Int. J. Math. And Appl., 8(3)(2020), 141–146
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

The Lattice Structure of the Subgroups of Order 42 and 48 in the Subgroup Lattices of 2×2 Matrices Over Z_7

V. Durai Murugan[1,*], R. Seethalakshmi[2] and P. Namasivayam[3]

1 Department of Mathematics, St. Joseph College of Arts and Science, Vaikalipatti, Tenkasi, Tamilnadu, India.

2 Research Scholar, Department of Mathematics, The MDT Hindu College (Affiliated to Manonmaniam Sundaranar University, Tirunelveli), Pettai, Tamilnadu, India.

3 Department of Mathematics, The MDT Hindu College, Tirunelveli, Tamilnadu, India.

Abstract: Let $\mathcal{G} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_p, ad - bc \neq 0\right\}$. Then $\mathcal{G}$ is a group under matrix multiplication modulo p. Let $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} : ad - bc = 1\right\}$. Then G is a subgroup of $\mathcal{G}$. We have, $o(\mathcal{G}) = p(p^2-1)(p-1)$ and $o(G) = p(p^2-1)$. Let $L(G)$ denotes the lattice of subgroups of G, where G is the group of 2×2 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number. In this paper, we give the structure of the subgroups of order 42 and 48 of $L(G)$ in the case when $p = 7$.

Keywords: Matrix group, subgroups, Lagrange's theorem, Lattice, Atom.

1. Introduction

Let

$$\mathcal{G} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_p, ad - bc \neq 0\right\}.$$

Then $\mathcal{G}$ is a group under matrix multiplication modulo p. Let

$$G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} : ad - bc = 1\right\}.$$

Then G is a subgroup of $\mathcal{G}$. We have, $o(\mathcal{G}) = p(p^2-1)(p-1)$ [6] and $o(G) = p(p^2-1)$ [6]. In this paper, we give the structure of the subgroups of order 42 and 48 of $L(G)$ in the case when $p = 7$.

2. Preliminaries

In this section we give the definition needed for the development of the paper.

* E-mail: vvndurai@gmail.com

Definition 2.1. *A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair $(P, \leq)$ is called a **partially ordered set or poset**. A poset $(P, \leq)$ is totally ordered if every $x, y \in P$ are comparable, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.*

Definition 2.2. *Let $(P, \leq)$ be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the **supremum or join** of S. A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the **infimum or meet** of S.*

Definition 2.3. *Poset $(P, \leq)$ is called a **lattice** if every pair x, y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.*

Definition 2.4. *For two elements a and b in P, a is said to cover b or b is said to be covered by a (in notation, $a \succ b$ or $b \prec a$) if and only if $b < a$ and, for no $x \in P$, $b < x < a$.*

Definition 2.5. *An element $a \in P$ is called an **atom**, if $a \succ 0$ and it is a dual atom, if $a \prec 1$.*

Theorem 2.6. *If G is a finite group and H is a subgroup of G, then the order of H is a divisor of the order of G.*

Theorem 2.7. *If G is a finite group and $a \in G$, then the order of 'a' is a divisor of the order of G.*

Theorem 2.8. *Let G be a finite group and let p be any prime number that divides the order of G. Then G contains an element of order p.*

Theorem 2.9. *If p is a prime number and $p^\alpha \mid o(G)$, $p^{\alpha+1} \nmid o(G)$, then G has a subgroup of order p^α, called a p-sylow subgroup.*

Theorem 2.10. *The number of p-sylow subgroups in G, for a given prime p, is of the form $1 + kp$.*

3. Arrangement of Elements of G According to their Orders

The number of element of order 2 is 1. The number of elements of order 3 is 56. The number of elements of order 4 is 42. The number of elements of order 6 is 56. The number of elements of order 7 is 48. The number of elements of order 8 is 84. The number of elements of order 14 is 48.

