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Degree Sum Exponent Distance Index of a Graph

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Abstract: In this paper, we determine degree sum exponent distance $\chi_{dist}(G)$ for some standard graphs and some graphs arising from complete graph, by taking it as a particular case of general sum-connectivity index.

MSC: 05C12.

Keywords: Degree, Distance, Degree sum distance exponent index.

1. Introduction

The generalized version of the first Zagreb index was introduced by Li and Zheng [2] and is given by,

$$M_1^{\alpha}(G) = \sum_{u \in V(G)} (d_G(u))^{\alpha} \tag{1}$$

If $\alpha = 2$, from Equation (1) we get a particular case which is well known as first Zagreb index [3] which is defined as,

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2$$

Several Zagreb index is very useful in QSPR and QSAR see [15–18]. In 1998 B. Bolloba's and P. Erdos [20], introduced the generalized version of the Randic index as,

$$R^{\alpha}(G) = \sum_{u,v \in V(G)} (d_G(u)d_G(v))^{\alpha} \tag{2}$$

If $\alpha = -1/2$, from Equation (2) we get a particular case which is known as Randic index. The Randic connectivity index, proposed by Randic in 1975 [1], is the most used molecular descriptor in QSPR and QSAR. Milan Randic invented a molecular structure descriptor defined as [21],

$$R(G) = \sum_{uv \in E(G)} (d_G(u)d_G(v))^{-1/2}$$

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This concept was extended to the general sum-connectivity index in [4], which is defined as,

$$\chi_\alpha(G) = \sum_{u,v \in E(G)} (d(u) + d(v))^\alpha \tag{3}$$

Several properties of degree sum connectivity index are studied in [4–7]. If $\alpha = 1$, from Equation (3) we get a particular case which is well known as first Zagreb index [3] which is defined as,

$$M_2(G) = \sum_{u,v \in V(G)} (d(u) + d(v))^1.$$

If $\alpha = -1/2$, from we get a particular case as sum connectivity index was initiated by Zhou and Trinajstic and defined as [27, 28],

$$S(G) = \sum_{u,v \in E(G)} (d(u) + d(v))^{-1/2}.$$

Graphs based on distance is widely used in mathematical chemistry [34], distance based graph invariant with connected graph G is the Wiener number $W(G)$, is also known as Wiener index given by Harold Wiener which is defined by [35] in 1947,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v),$$

where $d_G(u, v)$ denotes the distance between the vertices u and v in graph G. Eventually modification of Wiener index were proposed, which are known as degree distance and Schultz molecular topological index. In 1989, Harry P. Schultz introduced Schultz index which is defined by [23],

$$Sc(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d(u) + d(v))d(u, v)$$

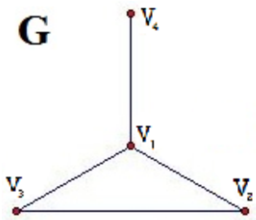
Several properties of Schultz molecular topological index are studied in [29–33]. In 1994, Dobrynin and Kochetova studied degree distance [8] which is defined by,

$$D'(G) = \sum_{v \in V(G)} d(v)d(u, v)$$

and several properties of the degree distance of connected graphs are studied in [9–12]. The purpose of this paper is to study the new index, called the degree sum exponent distance index. As we see distance play important role in degree distance index and Schultz index so we take distance exponent in our study. Let G be a connected graph of order n with vertex set $V(G) = (v_1, v_2, ..., v_n)$. We denote $d(u)$ as the degree of a vertex u, which is the number of edges incident on it and $d(u, v)$ as the distance between two vertices u and v which is the length of the shortest path joining them. Replacing α by $d(u, v)$ in equation (3) (the definition of general sum connectivity index), we define the degree sum exponent distance index of a connected graph G as,

$$\chi_{dist}(G) = \sum_{u,v \in V(G)} (d(u) + d(v))^{d(u,v)}$$

For example, for the graph shown below,



In G there are four pairs at distance one, with two pairs with degree sum 4 each and two pairs with degree sum 5 each, there are two pairs of vertices at distance two, with degree sum 3 each. Hence the degree sum exponent distance of G is, $\chi_{dist}(G) = 8 + 10 + 18 = 36$.

