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The Rook Class Partition Algebras

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Abstract: The Class partition algebras $P_k(n,m)$ have been studied independently in [6] by Kennedy and also studied in [11] by Martin and Elgamal. We introduce the rook version of class partition algebras $P_k(n,m)$, which is subalgebra of the algebra $P_k(n) \otimes P_k(m)$ tensor product partition algebra. We also find corresponding schur–weyl dualities.

MSC: 16S20, 16S99.

Keywords: Wreath product , Symmetric group, Partition algebra, Centralizer algebra.

1. Introduction

The partition algebras $P_k(x)$ have been defined in [10] by Martin and in [5] by Jones independently. The algebra studied as Potts model in statistical mechanics and generalization of the Temperley–Lieb algebras. In ([8]; [9]) the algebra appears implicity and in [10] appears explicitly. Jones considered the algebra $P_k(n)$, as the symmetric group's centralizer algebra on $V^{\otimes k}$ (see [5]). The "G-Colored Partition Algebras $P_k(n;G)$" introduced by Bloss in [1] which is an edge coloring of the partition algebra $P_k(n)$. The algebra have been realized as centralizer algebra of $G \wr S_n$ on $W^{\otimes k}$, where $G \wr S_n$ is wreath product group of order $(|G|)^n n!$ and $W = \mathbb{C}^{n|G|}$.

The G-vertex colored partition algebras $P_k(n,G)$ has been recently introduced in [14] by Parvathi and Kennedy. The algebra $P_k(n,G)$ realized as the centralizer algebras of the direct product group $G \times S_n$ which is subgroup of $G \wr S_n$ on $W^{\otimes k}$, where $W = \mathbb{C}^{n|G|}$. The extended vertex colored partition algebras $\widehat{P}_k(n,G)$ have been recently studied in [15] by Parvathi and Kennedy and the algebra is centralizer algebra of the symmetric group S_n which is subgroup of $G \times S_n$.

The Class partition algebra $P_k(n,m)$ have been studied recently in [6] by Kennedy and also studied in [11] by Martin and Elgamal. The algebra $P_k(n,m)$ realized as the centralizer algebra of $S_m \wr S_n$ act on $W^{\otimes k}$, where $W = \mathbb{C}^{nm}$ and W is permutation module for S_{nm}. In [6] studied explicitly, the inclusion of groups $S_n \subseteq G \times S_n \subseteq S_m \times S_n \subseteq S_m \wr S_n \subset S_{nm}$ induces the reverse inclusion naturally of related centralizer algebras as $P_k(nm) \subseteq P_k(n,m) \subseteq P_k(n) \otimes P_k(m) \subseteq P_k(n,G)$ if $n \geq 2k$ and $|G| = m$.

In [12], The (half) rook partition algebras has been studied by Martin and Rollet and also, the algebra studied using different kind of notations in [4] by Halverson and Ram and in [2] by Grood . The rook colored partition algebras $P_{k+\frac{1}{2}}(x,G)$ and $\widehat{P}_{k+\frac{1}{2}}(x,G)$ have been studied in [7]. The algebras have been realized as the centralizer algebras of direct product group $G \times S_{n-1}$ and symmetric group S_{n-1} respectively. The notation $P_{k+\frac{1}{2}}(x)$ is used for the rook partition algebra by Kennedy

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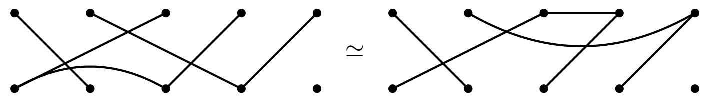
and Muniasamy in [7]. The rook partition algebra $P_{k+\frac{1}{2}}(x)$ is the centralizer algebra of S_{n-1} on $V^{\otimes k}$, where $V = \mathbb{C}^n$ is the permutation module for S_n. In this paper, we introduce the rook version of class partition algebras $P_k(n, m)$, which is subalgebra of the algebra $P_k(n) \otimes P_k(m)$ tensor product partition algebra. We also find corresponding schur–weyl dualities.

2. Preliminaries

We recall some of the results which are required for our motive.

2.1. The partition algebra $P_k(x)$ and rook version

A k-partition diagram is a simple graph of one above the other of two lines of k-vertices. The $2k$ vertices partitioned into l subsets, $1 \leq l \leq 2k$ by the connected components of a k-partition diagram. We state that two diagrams are equivalent when they determines the same partition of $2k$ vertices. For example,



When we are talking about diagrams, we are really speak about the associated equivalence classes. Number the vertices $1, 2, ..., k$ in the upper line from left to right and $k+1, k+2, ..., 2k$ in the lower line from left to right in a k-partition diagram.

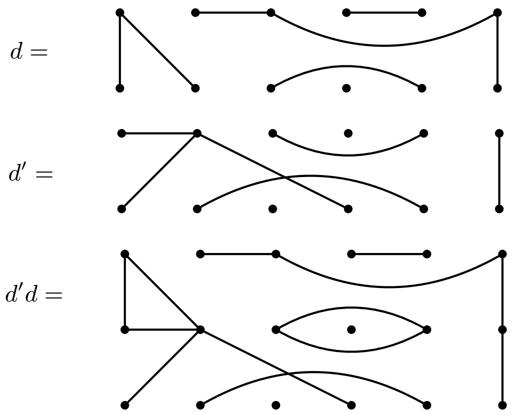
The field F will always represent a field of characteristic is arbitrary throughout this paper and x will represent an field element of the field F. The following is known as the product of two diagrams d and d':

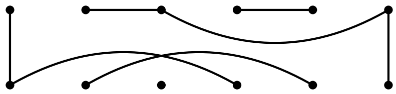
1. Set d at the top and d' below it so that the lower line of d coincides with the upper line of d'.

2. Now, we have a diagram with upper line, middle line and lower line of vertices. This diagram is named as attachment of d and d'. Let the number of components that lie completely in the middle line is λ.

3. Make a new diagram d'' by deleting the vertices in the middle line but keeping the lower line and upper line and maintaining the connections between them. Replacing every "component" contained in the middle line with the variable x. That is, $d'd = x^{\lambda} d''$.

For example,



$d'd = x^2$ 

This product is associative and well defined up to equivalence. Linearly extending this product makes the algebra $P_k(x)$ an associative algebra with identity.

The *partition algebra* $P_k(x)$ is the F-span of all k-partition diagrams for every x in the field F and a natural number k. The identity element is given by the partition diagram with every vertex in the upper line connected only to the vertex below it in the lower line. The dimension of the partition algebra $P_k(x)$ is the Bell number $B(2k)$, where

$$B(2k) = \sum_{l=1}^{l=2k} S(2k,l) \tag{1}$$

and where the number of equivalence relations with exactly l parts for a set of $2k$ elements is Stirling number $S(2k,l)$ (see,[16]). By convention, $P_0(x) = F$. Replacing the variable x by complex number ξ, we obtain a F-algebra $P_k(\xi)$.