4. Subgroups of G of Different Orders

The number of subgroups of order 2 is 1. The number of subgroups of order 3 is 27. The number of subgroups of order 4 is 21. The number of subgroups of order 6 is 28. The number of subgroups of order 7 is 8. The number of subgroups of order 8 is 21. The number of subgroups of order 12 is 18. The number of subgroups of order 14 is 8. The number of subgroups of order 16 is 21. The number of subgroups of order 21 is 8. The number of subgroups of order 42 is 8. The number of subgroups of order 48 is 14.

5. Lattice Structure of Some Lower Intervals of Subgroups of Order 42 in L(G) Over Z_7

Let U be an arbitrary subgroup of order 42. Then the elements of U must have orders 1,3,6,7 or 14. We tabulate the subgroups of order 42 in $L(G)$.

Table 1. **Intervals $[\{e\}, U_i]$ in $L(G)$, $i = 1, 2, \ldots, 8$**

Order	Subgroups	Order	Subgroups
42	U_1	42	U_2
21	T_1	21	T_2
14	R_1	14	R_2
7	N_1	7	N_2
6	M_1, M_{12}, M_{14}, M_{16}, M_{17}, M_{21}, M_{23}	6	M_1, M_{11}, M_{13}, M_{15}, M_{18}, M_{22}, M_{24}
3	K_1, K_{12}, K_{14}, K_{16}, K_{17}, K_{21}, K_{23}	3	K_1, K_{11}, K_{13}, K_{15}, K_{18}, K_{22}, K_{24}

Order	Subgroups	Order	Subgroups
42	U_3	42	U_4
21	T_3	21	T_4
14	R_3	14	R_4
7	N_3	7	N_4
6	M_4, M_6, M_9, M_{17}, M_{20}, M_{24}, M_{26}	6	M_4, M_5, M_{10}, M_{18}, M_{19}, M_{23}, M_{25}
3	K_4, K_6, K_9, K_{17}, K_{20}, K_{24}, K_{26}	3	K_4, K_5, K_{10}, K_{18}, K_{19}, K_{23}, K_{25}

Order	Subgroups	Order	Subgroups
42	U_5	42	U_6
21	T_5	21	T_6
14	R_5	14	R_6
7	N_5	7	N_6
6	M_3, M_6, M_8, M_{16}, M_{19}, M_{22}, M_{28}	6	M_3, M_5, M_7, M_{15}, M_{20}, M_{21}, M_{27}
3	K_3, K_6, K_8, K_{16}, K_{19}, K_{22}, K_{28}	3	K_3, K_5, K_7, K_{15}, K_{20}, K_{21}, K_{27}

Order	Subgroups	Order	Subgroups
42	U_7	42	U_8
21	T_7	21	T_8
14	R_7	14	R_8
7	N_7	7	N_8
6	M_2, M_8, M_{10}, M_{12}, M_{13}, M_{26}, M_{27}	6	M_2, M_7, M_9, M_{11}, M_{14}, M_{25}, M_{28}
3	K_2, K_8, K_{10}, K_{12}, K_{13}, K_{26}, K_{27}	3	K_2, K_7, K_9, K_{11}, K_{14}, K_{25}, K_{28}

We display one typical interval $[\{e\}\ , U_1]$ of $L(G)$ in the following diagram.