2. Degree Sum Exponent Distance Index of Some Graphs

Theorem 2.1. *The degree sum exponent distance index of complete graph K_n is, $\chi_{dist}(K_n) = n(n-1)^2$.*

Proof. Consider a complete graph K_n with n vertices. All the vertices have degree $n-1$ and they are at distance 1 hence taking the entire contribution,the degree sum exponent distance index of K_n is given by,

$$\chi_{dist}(K_n) = \sum_{u,v \in V(G)} [deg(u) + deg(v)]^{d(u,v)} = {}^{n}C_2.2(n-1) = n(n-1)^2$$

□

Theorem 2.2. *If $K_{m,n}$ is complete bipartite graph then degree sum exponent distance index is, $\chi_{dist}(K_{m,n}) = mn(4mn - m - n)$.*

Proof. Consider a complete bipartite graph $K_{m,n}$ with m vertices have degree n and n vertices have degree m. Taking $V^{'}$ and $V^{''}$ as the vertex partitions we have degree sum exponent distance index of $K_{m,n}$ given by,

$$\begin{aligned} \chi_{dist}(K_{m,n}) &= \sum_{u,v \in V(G)} [deg(u) + deg(v)]^{d(u,v)} \\ &= \sum_{u \in V', v \in V''} [n+m]^1 + \sum_{u,v \in V'} [n+n]^2 + \sum_{u,v \in V''} [m+m]^2 \\ &= mn(n+m) + {}^{m}C_2.\, 4n^2 + {}^{n}C_2.\, 4m^2 \end{aligned}$$

On simplification we get the desired result. □

Corollary 2.3. *If $K_{1,n}$ is star graph then degree sum exponent distance index of $K_{1,n}$ is, $\chi_{dist}(K_{1,n}) = n(3n-1)$.*

Proof. We put m=1 and n=$n-1$ in Theorem 2.2 to get the result. □

Corollary 2.4. *If $K_{n,n}$ is equi-bipartite graph then degree sum exponent distance index is, $\chi_d(K_{n,n}) = 2n^3(2n-1)$.*

Proof. We put $m = n$ in Theorem 2.2 to obtain the desired result. □

Theorem 2.5. *The degree sum exponent distance index of cocktail party graph $CP(n)$ is, $\chi_{dist}(CP(n)) = 24n(n-1)^2$, for $n \geq 2$.*

Proof. Consider a cocktail party graph $CP(n)$ obtained from complete graph of order $2n$ by removing a matching. The cocktail party graph is $2n-2$ regular graph. There are exactly n pairs of vertices at distance 2 and remaining ${}^{2n}C_2 - n$ at distance 1. Hence, the degree sum exponent distance index of $CP(n)$ is given by,

$$\chi_{dist}(CP(n)) = n(2n-2)^2 + 2(2n-2)[\,{}^{2n}C_2 - n]$$

On simplification we get the desired result. □

Theorem 2.6. *The degree sum exponent distance index of crown graph S_n^0 is, $\chi_{dist}(S_n^0) = 2n(n-1)^2(6n-5)$.*

Proof. Consider a crown graph S_n^0, obtained from $K_{n,n}$ by removing a matching $(n \geq 2)$. It's $n-1$ regular graph with ${}^nC_2$ pairs of vertices at distance 1 two times, ${}^nC_2$ pairs of vertices at distance 2 two times and n pairs of vertices at distance 3. So the degree sum exponent distance index of S_n^0 is given by,

$$\chi_{dist}(S_n^0) = {}^nC_2.8(n-1)^2 + {}^nC_2.4(n-1) + 8(n-1)^3 n$$

On simplification we get the desired result. □

Theorem 2.7. *The degree sum exponent distance of even cycle and odd cycle are given by,*

$$\chi_{dist}(C_{2n}) = \frac{8n}{3}(4^{n-1}-1) + n4^n \quad (4)$$
$$\chi_{dist}(C_{2n+1}) = \frac{4(2n+1)}{3}(4^n-1) \quad (5)$$

Proof. A cycle is 2 regular graph hence degree sum is always 4 for any pair of vertices. We discuss for even cycle and odd cycle separately.