2.2. Schur–Weyl Duality

We follow the notations, as given in ([3], (2001)). Let $V = \mathbb{C}^n$, where V is the permutation module for S_n with a standard basis $v_1, v_2, ..., v_n$. Then $\pi(v_i) = v_{\pi(i)}$, for $\pi \in S_n$ and $1 \leq i \leq n$. For every positive integer k, the tensor product space $V^{\otimes k}$ is a module for S_n with a standard basis given by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, where $1 \leq i_j \leq n$. The action of $\pi \in S_n$ on a basis vector is given by

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\pi(i_1)} \otimes v_{\pi(i_2)} \otimes \cdots \otimes v_{\pi(i_k)}. \tag{2}$$

For every diagram d and every integer sequence $i_1, i_2, ..., i_{2k}$ with $1 \leq i_r \leq n$, define

$$\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} = \begin{cases} 1 & \text{if } i_s = i_r \text{ whenever vertices } s \text{ and } r \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Define the action of $d \in P_k(n)$ on $V^{\otimes k}$ by defining it on the standard basis by

$$d(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \leq i_{k+1},\ldots,i_{2k} \leq n} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}} v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{2k}}. \tag{4}$$

Theorem 2.1 ([5])**.** *The algebras $\mathbb{C}[S_n]$ and $P_k(n)$ generate full centralizers of each other in $End(V^{\otimes k})$. In particular, for $n \geq 2k$,*

(i). $P_k(n) \cong End_{S_n}(V^{\otimes k})$

(ii). S_n generates $End_{P_k(n)}(V^{\otimes k})$.

The rook partition algebra $P_{k+\frac{1}{2}}(n)$ is the centralizer algebra of S_{n-1} which is subgroup of all permutation which fix the n^{th} symbol in S_n. Hence, we have $P_k(n) \subseteq P_{k+\frac{1}{2}}(n)$. The rook partition algebra $P_{k+\frac{1}{2}}(n)$ realized as a subalgebra of $P_{k+1}(n)$. The partition algebra $P_{k+1}(n)$ is spanned by set of all diagrams in which the last two vertices ($(k+1)^{th}$ and $2(k+1)^{th}$) are in a same class, see [2, 4, 12].

2.3. The class partition algebras $P_k(x,y)$

Let $P_k(x)\otimes P_k(y)$ be the tensor product algebra of $P_k(x)$ and $P_k(y)$ with dimension $[B(2k)]^2$ where $P_k(x)$ and $P_k(y)$ are partition algebras. Define the standard basis for this algebra is as follows:

$$\mathscr{B}_k := \{(d\otimes d') \mid d' \text{ and } d \text{ are the } k\text{-partition diagrams}\}$$

Definition 2.2. Let d and d' be partitions of $[2k]$ where $[2k]$ is set of $2k$ elements. We state that d' is coarser than d if any class of d is contained in some class of d'. We denote it as $d'\leq d$.

Let $\mathscr{B}'_k := \{(d\otimes\theta d)\in\mathscr{B}_k \mid d\leq\theta d\}$. A class partition diagram $(d\otimes\theta d)\in\mathscr{B}'_k$ can be considered as colored partition diagram with the fundamental partition diagram d colored by θd. Furthermore, since $d\leq\theta d$, we can find $(d\otimes\theta d)\in\mathscr{B}'_k$ as a colored partition diagram with fundamental diagram d where every class E of d is colored by a partition of E.
Let $(d_1\otimes\theta d_1),(d_2\otimes\theta d_2)\in\mathscr{B}'_k$ be two class partition diagrams with $d_1\leq\theta d_1$, $d_2\leq\theta d_2$. In [6], the multiplication on two class partition diagrams defined as follows: $(d_1\otimes\theta d_1),(d_2\otimes\theta d_2)\in\mathscr{B}'_k$ and $d_2d_1=x^{\lambda}d$ in $P_k(x)$. Then clearly, $(\theta d_2)(\theta d_1)=y^{\lambda+\omega}\theta d$ in $P_k(y)$, for some $\theta d\geq d$ and $\omega\geq 0$. That is, $(d_2\otimes\theta d_2)(d_1\otimes\theta d_1)=x^{\lambda}y^{\lambda+\omega}(d\otimes\theta d)$. Therefore, the multiplication of two elements of $\mathscr{B}'_k$ is a scalar product of some element in $\mathscr{B}'_k$. Thus, the span of all elements in the set $\mathscr{B}'_k$ is a subalgebra of the algebra $P_k(x)\otimes P_k(y)$ with identity element denoted as $P_k(x,y)$, called the *class partition algebra*. The dimension for the algebra $P_k(x,y)$ is

$$|\mathscr{B}'_k| = \sum_{\lambda\vdash 2k} S(2k,\lambda)\prod_{r=1}^{r=l} B(\lambda_r) \tag{5}$$

where, the number of partitions of $2k$ elements of the form $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_l)\vdash 2k$ is $S(2k,\lambda)$ and $B(\lambda_r)$ is the Bell number of λ_r. The Bell number defined in (1) can also be equivalently defined as

$$B(2k) = \sum_{\lambda\vdash 2k} S(2k,\lambda). \tag{6}$$

Remark 2.3. For the algebra $P_k(x,y)$ we have another dimension formula as follows:

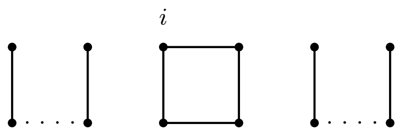
$$|\mathscr{B}'_k| = \sum_{l=1}^{2k} S(2k,l)B(l) \tag{7}$$

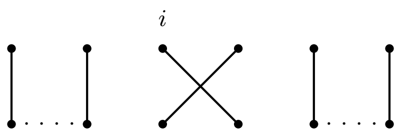
Theorem 2.4 ([6])**.** *The algebra $P_k(x,y)$ is generated by the set of all elements*

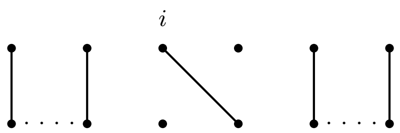
$$\mathbb{G}_k := \{(a_i\otimes a_i),(a_i\otimes b_i),(b_i\otimes b_i),(b_i\otimes c_i),(c_i\otimes c_i)\}$$

where $\{a_i,b_i,c_i \mid 1\leq i\leq k-1\}$ is the generating set for the algebra $P_k(x)$.

For $1\leq i\leq k-1$ define the elements as follows:

$a_i =$


$b_i =$


$c_i =$


Remark 2.5.

1. *The set of all elements* $\{(d\otimes d)|(d\otimes d)\in\mathscr{B}'_k\}$ *span a algebra which is subalgebra of the algebra* $P_k(x,y)$ *generated by the set* $\{(a_i\otimes a_i),(b_i\otimes b_i),(c_i\otimes c_i)\}$ *and isomorphic to* $P_k(xy)$.
2. *The subalgebra of the algebra* $P_k(x,y)$ *spanned by the set of elements* $(d\otimes\theta d)$ *in* $\mathscr{B}'_k$ *such that the* k^{th} *and* $2k^{th}$ *vertices of* d *and* θd *are isolated vertical edges is isomorphic to the algebra* $P_{k-1}(x,y)$.