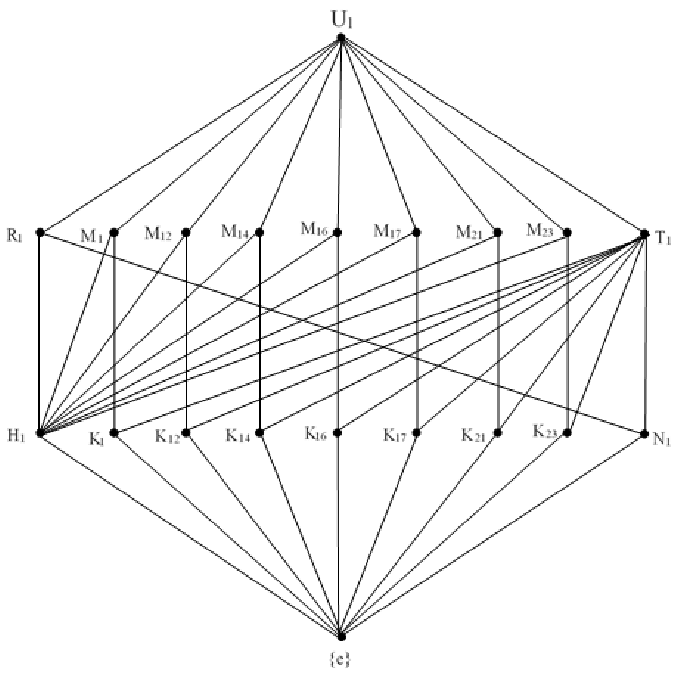

Figure 1. **The Interval $[\{e\}, U_1]$**

6. Lattice Structure of Some Lower Intervals of Subgroups of Order 48 in L(G) Over Z_7

Let V be an arbitrary subgroup of order 42. Then the elements of V must have orders 1,3,4,6 or 8. We tabulate the subgroups of order 48 in $L(G)$.

Table 2. **Intervals** $[\{e\}, V_i]$ **in** $L(G)$, $i = 1, 2, \ldots, 14$

Order	Subgroups	Order	Subgroups
48	V_1	48	V_2
16	S_{12}, S_{16}, S_{17}	16	S_{13}, S_{18}, S_{19}
12	Q_1, Q_4, Q_7, Q_8	12	Q_1, Q_3, Q_9, Q_{10}
8	P_{12}, P_{16}, P_{17}	8	P_{13}, P_{18}, P_{19}
6	M_1, M_4, M_7, M_8	6	M_1, M_3, M_9, M_{10}
4	$L_1, L_2, L_3, L_{10}, L_{11}, L_{12}, L_{14}, L_{16}, L_{17}$	4	$L_1, L_2, L_3, L_8, L_9, L_{13}, L_{15}, L_{18}, L_{19}$
3	K_1, K_4, K_7, K_8	3	K_1, K_3, K_9, K_{10}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_3	48	V_4
16	S_4, S_5, S_{15}	16	S_6, S_7, S_{14}
12	Q_2, Q_4, Q_{15}, Q_{16}	12	Q_2, Q_3, Q_{17}, Q_{18}
8	P_4, P_5, P_{15}	8	P_6, P_7, P_{14}
6	M_2, M_4, M_{15}, M_{16}	6	M_2, M_3, M_{17}, M_{18}
4	$L_1, L_4, L_5, L_{10}, L_{11}, L_{13}, L_{15}, L_{20}, L_{21}$	4	$L_1, L_6, L_7, L_8, L_9, L_{12}, L_{14}, L_{20}, L_{21}$
3	K_2, K_4, K_{15}, K_{16}	3	K_2, K_3, K_{17}, K_{18}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_5	48	V_6
16	S_5, S_{10}, S_{21}	16	S_4, S_{11}, S_{20}
12	Q_5, Q_9, Q_{12}, Q_{22}	12	$Q_6, Q_{10}, Q_{11}, Q_{21}$
8	P_5, P_{10}, P_{21}	8	P_4, P_{11}, P_{20}
6	M_5, M_9, M_{12}, M_{22}	6	$M_6, M_{10}, M_{11}, M_{21}$