Case (i): When Cycle is of even order say $2n$. There are $2n$ pair's of vertices, at distance 1 (corresponding to edges), at distance 2, at distance 3 so on till $n-1$ and exactly n pair's of vertices at distance n. Adding them, the total contribution is,

$$\chi_{dist}(C_{2n}) = 2n \times 4^1 + 2n \times 4^2 + 2n \times 4^3 + \ldots + 2n \times 4^{n-1} + n \times 4^n$$

On simplification we get the desired result.

Case (ii): When Cycle is of odd order say $2n+1$. Then there are $2n+1$ pair's of vertices at distance 1, distance 2, distance 3 till distance n. Adding all the contributions we get,

$$\chi_{dist}(C_{2n+1}) = (2n+1)4^1 + (2n+1)4^2 + (2n+1)4^3 + \ldots + (2n+1)4^n$$

On simplification we get the desired result. □

Theorem 2.1. *If P_n is path graph with $n \geq 4$ then degree sum exponent distance index is,*

$$\chi_{dist}(P_n) = 2^{n-1} + 2\sum_{i=1}^{n-2} 3^i + \sum_{i=n-3}^{1}\sum_{j=1}^{n-3} i4^j$$

Proof. Consider a path graph P_n with n vertices. There are only two distinct degrees in P_n i.e, 1 and 2, corresponding to pendant and non pendant vertices respectively. Hence, three possible degree sums are 2, 3 and 4 corresponding to pair of pendent vertices, pendent with middle and middle vertices respectively. There are

(i). $(n-1)$ pairs at distance 1. Out of which two pairs with degree sum 3 and remaining $(n-3)$ with degree sum 4. The contribution for degree sum exponent distance is $6+4(n-3)$.

(ii). $(n-2)$ pairs at distance 2. Out of which two pairs with degree sum 3 and remaining $(n-4)$ with degree sum 4. The contribution for degree sum exponent distance is $2\times 3^2+4^2(n-3)$.

Continuing this way finally we are left with a pair of pendent vertices at distance $n-1$ giving the corresponding contribution 2^{n-1}. Adding all the contributions we get,

$$\chi_{dist}(P_n)=2^{n-1}+(2)3^1+(2)3^2+\ldots+(2)3^{n-2}+(n-3)4^1+(n-4)4^2+\ldots+(1)4^{n-3}$$

On simplification we get the desired result. □

Theorem 2.8. *The degree sum exponent distance index of a book graph B_m is, $\chi_{dist}(B_m)=2m^3+94m^2-50m+2$.*

Proof. The graph $S_m\times P_2$ is called a Book graph B_m, where S_m is a star graph with $m+1$ vertices. B_m is a graph of order $2(m+1)$. There are $2m$ pairs of vertices having degree sum $m+3$, one pair with degree sum $2m+2$ and m pair of vertices with degree sum 4 here all pairs are at distance 1. Now for distance 2, there are $2m$ pair of vertices having degree sum $m+3$ and $2m$ pairs of vertices with degree sum 4. At distance 3, there are $2m$ pairs of vertices with degree sum 4. So the degree sum exponent distance index of B_m is given by,

$$\chi_{dist}(B_m)=[2m+2]+2m[m+3]+2m[m+3]^2+4m+4^2(2.\,{}^{m}C_2)+4^3(2.\,{}^{m}C_2)$$

On simplification we get the desired result. □

Theorem 2.9. *The degree sum exponent distance index of B_t is a book graph with triangular pages of order $(t+2)$ and size $(2t+1)$ is, $\chi_{dist}(B_t)=10t^2+2$.*

