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, Tx_{n_{k+1}}), d(x_{m_k}, Sx_{m_{k+1}})\right\}) \\
d(x_{m_{k+1}}, Tx_{n_{k+1}}) &\leq \max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_{k+1}})\right\} \\
&- \phi(\max\left\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{m_{k+1}})\right\})
\end{aligned}$$

From (4), (5), (7) and (8) and letting $k \to \infty$ in the above inequalities and using the conditions of ψ and ϕ, we have $\epsilon \leq \epsilon - \phi(\epsilon)$ which is contraction by virtue of property ϕ. Hence (x_n) is a cauchy sequence. Since $(x_n) \subset A$ and A is a closed subset of the complete metric space (E, d), there exists x^* in A such that $x_n \to x^*$. Putting $x = x_n$ and $y = x^*$ in (1) and

$$\begin{aligned}
d(x_n, Tx^*) &= d(x_n, Sx^*) \leq d(x_n, x^*) + d(x^*, Tx_n) \quad \textit{and} \\
d(x^*, Tx_n) &= d(x^*, Sx_n) \leq d(x^*, Tx^*) + d(Tx^*, Tx_n)
\end{aligned}$$

We have

$$\begin{aligned}
d(x_{n+1}, Tx^*) - d(A,B) &= d(x_{n+1}, Sx^*) - d(A,B) \\
&\leq \max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*) - d(A,B), d(x^*, Sx^*) - d(A,B), \frac{1}{2}[d(x_n, Tx^*) + d(x^*, Tx_n)] - d(A,B), \\
&\frac{1}{2}[d(x_n, Sx^*) + d(x^*, Sx_n)] - d(A,B)\Big\} \\
&- \phi(\max\{d(x_n, x^*), d(x^*, Tx^*) - d(A,B), d(x^*, Sx^*) - d(A,B)\})
\end{aligned}$$

Taking the limits as $n \to \infty$ in the above inequalities and using the conditions of ϕ, we have

$$d(x^*, Tx^*) - d(A,B) = d(x^*, Sx^*) - d(A,B) \leq d(x^*, Tx^*) - d(A,B) - \phi(d(x^*, Tx^*) - d(A,B))$$

which implies that $d(x^*, Tx^*) = d(x^*, Sx^*) = d(A,B)$. Hence x^* is a common best proximity point of T.

For the uniqueness: Let u and v be two best proximity point and suppose that $u \neq v$, then putting $x = u$ and $y = v$ in (1) we obtain

$$\begin{aligned}
d(T_u, T_v) &= d(S_u, S_v) \\
&\leq \max\Big\{ d(u,v), d(u, T_u) - d(A,B), d(v, T_v) - d(A,B), d(u, S_u) - d(A,B), d(v, S_v) - d(A,B),
\end{aligned}$$

$$\frac{1}{2}[d(u,T_v)+d(v,T_u)]-d(A,B),\frac{1}{2}[d(u,S_v)+d(v,S_u)]-d(A,B)\Big\}$$
$$-\phi(max\{d(u,v),d(v,T_v)-d(A,B),d(u,S_u)-d(A,B)\})$$

that is $d(u,v)\le d(u,v)-\phi(d(u,v))$, which is contradiction by virtue of a property of ϕ. There $u=v=Tu=Su=Tv=Sv$. Hence $u=v$. The proof is complete. □

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