The wreath product of S_m with S_n denoted by $S_m\wr S_n$, is a group of order $(m!)^n n!$ and the elements of the group is of the form $\pi_{(\alpha_1,\alpha_2,...,\alpha_n)}$, where $\pi\in S_n$ and $\alpha_1,\alpha_2,...,\alpha_n\in S_m$. The multiplication in $S_m\wr S_n$ is given by

$$\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}\pi'_{(\alpha'_1,\alpha'_2,\ldots,\alpha'_n)}=(\pi\pi')_{(\alpha_{\pi'(1)}\alpha'_1,\alpha_{\pi'(2)}\alpha'_2,\ldots,\alpha_{\pi'(n)}\alpha'_n)}.$$

Let

$$W=\mathrm{Span}_{\mathbb{C}}\{v_{(i,j)}\mid 1\le i\le n\text{ and }1\le j\le m\}.$$

In [6], the action of $S_m\wr S_n$ on the standard basis of $W^{\otimes k}$ is defined as follows:

$$\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}(v_{(i_1,j_1)}\otimes\cdots\otimes v_{(i_k,j_k)})=v_{(\pi(i_1),\alpha_{i_1}(j_1))}\otimes\cdots\otimes v_{(\pi(i_k),\alpha_{i_k}(j_k))}.$$

Also, defined a map $\tilde\phi:P_k(n)\otimes P_k(m)\longrightarrow End(W^{\otimes k})$ by defining it on a basis element $(d\otimes d')$, as follows:

$$\begin{aligned}\tilde\phi(d\otimes d')&=\left(\tilde\phi(d\otimes d')^{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\ldots,(i_{2k},j_{2k})}\right)\\&=\left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}\psi(d')^{j_1,j_2,\ldots,j_k}_{j_{k+1},j_{k+2},\ldots,j_{2k}}\right)\\&=\sum_{\substack{p\sim q\text{ in }d\Rightarrow i_p=i_q\\ p\sim q\text{ in }d'\Rightarrow j_p=j_q}}E^{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\ldots,(i_{2k},j_{2k})}\end{aligned}$$

where $\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}$ is defined as in (3). Also, the action of $P_k(n)\otimes P_k(m)$ on $W^{\otimes k}$ defined by,

$$(d\otimes d')\cdot(v_{(i_1,j_1)}\otimes v_{(i_2,j_2)}\otimes\cdots\otimes v_{(i_k,j_k)})=\sum_{\substack{1\le i_{k+1},\ldots,i_{2k}\le n\\ 1\le j_{k+1},\ldots,j_{2k}\le m}}\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},\ldots,i_{2k}}\psi(d')^{j_1,j_2,\ldots,j_k}_{j_{k+1},\ldots,j_{2k}}v_{(i_{k+1},j_{k+1})}\otimes\cdots\otimes v_{(i_{2k},j_{2k})}.$$

Note that $\{\pi_\alpha:=\pi_{(\alpha,\alpha,\ldots,\alpha)}\in S_m\wr S_n\mid\pi\in S_n\text{ and }\alpha\in S_m\}$ form a subgroup of the wreath product $S_m\wr S_n$, which is isomorphic to the group $S_m\times S_n$. The restricted action of $S_m\times S_n$ on W defined as $\pi_\alpha(i,j)=(\pi(i),\alpha(j))$. Hence, the map $\tilde\phi$ on $P_k(n)\otimes P_k(m)$ is an algebra homomorphism onto $End_{S_m\times S_n}(W^{\otimes k})$ such that the restriction of $\tilde\phi$ on $P_k(n,m)$ is onto $End_{S_m\wr S_n}(W^{\otimes k})$ (see, [6]).

Theorem 2.6 ([6])**.** *The algebras* $\mathbb{C}[S_m\times S_n]$ *and* $P_k(n)\otimes P_k(m)$ *generates full centralizers of each other in* $End(W^{\otimes k})$. *In particular, for* $n,m\ge 2k$,

(i). $P_k(n)\otimes P_k(m)\cong End_{S_m\times S_n}(W^{\otimes k})$

(ii). $S_m\times S_n$ *generates* $End_{P_k(n)\otimes P_k(m)}(W^{\otimes k})$.

Theorem 2.7 ([6])**.** *The algebras* $\mathbb{C}[S_m\wr S_n]$ *and* $P_k(n,m)$ *generates full centralizers of each other in* $End(W^{\otimes k})$. *In particular, for* $n,m\ge 2k$,

(i). $P_k(n,m)\cong End_{S_m\wr S_n}(W^{\otimes k})$

(ii). $S_m\wr S_n$ *generates* $End_{P_k(n,m)}(W^{\otimes k})$.

3. The Class Partition Algebra's Rook Version

We introduce the class partition algebra's rook version and study about its structure in this section.

3.1. The action of $S_{m-1} \wr S_{n-1}$ on $W^{\otimes k}$

We describe two bases of $End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$ in this section, where $W = \mathbb{C}^{nm}$ and the action of the wreath product $S_{m-1} \wr S_{n-1}$ on $W^{\otimes k}$ is defined as follows:

Let W be a finite dimensional vector space with dimension nm. We may identify W as $\text{Span}_{\mathbb{C}}\{v_{(i,j)} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Define the action of $S_{m-1} \wr S_{n-1}$ on W as

$$\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(v_{(i,j)}) = v_{(\pi(i),\alpha_i(j))}, \forall\ \pi \in S_{n-1} \text{ and } \alpha_1, \alpha_2, ..., \alpha_{n-1} \in S_{m-1}. \tag{8}$$

Note that when $m = 1$, W specializes to V, where V is permutation representation of S_n. A subgroup of S_n isomorphic to S_{n-1} is the set of all permutation in S_n fixing the nth symbol.

Proposition 3.1. *The action of $S_{m-1} \wr S_{n-1}$ on W defined as in (8) is permutation action with respect to $\mathrm{K} := \{\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1} \mid \pi \in S_{n-2}, \alpha_{n-1} \in S_{m-2}\}$, where K is subgroup of $S_{m-1} \wr S_{n-1}$.*

Proof. For $1 \leq i \leq n-1$, $1 \leq j \leq m-1$, define the elements $\sigma_{i,j} := (i, n-1)_{((j,m-1),(j,m-1),\dots,(j,m-1))} \in S_{m-1} \wr S_{n-1}$, where $(i, n-1)$ and $(j, m-1)$ are the transpositions (resp. identities) of S_{n-1} and S_{m-1} respectively, if $i \neq n-1$ and $j \neq m-1$ (resp. if $i = n-1$ and $j = m-1$). Thus, $\sigma_{i,j}\mathrm{K} = \sigma_{i',j'}\mathrm{K}$ iff $i' = i$ and $j' = j$. Therefore, we have $(n-1)(m-1)$ coset representatives $\sigma_{i,j}$ for $(S_{m-1} \wr S_{n-1})/\mathrm{K}$. Define elements $\sigma_{\pi(i),\alpha_i(j)}\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}\sigma_{i,j} := \pi'_{(\alpha'_1,\alpha'_2,\dots,\alpha'_{n-1})}$, for every $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1}$. Then $\pi'(n-1) = [(\pi(i), n-1)\pi(i, n-1)](n-1) = n-1$ and $\alpha'_{n-1}(m-1) = [(\alpha'_i(j), m-1)\alpha'_i(j, m-1)](m-1) = m-1$. Therefore, $\sigma_{\pi(i),\alpha_i(j)}\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}\sigma_{i,j} \in \mathrm{K}$. Hence $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}\sigma_{i,j}\mathrm{K} = \sigma_{\pi(i),\alpha_i(j)}\mathrm{K}$. Thus the action of $S_{m-1} \wr S_{n-1}$ on W by $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(i,j) = (\pi(i), \alpha_i(j))$ is permutation action of $S_{m-1} \wr S_{n-1}$ with respect to the subgroup K. □