Proof. The graph obtained by joining both vertices of P_2 to t- isolated vertices is called triangular book graph and we denote such a graph of order $t+1$ as B_t. The book graph B_t with triangular pages has two set of vertices one with t vertices are of degree 2 and the remaining 2 vertices are of degree $t+1$. It's two vertices are of degree $t+1$ and remaining vertices of degree 2. So the degree sum exponent distance index of B_t is given by,

$$\chi_{dist}(B_t)=2[t+1]+2t[t+3]+4^2.\,{}^{t}C_2$$

On simplification we get the desired result. □

Theorem 2.10. *The degree sum exponent distance index of fan graph F_n is, $\chi_{dist}(F_n)=12n(3n-2)$.*

Proof. Consider a fan graph F_n, obtained from joining n copies of cycle graph C_3 with a common vertex of degree $2n$ and remaining of degree 2. So the degree sum exponent distance index of F_n is given by,

$$\chi_{dist}(F_n)=2n[2n+2]+4n+4^2.(\,{}^{2n}C_2-n)$$

On simplification we get the desired result. □

Theorem 2.11. *The degree sum exponent distance index of double star graph $D_{n,m}$ is,* $\chi_{dist}(D_{n,m}) = n^2(m+3)+m^2(n+3)+16mn+5(m+n)+2$.

Proof. Consider a double star graph $D_{n,m}$, obtained by joining the center of two stars $K_{1,n}$ and $K_{1,m}$ with an edge, where pair of vertices have degree $n+1$ and $m+1$ respectively with remaining vertices of degree 1. So the degree sum exponent distance index of $D_{n,m}$ graph is given by,

$$\chi_{dist}(D_{n,m}) = (n+m+2)+n(n+2)+m(n+2)^2+n(m+2)^2+m(m+2)+2^2.\,{}^nC_2+2^2.\,{}^mC_2+2^3mn$$

On simplification we get the desired result. □

Theorem 2.12. *The degree sum exponent distance index of wheel graph W_n is,* $\chi_{dist}(W_n) = 19n^2-45n$.

Proof. Let $(v_0, v_1, ...v_n)$ be the vertex set of wheel W_n with v_0 as the central vertex and $v_i, i = 1, 2, ...n$ be the peripheral vertices of degree 3 each. The contribution for degree sum exponent distance by central vertex with remaining n vertices is $n(n+3)$. Also, among the peripheral vertices there are n pairs of vertices of degree 3 at distance 1 namely v_i, v_{i+1} i=1,2...n under mod n. Thus total contribution for distance 1 being $n(n+3)+6n$. Now for distance 2 there are n possible pairs v_i, v_{i+2} $i = 1, 2...n$ under mod n contributing $6^2n = 36n$. Hence,

$$\chi_{dist}(W_n) = n[n+3]+n[6]^1+({}^{n-1}C_2-1)[6]^2.$$

On simplification we get the desired result. □

Theorem 2.13. *The degree sum exponent distance index of the Broom graphs $B_{n,m}$ is given by,*

$$\chi_{dist}(B_{n,m}) = n(2^m+3n)+(n+2)^{m-2}+\frac{(n+3)}{(n+2)}((n+3)^{m-2}-1)+\frac{3}{2}[n(3^{m-1}-3)+3^{m-2}-1]+\sum_{i=m-3}^{1}\sum_{j=1}^{m-3} i.4^j$$

Proof. The $B_{n,m}$ broom graph is the graph obtained by joining a star graph $K_{1,n}$ to a path graph P_m with a bridge. It is a coalescence of a star graph and a path graph with pendant vertex, where the $K_{1,n}$ having n vertices of degree 1 and one vertex with degree n which is a vertex coalescence, now for P_m having one vertex with degree 1 and remaining vertices with degree 2. For distance 1 corresponding to edges of $K_{1,n}$, we have $n+1$ pairs with degree sum $(n+1)$ and for distance 2, ${}^nC_2$ pairs with degree sum 2. Now for remaining vertices of P_m only two distinct degree i.e, 1 and 2, corresponding to pendant and non pendant vertices respectively. There are,

(i). $(m-2)$ pair at distance 1, out of which one pair of vertices have degree sum 3 and other pair of vertices have degree sum 4.