4	$L_3, L_4, L_5, L_7, L_8, L_{10}, L_{15}, L_{16}, L_{21}$	4	$L_2, L_4, L_5, L_6, L_9, L_{11}, L_{15}, L_{17}, L_{20}$
3	K_5, K_9, K_{12}, K_{22}	3	$K_6, K_{10}, K_{11}, K_{21}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_7	48	V_8
16	S_6, S_9, S_{20}	16	S_7, S_8, S_{21}
12	Q_5, Q_8, Q_{14}, Q_{24}	12	Q_6, Q_7, Q_{13}, Q_{23}
8	P_6, P_9, P_{20}	8	P_7, P_8, P_{21}
6	M_5, M_8, M_{14}, M_{24}	6	M_6, M_7, M_{13}, M_{23}
4	$L_3, L_4, L_6, L_7, L_9, L_{11}, L_{14}, L_{19}, L_{20}$	4	$L_2, L_5, L_6, L_7, L_8, L_{10}, L_{14}, L_{18}, L_{21}$
3	K_5, K_8, K_{14}, K_{24}	3	K_6, K_7, K_{13}, K_{23}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_9	48	V_{10}
16	S_3, S_{10}, S_{16}	16	S_2, S_{11}, S_{17}
12	$Q_{11}, Q_{17}, Q_{19}, Q_{27}$	12	$Q_{12}, Q_{18}, Q_{20}, Q_{28}$
8	P_3, P_{10}, P_{16}	8	P_2, P_{11}, P_{17}
6	$M_{11}, M_{17}, M_{19}, M_{27}$	6	$M_{12}, M_{18}, M_{20}, M_{28}$
4	$L_3, L_5, L_9, L_{10}, L_{12}, L_{16}, L_{17}, L_{19}, L_{21}$	4	$L_2, L_4, L_8, L_{11}, L_{12}, L_{16}, L_{17}, L_{18}, L_{20}$
3	$K_{11}, K_{17}, K_{19}, K_{27}$	3	$K_{12}, K_{18}, K_{20}, K_{28}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_{11}	48	V_{12}
16	S_2, S_8, S_{18}	16	S_3, S_9, S_{19}
12	$Q_{14}, Q_{15}, Q_{19}, Q_{26}$	12	$Q_{13}, Q_{16}, Q_{20}, Q_{25}$
8	P_2, P_8, P_{18}	8	P_3, P_9, P_{19}
6	$M_{14}, M_{15}, M_{19}, M_{26}$	6	$M_{13}, M_{16}, M_{20}, M_{25}$
4	$L_2, L_7, L_8, L_{11}, L_{13}, L_{17}, L_{18}, L_{19}, L_{21}$	4	$L_3, L_6, L_9, L_{10}, L_{13}, L_{16}, L_{18}, L_{19}, L_{20}$
3	$K_{14}, K_{15}, K_{19}, K_{26}$	3	$K_{13}, K_{16}, K_{20}, K_{25}$