Remark 3.2. *The set $\{\pi_\alpha := \pi_{(\alpha,\alpha,\dots,\alpha)} \in S_{m-1} \wr S_{n-1} | \ \pi \in S_{n-1}$ and $\alpha \in S_{m-1}\}$ form a subgroup which is isomorphic to $S_{m-1} \times S_{n-1}$. The restricted action of the subgroup $S_{m-1} \times S_{n-1}$ on W is defined as $\pi_\alpha(i,j) = (\pi(i), \alpha(j))$.*

Extend the action of the wreath product $S_{m-1} \wr S_{n-1}$ on W diagonally to an action of $S_{m-1} \wr S_{n-1}$ on $W^{\otimes k}$ as follows:

$$\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(v_{(i_1,j_1)} \otimes \dots \otimes v_{(i_k,j_k)}) = v_{(\pi(i_1),\alpha_{i_1}(j_1))} \otimes \dots \otimes v_{(\pi(i_k),\alpha_{i_k}(j_k))}, \tag{9}$$

where $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1}$. We will write the above action as $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(v_I) = v_{\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(I)}$.

Let $A \in End(W^{\otimes k})$. Define $A(v_J) = \sum_I A_I^J(v_I)$, where v_I is a basis element of $W^{\otimes k}$ and $A_I^J \in \mathbb{C}$ is the $(I,J)^{th}$ entry of A and $I, J \in \mathbb{S}^k$ ($\mathbb{S} = [n] \times [m]$). We have $End_{S_m \wr S_n}(W^{\otimes k}) \subseteq End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k}) \subseteq End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$. The following is Jones result analogue.

Lemma 3.3.

(i). $A \in End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(I)}^{\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})}(J)}$, for all $\pi_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1}$.

(ii). $A \in End_{S_{m-1} \times S_{n-1}}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{\pi_\alpha(I)}^{\pi_\alpha(J)}$, for all $\pi_\alpha \in S_{m-1} \times S_{n-1}$.

Proof. (i). We have $A \in End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$

$$
\begin{aligned}
&\Leftrightarrow \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} A = A\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}, \quad \text{for all } \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1} \\
&\Leftrightarrow \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} A(v_J) = A\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(v_J), \quad \text{for all } v_J \\
&\Leftrightarrow \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} \sum_I A_I^J(v_I) = A(v_{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(J)}) \\
&\Leftrightarrow \sum_I A_I^J \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(v_I) = \sum_I A_I^{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(J)}(v_I) \\
&\Leftrightarrow \sum_I A_I^J (v_{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(I)}) = \sum_I A_{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(I)}^{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(J)} (v_{\pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(I)})
\end{aligned}
$$

Since the wreath product $S_{m-1} \wr S_{n-1}$ action is defined by the permutation representation. The result (i) obtained from equating the scalars and linear independence . The result (ii) is same as that of (i). □

Lemma 3.4.

(i).

$$
dim\ End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k}) = \sum_{\substack{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_i)\vdash 2k+1 \\ i \le n}} S(2k+1,\lambda) \prod_{r=1}^{r=i} \sum_{j=1}^{j=m} S(\lambda_r, j) \tag{10}
$$

When $n-1, m-1 \ge 2k$,

$$
dim\ End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k}) = \sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_i)\vdash 2k+1} S(2k+1,\lambda) \prod_{r=1}^{r=i} B(\lambda_r) \tag{11}
$$

or, equivalently

$$
dim\ End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k}) = \sum_{l=1}^{2k+1} S(2k+1,l)B(l). \tag{12}
$$

(ii). $dim\ End_{S_{m-1} \times S_{n-1}}(W^{\otimes k}) = \sum_{i=1,j=1}^{i=n,j=m} S(2k+1,i)S(2k+1,j)$. When $n-1, m-1 \ge 2k$, $dim\ End_{S_{m-1} \times S_{n-1}}(W^{\otimes k}) = [B(2k+1)]^2$.

Proof. (i). By Lemma 3.3, A commutes with the $S_{m-1} \wr S_{n-1}$-action on $W^{\otimes k}$ if and only if the matrix entries of A are equal on $S_{m-1} \wr S_{n-1}$-orbits. Thus dim $End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$ is the number of $S_{m-1} \wr S_{n-1}$-orbits on $\mathbb{S}^{2k}$. Fix a tuple of indices (I, J) =$((i_1,j_1),(i_2,j_2),\ldots,(i_{2k},j_{2k}))\in \mathbb{S}^{2k}$. This tuple of indices determines the partitions $d := d(i_1, i_2, \ldots, i_{2k})$ and $\theta d := d((i_1, j_1),(i_2, j_2), \ldots,(i_{2k}, j_{2k}))$ of $[2k]$ set of $2k$ elements according to those that have an equal value. Remember that (a) the partition d has n classes at most, (b) the partition θd satisfies that $d \le \theta d$ and every class of d is partitioned into m classes at most of θd.

Let $[(I, J)]$ be the orbit of $(I, J) \in \mathbb{S}^{2k}$. Then

$$
\begin{aligned}
(I', J') \in [(I, J)] \Leftrightarrow\ & (I', J') = \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(I, J), \quad \text{for some } \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} \in S_{m-1} \wr S_{n-1} \\
\Leftrightarrow\ & (i'_r, j'_r) = \pi_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(i_r, j_r), \quad \text{for all } r \text{ with } 1 \le r \le 2k,
\end{aligned}
$$

where (i'_r, j'_r) and (i_r, j_r) are the r^{th} component of (I', J') and (I, J) respectively.

$$
\begin{aligned}
\Leftrightarrow\ & (i'_r, j'_r) = (\pi(i_r), \alpha_{i_r}(j_r)) \\
\Leftrightarrow\ & i'_r = \pi(i_r) \quad \text{and} \quad j'_r = \alpha_{i_r}(j_r)
\end{aligned}
$$

$$\Leftrightarrow\ [i'_p = i'_q \text{ iff } i_p = i_q\ (1 \le p, q \le 2k)], [i'_p = n \text{ iff } i_p = n\ (1 \le p \le 2k)] \text{ and}$$
$$[(i'_p, j'_p) = (i'_q, j'_q) \text{ iff } (i_p, j_p) = (i_q, j_q)], [(i'_p, j'_p) = (n, m) \text{ iff } (i_p, j_p) = (n, m)], \quad (1 \le p, q \le 2k)$$
$$\Leftrightarrow\ d(i'_1, i'_2, ..., i'_{2k}) = d(i_1, i_2, ..., i_{2k}), [i'_p = n \text{ iff } i_p = n\ (1 \le p \le 2k)] \text{ and}$$
$$d((i'_1, j'_1), (i'_2, j'_2), ..., (i'_{2k}, j'_{2k})) = d((i_1, j_1), (i_2, j_2), ..., (i_{2k}, j_{2k})),$$
$$[(i'_p, j'_p) = (n, m) \text{ iff } (i_p, j_p) = (n, m)\ (1 \le p \le 2k)].$$