(ii). $(m-3)$ pair at distance 2, out of which one pair with degree sum 3, remaining vertices with degree sum 4.

Continuing this way finally we are left with a pair of pendent vertices at distance $n-1$. Adding we get,

$$\chi_{dist}(B_{n,m}) = [n+2]n+[n+3]^2+\ldots[n+3]^{m-2}+[n+2]^{m-1}+\,{}^nC_2.2^2+(3^2+3^3+\ldots3^{m-2})n$$
$$+2^mn+3^1+3^2+\ldots3^{m-2}+4^1(m-3)+4^2(m-4)+\ldots4^{m-4}.2+4^{m-3}.1$$

On simplification we get the desired result. □

3. Degree Sum Exponent Distance Index of Some Graphs Derived from Complete Graphs

In this section we discuss few graphs derived from complete graph through some operations and evaluate their degree sum exponent index. First we will consider some graphs derived from single complete graph. Let $K_n - e$ denote the graph obtained from complete graph by deleting any edge.

Theorem 3.1. *The degree sum exponent distance index of $K_n - e$ is, $\chi_{dist}(K_n - e) = (n-2)(n^2+4n-11)$.*

Proof. In $K_n - e$ there are two vertices of degree $n-2$ and remaining $n-2$ of degree $n-1$. So possible degree sums $2(n-2), 2(n-1)$ and $(2n-3)$. For distance 1 (corresponding to edges) there are ${}^{(n-2)}C_2$ with pairs of vertices with degree sum $2(n-1)$ and $2(n-2)$ pairs of vertices with degree sum $(2n-3)$. Hence, the contribution is

$$^{(n-2)}C_2 \times 2(n-1) + 2(n-2) \times (2n-3).$$

For distance 2 there is only one pair of vertices with degree sum $2(n-2)$ giving the contribution $2(n-2)^2$. Adding the contributions we get the required result. □

Definition 3.2. *The thorn of complete graph K_n^{+p} is obtained from the complete graph K_n by adding p number of pendent vertices to each vertex. This concept of thorn graphs was introduced by I. Gutman in 1998 see [13] and found variety of chemical applications.*

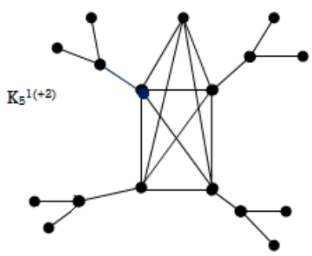
Theorem 3.3. *The degree sum exponent distance index of K_n^{+p} is,*

$$\chi_{dist}(K_n^{+p}) = np(n+p) + (n+p-1)\frac{n(n-1)}{2} + 2np(p-1)^2 + 8np(n-1).$$

Proof. The graph K_n^{+p} is of diameter 3 with two distinct vertex degrees $n+p-1$ corresponding to internal n vertices and 1 corresponding to pendent vertices. Hence possible degree sums are $n+p$ and 2. For distance 1 there are ${}^nC_2$ pairs of internal vertices with degree $n+p-1$ each. Also each of the internal vertices is adjacent to p pendent vertices with degree sum $n+p$. Hence, the contribution for distance 1 is $(n+p-1)({}^nC_2) + np$.
For distance 2 we have $n \times {}^pC_2$ pairs of vertices at distance two incident on a single vertex. The contribution is $n \times {}^pC_2 \times 2^2 = 4n\frac{p(p-1)}{2} = 2np(p-1)$.
Finally, for distance 3 we have $p(n-1+n-2+n-3+...+1) = 4p\frac{n(n-1)}{2} = np(n-1)$ pairs of vertices so the contribution is $np(n-1) \times 2^3 = 8np(n-1)$. Adding all the contributions we get the required result. □