Order	Subgroups	Order	Subgroups
48	V_{13}	48	V_{14}
16	S_1, S_{12}, S_{14}	16	S_1, S_{13}, S_{15}
12	Q_{21}, Q_{22}, Q_{25}, Q_{26}	12	Q_{23}, Q_{24}, Q_{27}, Q_{28}
8	P_1, P_{12}, P_{14}	8	P_1, P_{13}, P_{15}
6	M_{21}, M_{22}, M_{25}, M_{26}	6	M_{23}, M_{24}, M_{27}, M_{28}
4	L_1, L_6, L_7, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}	4	L_1, L_4, L_5, L_{12}, L_{13}, L_{14}, L_{15}, L_{18}, L_{19}
3	K_{21}, K_{22}, K_{25}, K_{26}	3	K_{23}, K_{24}, K_{27}, K_{28}

We display one typical interval $[\{e\}, V_1]$ of $L(G)$ in the following diagram.

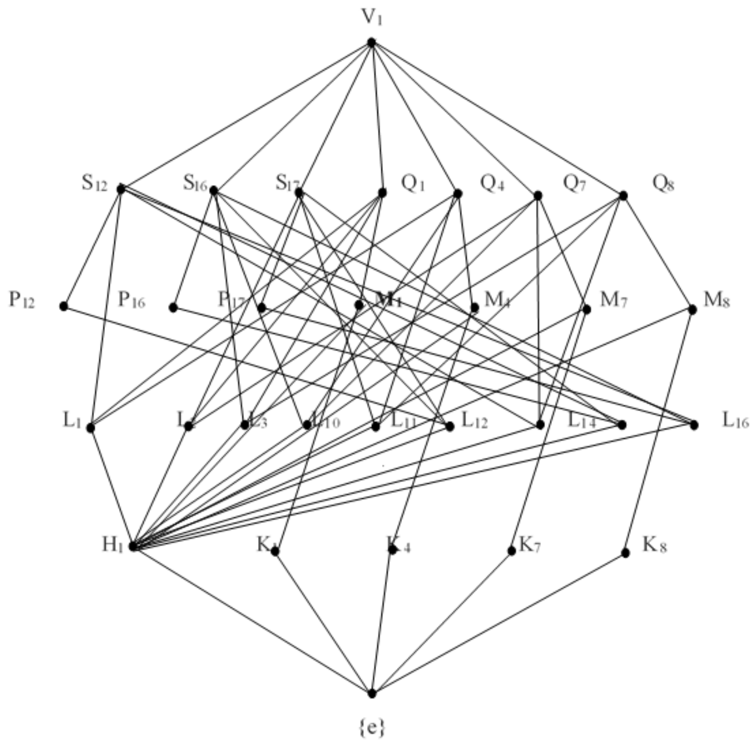

Figure 2. **The Interval** $[\{e\}, V_1]$

7. Conclusion

In this paper, we produced the lattice structure of subgroups of order 42 and 48 in the subgroup lattices of 2×2 matrices over Z_7.

References

[1] N. Bourbaki, *Elements of Mathematics, Algebra I*, Springer Verlag Berlin Heidelberg, New York, London Paris Tokio, (1974).

[2] J. B. Fraleigh, *A first course in Abstract Algebra*, Addison-Wesley,London, (1992).

[3] C. F. Gardiner, *A first course in group theory*, Springer-Verlag, Berlin, (1997).

[4] G. Gratzer, *General Lattice theory*, Birkhauser Veslag, Basel, (1998).

[5] I. N. Herstien, *Topics in Algebra*, John Wiley and Sons, New York, (1975).

[6] D. Jebaraj Thiraviam, *A Study on some special types of lattices*, Ph.D thesis, Manonmaniam Sundaranar University, (2015).

[7] A. Vethamanickam and D. Jebaraj Thiraviam, *On Lattices of Subgroups*, Int. Journal of Mathematical Archiv, 6(9)(2015), 1-11.

[8] A. Vethamanickam and V. Durai Murugan, *On The Lattice of Subgroups of* 3×3 *Matrices over* Z_2, Int. Journal of Scientific Research and Reviews, 8(2)(2019), 4107-4128.

[9] Bashir Humera and Zahid Raza, *On subgroups lattice of Quasidihedral group*, International Journal of Algebra, 6(25)(2012), 1221-1225.

[10] I. Chajda and S. Radeleczki, *O-conditions and Tolerance schemes*, Acta Math. Univ. Comenianae, LXX11(2)(2003), 177-184.

[11] R. Dedekind. *Über die Anzahl der Ideal –classen in den verschiedenen Ordnungen eines endlichen körpers*, Festschrift zur Saecularfeier des Geburtstages von C.F. Gauss, Vieweg, Braunschweig, (1877), 1-55; see Ges. Werke, Band I, Vieweg, Braunschweig, (1930), 105-157.

[12] Karen M. Gragg and P. S. Kung, *Consistent Dually semimodular lattices*, J. Combinatorial Theory, Ser. A, 60(1992), 246-263.

[13] Michio Suzuki, *On the lattice of subgroups of finite groups*, Tokyo university, Tokyo, Japan, 345-371.

[14] O. Ore, *Structures and group theory, I-II*, Duke Math. J., 3(1937), 149-174; 4(1938), 247-269.

[15] P. Pudlak and J. Tuma, *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis, 10(1980), 74-95.

[16] A. Rosenfeld, *Fuzzy groups*, J. Math Anal. and App., 35(1971), 512-517.

[17] A. Rottlaender, *Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen*, Math. Z., 28(1928), 641-653.

[18] R. Sulaiman, *Subgroup lattice of the symmetric group* S_4, International Journal of Algebra, 6(1)(2012), 29-35.

[19] A. Vethamanickam, *Topics in universal Algebra*, Ph.D Thesis, Madurai Kamaraj University, (1994).

[20] Vijay K. Khanna and S. K. Bhambri, *Course in Abstract Algibra*, Fifth Edition, Vikas Publishing House Pvt. Ltd., New Delhi.