Thus every $S_{m-1} \wr S_{n-1}$-orbit determines the partitions d and θd of $[2k]$ set of $2k$ elements, which satisfies (a), (b) and the classes $N = \{p \in [2k] \mid i_p = n\}$ of d, $L = \{p \in [2k] \mid (i_p, j_p) = (n, m)\}$ of θd (i.e. $(d_N \otimes \theta d_L)$) and vice versa.
If d and $\bar{d}$ be two $k + \frac{1}{2}$-partition diagrams of the same form (see, (5)) $\lambda = (\lambda_1, \lambda_2, ..., \lambda_i) \vdash 2k + 1$ $(i \le n)$, then the number of θd and $\theta \bar{d}$ satisfying the condition (b) are same and that is equal to $\prod_{r=1}^{r=i} \sum_{j=1}^{j=m} S(\lambda_r, j)$. Thus, we get (10).
When $m-1, n-1 \ge 2k$, then the number of θd and $\theta \bar{d}$ satisfying condition (b) are equal to $\prod_{r=1}^{r=i} B(\lambda_r)$. Thus, we get (11). If we vary over d by fixing a θd such that $d \le \theta d$, then the number of such d is $B(l)$ Bell number of l, where l is the number of classes in θd. Thus, we get (12).
(ii) Let $[(I, J)]$ be the $S_{m-1} \times S_{n-1}$-orbit of $(I, J) \in \mathbb{S}^{2k}$. Then $(I', J') \in [(I, J)]$

$$\Leftrightarrow (I', J') = \pi_\alpha(I, J), \qquad \text{for some } \pi_\alpha \in S_{m-1} \times S_{n-1}$$
$$\Leftrightarrow (i'_r, j'_r) = \pi_\alpha(i_r, j_r) \qquad \text{for all } r \text{ with } 1 \le r \le 2k,$$

where (i'_r, j'_r) and (i_r, j_r) are the r^{th} component of (I', J') and (I, J) respectively.

$$\Leftrightarrow (i'_r, j'_r) = (\pi(i_r), \alpha(j_r))$$
$$\Leftrightarrow i'_r = \pi(i_r) \quad \text{and} \quad j'_r = \alpha(j_r)$$
$$\Leftrightarrow [i'_p = i'_q \text{ iff } i_p = i_q\ (1 \le p, q \le 2k)], [i'_p = n \text{ iff } i_p = n\ (1 \le p \le 2k)] \text{ and}$$
$$[j'_p = j'_q \text{ iff } j_p = j_q\ (1 \le p, q \le 2k)], [j'_p = m \text{ iff } j_p = m\ (1 \le p \le 2k)]$$
$$\Leftrightarrow d(i'_1, i'_2, ..., i'_{2k}) = d(i_1, i_2, ..., i_{2k}), [i'_p = n \text{ iff } i_p = n\ (1 \le p \le 2k)] \text{ and}$$
$$d(j'_1, j'_2, ..., j'_{2k}) = d(j_1, j_2, ..., j_{2k}), [j'_p = m \text{ iff } j_p = m\ (1 \le p \le 2k)].$$

Thus every $S_{m-1} \times S_{n-1}$-orbit determines the partitions $d, \bar{d}$ of $[2k]$ the set $2k$ elements, with classes at most n, m respectively and the classes $N = \{p \in [2k] \mid i_p = n\}$ of d, $M = \{p \in [2k] \mid j_p = m\}$ of $\bar{d}$ (i.e. $(d_N \otimes \bar{d}_M)$) and vice versa. Hence the result. □

For every $S_{m-1} \wr S_{n-1}$-orbit $(d_N \otimes \theta d_L) = [(i, j)]$ we define, a matrix $T^I_J \in End(W^{\otimes k})$ by $T^I_J = \sum_{(I', j') \in [(I, J)]} E^{I'}_{J'}$, where $E^{I'}_{J'}$ is the matrix unit, which has non-zero entry 1 in $(J', I')^{th}$ position. In fact, $T^I_J \in End(W^{\otimes k})$, since such a matrix satisfies the Lemma 3.3(i) condition: The entries of the matrix are equal on $S_{m-1} \wr S_{n-1}$-orbits. Using equation (**??**), we have that

$$T^{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)}_{(i_{k+1}, j_{k+1}), (i_{k+2}, j_{k+2}), ..., (i_{2k}, j_{2k})} = \sum E^{(i'_1, j'_1), (i'_2, j'_2), ..., (i'_k, j'_k)}_{(i'_{k+1}, j'_{k+1}), (i'_{k+2}, j'_{k+2}), ..., (i'_{2k}, j'_{2k})}, \tag{13}$$

where the sum is running over $i_p = i_q \Leftrightarrow i'_p = i'_q$, $(i_p, j_p) = (i_q, j_q) \Leftrightarrow (i'_p, j'_p) = (i'_q, j'_q), (1 \le p, q \le 2k)$ [i.e. $p \sim q$ in $d(i_1, i_2, ..., i_{2k}) \Leftrightarrow i'_p = i'_q$, $p \sim q$ in $d((i_1, j_1), (i_2, j_2), ..., (i_{2k}, j_{2k})) \Leftrightarrow (i'_p, j'_p) = (i'_q, j'_q)$] and $i_p = n \Leftrightarrow i'_p = n$, $(i_p, j_p) = (n, m)$ $\Leftrightarrow (i'_p, j'_p) = (n, m)$.

Since every matrix T^I_J is sum of different matrix units, the set $\{T^I_J \mid [(I,J)]$ is an $S_{m-1}\wr S_{n-1}$-orbit$\}$ is a linearly independent set. For $A \in End_{S_{m-1}\wr S_{n-1}}(W^{\otimes k})$, we obtain $A = \sum_{[(I,J)]} A^I_J T^I_J$ by using the lemma 3.3. Thus the matrices T^I_J span $End_{S_{m-1}\wr S_{n-1}}(W^{\otimes k})$, and so they are a basis for $End_{S_{m-1}\wr S_{n-1}}(W^{\otimes k})$.

Now, we define another basis for $End_{S_{m-1}\wr S_{n-1}}(W^{\otimes k})$ as follows: For each $S_{m-1}\wr S_{n-1}$-orbit $[(I,J)] = [(i_1,j_1),(i_2,j_2),....,(i_{2k},j_{2k})]$ define, the matrix $L^I_J = \sum T^{I'}_{J'}$, where the sum is running over $S_{m-1}\wr S_{n-1}$-orbit $[(I',J')] = [(i'_1,j'_1),\ (i'_2,j'_2),...,(i'_{2k},j'_{2k})]$ such that $d(i_1,i_2,\ ...,i_{2k}) \geq d(i'_1,i'_2,...,i'_{2k})$ and $d((i_1,j_1),(i_2,j_2),...,(i_{2k},j_{2k})) \geq d((i'_1,j'_1),$ $(i'_2,j'_2),\ ...,\ (i'_{2k},\ j'_{2k}))$. The matrix T^I_J can be expressed in terms of the matrix L^I_J by using Möbius inversion (see, [16]). So, they also span $End_{S_{m-1}\wr S_{n-1}}(W^{\otimes k})$. By using the equation (13), we get

$$L^{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k})} = \sum E^{(i'_1,j'_1),(i'_2,j'_2),...,(i'_k,j'_k)}_{(i'_{k+1},j'_{k+1}),(i'_{k+2},j'_{k+2}),...,(i'_{2k},j'_{2k})}, \tag{14}$$

where the sum is running over $i_p = i_q \Rightarrow i'_p = i'_q,\ (i_p,j_p) = (i_q,j_q) \Rightarrow (i'_p,j'_p) = (i'_q,j'_q),\ (1 \leq p,q \leq 2k)$ and $i_p = n \Rightarrow i'_p = n,$ $(i_p,j_p) = (n,m) \Rightarrow (i'_p,j'_p) = (n,m)$.