3.1. First level thorn graph of complete graph

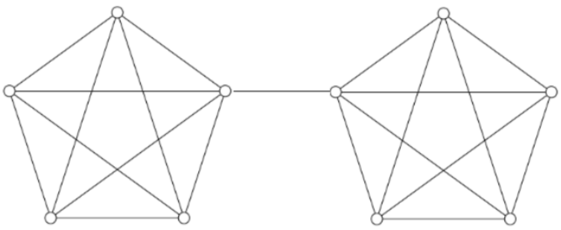
The first level thorn graph of a complete graph $K_n^{1(+p)}$ is obtained by joining an edge to each vertex of K_n and then joining 'p' pendent vertices to each pendant vertex of the new edges joined.



Theorem 3.4. *The degree sum exponent distance index of* $K_n^{1(+p)}$ *is,* $\chi_{dist}(K_n^{1(+p)}) = np(p+2) + n(n+p+1) + n^2(n-1) + np(n+1)^2 + n({}^pC_2)2^2 + n(n-1)(n+p+1)^2 + np(n-1)(n+1)^3 + {}^nC_2(2p+2)^2$.

Proof. The graph $K_n^{1(+p)}$ is of order $2n+np$ with $n + {}^nC_2 + np$ edges. There are n vertices of degree $n+1, n$ vertices of degree $p+1$ and np vertices of degree 1, the possible degree sums are ,$n+p+2$, $n+2$, $2(n+1)$, $2p+2$, and $p+2$.
For distance 1, (corresponding to edges) there are n pairs of vertices with degree sum $n+p+1$, np pairs of vertices with degree sum $p+2$, and ${}^nC_2$ pairs of vertices with degree sum '2n'.Hence contribution for distance 1 is, $n(n+p+1)+np(p+2)+{}^nC_2(2n)$.
For distance 2, there are $n({}^pC_2)$ pairs of vertices with degree sum 2, np pairs of vertices with degree sum $n+1$ and $n(n-1)$ pairs of vertices with degree sum $n+p+1$. Hence the contribution is $n({}^pC_2) \times 2^2 + np(n+1)^2 + n(n-1)(n+p+1)^2$.
For distance 3, there are $np(n-1)$ pairs of vertices with degree sum $n+1$ and ${}^nC_2$ pairs of vertices with degree sum $2p+2$, so that the contribution is $np(n-1)(n+1)^3 + ({}^nC_2)(2p+2)^3$.
For distance 4, there are, $p[(n-1)+(n-2)+\ldots 1] = \frac{pn(n-1)}{2}$ pairs of vertices with degree sum p+2. The contribution is $\frac{pn(n-1)}{2} \times (p+2)^4$.
Finally for distance 5 there are , $p^2[(n-1)+(n-2)+\ldots 1] = \frac{np^2(n-1)}{2}$ pairs of vertices with degree sum 2, so that the contribution is $\frac{np^2(n-1)}{2} \times 2^5$. Adding all we get the required result. Now we consider graphs derived from two complete graphs and obtain their degree sum exponent distance index. □

Bridge join of complete graphs:



Let K_n and K_m be complete graphs of order n and m respectively then bridge join of K_n and K_m denoted by, $K_nB_eK_m$ is the graph obtained by joining a vertex of K_m by a bridge with a vertex of K_n. It has diameter 3.

Theorem 3.5. *The degree sum exponent distance index of* $K_nB_eK_m$ *is,*

$$\chi_{dist}(K_nB_eK_m) = (m+n)+2(m-1)^2+2(n-1)^2+(m-1)(2m-1)+(n-1)(2n-1)+(m+n-2)(m+n-1)^2+(m-1)(n-1)(m+n-2)^3$$