Similarly, for each $S_{m-1}\times S_{n-1}$-orbit $[(I,J)]$ define, a matrix in $End_{S_{m-1}\times S_{n-1}}(W^{\otimes k})$ by using (**??**), as follows:

$$\widetilde{T}^{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k})} = \sum E^{(i'_1,j'_1),(i'_2,j'_2),...,(i'_k,j'_k)}_{(i'_{k+1},j'_{k+1}),(i'_{k+2},j'_{k+2}),...,(i'_{2k},j'_{2k})}, \tag{15}$$

where the sum is running over $i_p = i_q \Leftrightarrow i'_p = i'_q,\ j_p = j_q \Leftrightarrow j'_p = j'_q,\ (1 \leq p,q \leq 2k)$ and $i_p = n$ iff $i'_p = n,\ j_p = m$ iff $j'_p = m$.
Also, for each $S_{m-1}\times S_{n-1}$-orbit $[(I,J)]$ define, a matrix

$$\widetilde{L}^{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),...,(i_{2k},j_{2k})} = \sum E^{(i'_1,j'_1),(i'_2,j'_2),...,(i'_k,j'_k)}_{(i'_{k+1},j'_{k+1}),(i'_{k+2},j'_{k+2}),...,(i'_{2k},j'_{2k})}, \tag{16}$$

where the sum is running over $i_p = i_q \Rightarrow i'_p = i'_q,\ j_p = j_q \Rightarrow j'_p = j'_q,\ (1 \leq p,q \leq 2k)$ and $i_p = n \Rightarrow i'_p = n,\ j_p = m \Rightarrow j'_p = m$.
The matrices $\widetilde{T}^I_J$ and $\widetilde{L}^I_J$ form two different bases for $End_{S_{m-1}\times S_{n-1}}(W^{\otimes k})$.

Remark 3.5. If $[j_p = j_q \Rightarrow i_p = i_q]$ then (14) and (16) are equal.

3.2. The structure of the algebra $P_{k+\frac{1}{2}}(x,y)$

A $k+\frac{1}{2}$-partition diagram in the rook partition algebra is a k-partition diagram whose corresponding partition d has a special class N of d (the special class$N$ may be empty). Every $k+\frac{1}{2}$- partition diagram can be defined as d_N, where N is the unique special class of d and d is the k-partition diagram for the set $[2k]$ of $2k$ elements.

For each d_N, we may obtain a partition of $2k+1$ vertices by including a $(2k+1)$th vertex in N of d and vice versa. Therefore, the number of $k+\frac{1}{2}$-partition diagrams in a rook partition algebra is $B(2k+1)$. Thus, we may get every $k+\frac{1}{2}$-partition diagram as a k-diagram with an additional vertex on diagram's right side connected to the special class N.

Consider the algebra $P_{k+\frac{1}{2}}(x) \otimes P_{k+\frac{1}{2}}(y)$. The algebra $P_{k+\frac{1}{2}}(x) \otimes P_{k+\frac{1}{2}}(y)$ be the tensor product algebra of $P_{k+\frac{1}{2}}(x)$ and $P_{k+\frac{1}{2}}(y)$ with dimension $[B(2k+1)]^2$, where $P_{k+\frac{1}{2}}(x)$ and $P_{k+\frac{1}{2}}(y)$ are the rook partition algebras. The standard basis of this algebra is as follows:

$$\mathscr{K}_{k+\frac{1}{2}} := \{(d_N \otimes \bar{d}_M) \mid d_N, \bar{d}_M \text{ are } k+\tfrac{1}{2}\text{-partition diagrams}\}$$

Let $\mathscr{K}'_{k+\frac{1}{2}} := \{(d_N \otimes \theta d_L) \in \mathscr{K}_{k+\frac{1}{2}} \mid d_N \leq \theta d_L\}$. We may consider $(d_N \otimes \theta d_L) \in \mathscr{K}'_{k+\frac{1}{2}}$ as a colored partition diagram with corresponding diagram d with special class N of d, which is colored by θd with special class L of θd. Moreover, since $d_N \leq \theta d_L$, we may consider $(d_N \otimes \theta d_L) \in \mathscr{K}'_{k+\frac{1}{2}}$ as a colored diagram with corresponding partition diagram d_N, where every class F of d_N is colored by a partition of F.

Let $(d'_{N'} \otimes \theta d'_{L'}), (d''_{N''} \otimes \theta d''_{L''}) \in \mathscr{K}'_{k+\frac{1}{2}}$. The multiplication of two elements in $\mathscr{K}'_{k+\frac{1}{2}}$ defined as follows: $(d'_{N'} \otimes \theta d'_{L'}), (d''_{N''} \otimes \theta d''_{L''}) \in \mathscr{K}'_{k+\frac{1}{2}}$ and $d''_{N''} d'_{N'} = x^{\lambda} d_N$ in $P_{k+\frac{1}{2}}(x)$. Then clearly, $(\theta d''_{L''})(\theta d'_{L'}) = y^{\lambda+\beta} \theta d_L$ in $P_{k+\frac{1}{2}}(y)$, $\beta \geq 0$ and for some $\theta d_L \geq d_N$. That is, $(d''_{N''} \otimes \theta d''_{L''})(d'_{N'} \otimes \theta d'_{L'}) = x^{\lambda} y^{\lambda+\beta}(d_N \otimes \theta d_L)$. Therefore, in $\mathscr{K}'_{k+\frac{1}{2}}$ the multiplication of any two elements is a scalar product of certain element in $\mathscr{K}'_{k+\frac{1}{2}}$. Thus, the span of all elements in $\mathscr{K}'_{k+\frac{1}{2}}$ is a subalgebra of the algebra $P_{k+\frac{1}{2}}(x) \otimes P_{k+\frac{1}{2}}(y)$ with identity denoted as $P_{k+\frac{1}{2}}(x, y)$, called as *rook class partition algebra*. The dimension for this algebra is

$$|\mathscr{K}'_{k+\frac{1}{2}}| = \sum_{\lambda \vdash 2k+1} S(2k+1, \lambda) \prod_{r=1}^{r=l} B(\lambda_r),$$

where $B(\lambda_r)$ is the Bell number of the number λ_r and $S(2k+1, \lambda)$ is defined as in (5). Then the Bell number defined in equation (1) can also be equivalently defined as

$$B(2k+1) = \sum_{\lambda \vdash 2k+1} S(2k+1, \lambda),$$

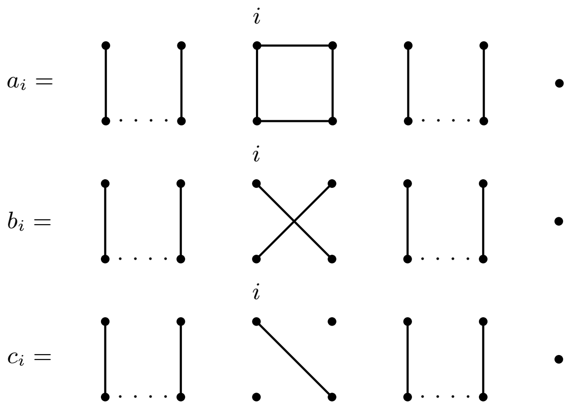
where the number of equivalence relations of $2k+1$ vertices is Bell number $B(2k+1)$.

Remark 3.6. For the algebra $P_{k+\frac{1}{2}}(x, y)$ we have an another formula for the dimension as follows:

$$|\mathscr{K}'_{k+\frac{1}{2}}| = \sum_{l=1}^{2k+1} S(2k+1, l) B(l),$$

where $S(2k+1, l)$ is defined as in equation (1).

Define the elements:



where $1 \leq i \leq k-1$. The algebra $P_{k+\frac{1}{2}}(x, y)$ is generated by the set of all elements $\{(a_i \otimes a_i), (a_i \otimes b_i), (b_i \otimes b_i), (b_i \otimes c_i), (c_i \otimes c_i)\}$, where $\{a_i, b_i, c_i \mid 1 \leq i \leq k-1\}$ is the generating set of the rook partition algebra $P_{k+\frac{1}{2}}(x)$.

3.3. Schur–Weyl duality

Diagonally we have an action of $S_{m-1} \wr S_{n-1}$ on $W^{\otimes k}$, where W is the permutation representation of the wreath product $S_{m-1} \wr S_{n-1}$. Also, we have an action of $P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)$ on $W^{\otimes k}$ defined as follows: Define a map $\phi : P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m) \longrightarrow End(W^{\otimes k})$ by defining it on a basis diagram $(d_N \otimes \bar{d}_M)$, as follows:

$$\phi(d_N \otimes \bar{d}_M) = \sum_{\substack{p \sim q \text{ in } d \Rightarrow i_p = i_q \\ p \sim q \text{ in } \bar{d} \Rightarrow j_p = j_q \\ p \in N \Rightarrow i_p = n, \; p \in M \Rightarrow j_p = m}} E^{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)}_{(i_{k+1}, j_{k+1}), (i_{k+2}, j_{k+2}), \ldots, (i_{2k}, j_{2k})} \quad (17)$$

Then we define an action of $P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)$ on $W^{\otimes k}$ as follows

$$(d_N \otimes \bar{d}_M)(v_J) = \phi(d_N \otimes \bar{d}_M)(v_J), \qquad \text{for all } J \in \mathbb{S}^k.$$

When $m = 1$, this action can be restricted to action of the rook partition algebra case on tensors.

Thus, we have an action of a basis element $(d_N \otimes \bar{d}_M) \in P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)$ on $W^{\otimes k}$ by defining it on the standard basis element by

$$(d_N \otimes \bar{d}_M) \cdot (v_{(i_1,j_1)} \otimes v_{(i_2,j_2)} \otimes \cdots \otimes v_{(i_k,j_k)}) = \sum_{\substack{1 \le i_{k+1},\dots,i_{2k} \le n \\ 1 \le j_{k+1},\dots,j_{2k} \le m \\ p \in N \Rightarrow i_p = n,\ p \in M \Rightarrow j_p = m}} \psi(d)^{i_1,i_2,\dots,i_k}_{i_{k+1},\dots,i_{2k}} \psi(\bar{d})^{j_1,j_2,\dots,j_k}_{j_{k+1},\dots,j_{2k}} v_{(i_{k+1},j_{k+1})} \otimes \cdots \otimes v_{(i_{2k},j_{2k})}.$$

Lemma 3.7.

(i). The map $\phi : P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m) \longrightarrow End(W^{\otimes k})$ is an algebra homomorphism onto on $End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$.

(ii). The restriction of ϕ on $P_{k+\frac{1}{2}}(n,m) \longrightarrow End(W^{\otimes k})$ is onto on $End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$.

Proof. From (17) we have,

$$\phi(d_N \otimes \bar{d}_M) = \sum_{\substack{d(i_1,i_2,\dots,i_{2k}) \le d \\ d(j_1,j_2,\dots,j_{2k}) \le \bar{d} \\ p \in N \Rightarrow i_p = n,\ p \in M \Rightarrow j_p = m}} E^{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\dots,(i_{2k},j_{2k})}, \tag{18}$$

where $1 \le i_1, i_2, \dots, i_{2k} \le n$, $1 \le j_1, j_2, \dots, j_{2k} \le m$.

$$\text{(i.e.)} \quad \phi(d_N \otimes \bar{d}_M) = \sum_{\substack{d(i_1,i_2,\dots,i_{2k}) \le d \\ d(j_1,j_2,\dots,j_{2k}) \le \bar{d} \\ p \in N \Rightarrow i_p = n,\ p \in M \Rightarrow j_p = m}} \widetilde{T}^{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\dots,(i_{2k},j_{2k})},$$

where the sum is running over one representative $(i_1, j_1), (i_2, j_2), \dots, (i_{2k}, j_{2k})$ for one $S_{m-1} \times S_{n-1}$-orbit. Thus, $\phi(d_N \otimes \bar{d}_M) \in End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$.

If $d_N \le \bar{d}_M$, then the equation (17) can be written as

$$\phi(d_N \otimes \bar{d}_M) = \sum_{\substack{p \sim q \text{ in } d \Rightarrow i_p = i_q \\ p \sim q \text{ in } \theta d \Rightarrow (i_p,j_p) = (i_q,j_q) \\ p \in N \Rightarrow i_p = n,\ p \in L \Rightarrow (i_p,j_p) = (n,m)}} E^{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\dots,(i_{2k},j_{2k})},$$

where $1 \le i_1, i_2, \dots, i_{2k} \le n$, $1 \le j_1, j_2, \dots, j_{2k} \le m$.

$$\text{(i.e.)} \quad \phi(d_N \otimes \bar{d}_M) = \sum_{\substack{d(i_1,i_2,\dots,i_{2k}) \le d \\ d((i_1,j_1),(i_2,j_2),\dots,(i_{2k},j_{2k})) \le \bar{d} \\ p \in N \Rightarrow i_p = n,\ p \in M \Rightarrow (i_p,j_p) = (n,m)}} T^{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)}_{(i_{k+1},j_{k+1}),(i_{k+2},j_{k+2}),\dots,(i_{2k},j_{2k})},$$

where the sum is running over one representative $(i_1, j_1), (i_2, j_2), \dots, (i_{2k}, j_{2k})$ for one $S_{m-1} \wr S_{n-1}$-orbit. Thus, $\phi(d_N \otimes \bar{d}_M) \in End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$, if $d_N \le \bar{d}_M$.

Claim: The map ϕ is an algebra homomorphism.

Let $(d'_{N'} \otimes \bar{d}'_{M'}), (d''_{N''} \otimes \bar{d}''_{M''}) \in P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)$ and $(d''_{N''} \otimes \bar{d}''_{M''})(d'_{N'} \otimes \bar{d}'_{M'}) = n^{\lambda} m^{\bar{\lambda}} (d_N \otimes \bar{d}_M)$, where $d''_{N''} d'_{N'} = n^{\lambda} d_N$ in $P_{k+\frac{1}{2}}(n)$ and $\bar{d}''_{M''} \bar{d}'_{M'} = m^{\bar{\lambda}} \bar{d}_M$ in $P_{k+\frac{1}{2}}(m)$. From (18) we have,

$\phi(d''_{N''} \otimes \bar{d}''_{M''})\phi(d'_{N'} \otimes \bar{d}'_{M'})$

$$= \sum_{\substack{d(i''_1, i''_2, \ldots, i''_{2k}) \leq d'' \\ d(j''_1, j''_2, \ldots, j''_{2k}) \leq \bar{d}'' \\ p \in N'' \Rightarrow i''_p = n,\ p \in M'' \Rightarrow j''_p = m}} E^{(i''_1, j''_1),(i''_2, j''_2),\ldots,(i''_k, j''_k)}_{(i''_{k+1}, j''_{k+1}),\ldots,(i''_{2k}, j''_{2k})} \sum_{\substack{d(i'_1, i'_2, \ldots, i'_{2k}) \leq d' \\ d(j'_1, j'_2, \ldots, j'_{2k}) \leq \bar{d}' \\ p \in N' \Rightarrow i'_p = n,\ p \in M' \Rightarrow j'_p = m}} E^{(i'_1, j'_1),(i'_2, j'_2),\ldots,(i'_k, j'_k)}_{(i'_{k+1}, j'_{k+1}),\ldots,(i'_{2k}, j'_{2k})},$$

where $1 \leq i''_z, i'_z \leq n, 1 \leq j''_z, j'_z \leq m$, and $1 \leq z \leq 2k$.

$$= \sum_{\substack{d(i''_1, i''_2, \ldots, i''_{2k}) \leq d'', d(j''_1, j''_2, \ldots, j''_{2k}) \leq \bar{d}'' \\ d(i'_1, i'_2, \ldots, i'_{2k}) \leq d', d(j'_1, j'_2, \ldots, j'_{2k}) \leq \bar{d}' \\ p \in N'' \Rightarrow i''_p = n, p \in M'' \Rightarrow j''_p = m,\ p \in N' \Rightarrow i'_p = n, p \in M' \Rightarrow j'_p = m}} \delta^{(i''_1, j''_1),(i''_2, j''_2),\ldots,(i''_k, j''_k)}_{(i'_{k+1}, j'_{k+1}),\ldots,(i'_{2k}, j'_{2k})} E^{(i'_1, j'_1),(i'_2, j'_2),\ldots,(i'_k, j'_k)}_{(i''_{k+1}, j''_{k+1}),\ldots,(i''_{2k}, j''_{2k})},$$

since $E^q_p E^s_r = \delta_{qr} E^s_p$, where δ_{qr} is the Kronecker delta.

$$= \sum_{\substack{d(i''_1, i''_2, \ldots, i''_{2k}) \leq d'', d(j''_1, j''_2, \ldots, j''_{2k}) \leq \bar{d}'' \\ d(i'_1, i'_2, \ldots, i'_{2k}) \leq d', d(j'_1, j'_2, \ldots, j'_{2k}) \leq \bar{d}' \\ p \in N'' \Rightarrow i''_p = n, p \in M'' \Rightarrow j''_p = m,\ p \in N' \Rightarrow i'_p = n, p \in M' \Rightarrow j'_p = m}} \delta^{i''_1, i''_2, \ldots, i''_k}_{i'_{k+1}, \ldots, i'_{2k}} \delta^{j''_1, j''_2, \ldots, j''_k}_{j'_{k+1}, \ldots, j'_{2k}} E^{(i'_1, j'_1),(i'_2, j'_2),\ldots,(i'_k, j'_k)}_{(i''_{k+1}, j''_{k+1}),\ldots,(i''_{2k}, j''_{2k})}$$

$$= n^{\lambda} m^{\bar{\lambda}} \sum_{\substack{d(i_1, i_2, \ldots, i_{2k}) \leq d \\ d(j_1, j_2, \ldots, j_{2k}) \leq \bar{d} \\ p \in N \Rightarrow i_p = n,\ p \in M \Rightarrow j_p = m}} E^{(i_1, j_1),(i_2, j_2),\ldots,(i_k, j_k)}_{(i_{k+1}, j_{k+1}),\ldots,(i_{2k}, j_{2k})}, \qquad \text{as in the rook partition algebra case.}$$

$$= n^{\lambda} m^{\bar{\lambda}} \phi(d_N \otimes \bar{d}_M)$$

$$= \phi((d''_{N''} \otimes \bar{d}''_{M''})(d'_{N'} \otimes \bar{d}'_{M'})).$$

Note that every matrix $\widetilde{L}^I_J$ has pre image such that

$$\phi(d(i_1, i_2, \ldots, i_{2k})_N \otimes d(j_1, j_2, \ldots, j_{2k})_M) = \widetilde{L}^{(i_1, j_1),(i_2, j_2),\ldots,(i_k, j_k)}_{(i_{k+1}, j_{k+1}),\ldots,(i_{2k}, j_{2k})}$$

where $N = \{p \in [2k] \mid i_p = n\}$ and $M = \{p \in [2k] \mid j_p = m\}$ and hence ϕ is onto $End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$. By the Remark (3.5) we have that, every matrix L^I_J has pre image in $P_{k+\frac{1}{2}}(n, m)$. Therefore, the restriction of the map ϕ on $P_{k+\frac{1}{2}}(n, m)$ is onto $End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$. Hence proved. □

Corollary 3.8. *The algebras $\mathbb{C}[S_{m-1} \times S_{n-1}]$ and $P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)$ generate full centralizers of each other in $End(W^{\otimes k})$. That is, for $n-1, m-1 \geq 2k$, we have*

(i). $P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m) \cong End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$

(ii). $S_{m-1} \times S_{n-1}$ generates $End_{P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m)}(W^{\otimes k})$.

Proof. Proof of (i): Since, $n-1, m-1 \geq 2k$, dim $P_{k+\frac{1}{2}}(n) \otimes P_{k+\frac{1}{2}}(m) =$ dim $End_{S_{m-1} \times S_{n-1}}(W^{\otimes k})$. Therefore (i) follows from the Lemma 3.7.

Proof of (ii): This follows from (i) and the Double Centralizer Theorem. □

Corollary 3.9. *The algebras $\mathbb{C}[S_{m-1} \wr S_{n-1}]$ and $P_{k+\frac{1}{2}}(n, m)$ generate full centralizers of each other in $End(W^{\otimes k})$. That is, for $n-1, m-1 \geq 2k$, we have*

(i). $P_{k+\frac{1}{2}}(n, m) \cong End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$

(ii). $S_{m-1} \wr S_{n-1}$ generates $End_{P_{k+\frac{1}{2}}(n,m)}(W^{\otimes k})$.

Proof. Proof of (i): Since, $n-1, m-1 \geq 2k$, dim $P_{k+\frac{1}{2}}(n,m) =$ dim $End_{S_{m-1} \wr S_{n-1}}(W^{\otimes k})$. Therefore (i) follows from the Lemma 3.7.

Proof of (ii): This follows from (i) and the Double Centralizer Theorem. □

As the centralizer of the semisimple group algebra $\mathbb{C}[S_{m-1} \wr S_{n-1}]$, the $\mathbb{C}$-algebra $P_{k+\frac{1}{2}}(n,m)$ is semisimple for $n-1, m-1 \geq 2k$.

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