Proof. The bridge join $K_nB_eK_m$ is of order $(m+n)$ with $({}^mC_2 + {}^nC_2 + 1)$ edges. It's diameter is 3. Three distinct degrees $m-1$, $n-1$ m and n are available. So possible degree sums are $m+n$, $m+n-2$, $2(m-1)$, $2(n-1)$, $2m-1$ and $2n-1$.
For distance 1, we have one pair with degree sum $m+n$, ${}^{m-1}C_2$ pairs with degree sum $m-2$, ${}^{n-1}C_2$ pairs with degree sum $(n-2)$, $(m-1)$ pairs with degree sum $(2m-1)$ and $(n-1)$ pairs with degree sum $(2n-1)$. Hence, contribution for degree sum exponent distance is

$$(m+n) + {}^{m-1}C_2.(m-2) + {}^{n-1}C_2.(n-2)(m-1)(2m-1) + (n-1)(2n-1).$$

For distance 2, there are total $(m+n-2)$ pairs of vertices joining a vertex of degree m or degree n with vertices of degree $(n-1)$ and $(m-1)$ respectively with same degree sum $(m+n-1)$. Hence, contribution is $(m+n-2)(m+n-1)^2$.

Finally for distance 3 there are $(m-1)(n-1)$ pairs of vertices with degree sum $(m+n-2)$ so that $(m-1)(n-1)(m+n-2)^3$ is the contribution. Adding all and simplifying we arrive at the desired result. □

By putting $m = 1$ in the above theorem we get degree sum exponent distance of $K_n + e$ (complete graph with an additional edge) as $\chi_{dist}(K_n + e) = (n+1) + (n-1)(n^2 + 4n - 5)$.

Definition 3.6 (Vertex Coalescence)**.** *If G_1 and G_2 are any two graphs then the graph obtained by gluing G_1 and G_2 at a point is v called vertex coalescence denoted by $G_1 O_v G_2$.*

Theorem 3.7. *The degree sum exponent distance index of vertex coalescence of $K_n O_v K_m$ is given by, $\chi_{dist}(K_n O_v K_m)$=$(m-1)^2(m-2) + (n-1)^2(n-2) + (2m+n-3)(m-1) + (2n+m-3)(n-1) + (m-1)(n-1)(m+n-2)^2$.*

Proof. The graph $K_n O_v K_m$ has diameter 2 with two sets of vertices, one at a distance 2 from each other and other at 1. We have ${}^{(m-1)}C_2$ pairs of vertices of degree $m-1$ each at distance 1, ${}^{(n-1)}C_2$ pairs of vertices of degree $n-1$ each at distance 1 and $m-1+n-1$ pairs combining with $m-1$ vertices of degree $m-1$ and $n-1$ vertices of degree $n-1$ giving $m+n-1+m-1$ and $m+n-1+n-1 = 2m+n-2+2n+m-2 = 3m+3n-4$. Hence total distance one contribution is,

$$2(m-1) \times {}^{m-1}C_2 + 2(n-1) \times {}^{(n-1)}C_2 + (2m+n-3)(m-1) + (2n+m-3)(n-1).$$

Also for distance 2 there are $(m-1)(n-1)$ pairs of vertices with degree sum $m+n-2$. Hence contribution for degree sum 2 is $(m-1)(n-1)(m+n-2)^2$. Adding we have,

$$\chi_{dist}(K_n O_v K_m) = (m-1)^2(m-2) + (n-1)^2(n-2) + (2m+n-3)(m-1) + (2n+m-3)(n-1) + (m-1)(n-1)(m+n-2)^2$$

□

Definition 3.8 (Edge Coalescence)**.** *If G_1 and G_2 are any two graphs then the graph obtained by merging G_1 and G_2 get on an edge 'e' is called edge coalescence denoted by $G_1 O_e G_2$.*

Theorem 3.9. *The degree sum exponent distance index of the edge coalescence of K_n and K_m is given by, $\chi_{dist}(K_n O_e K_m)$=* $2(n+m-3) + (m+n-4) + (3n+3m-8) + (n-1)(n-2)(n-3) + (m-1)(m-2)(m-3) + (n-2)(m-2)(n+m-2)^2$.

Proof. The graph $K_n O_e K_m$ has two sets of vertices one at a distance 2 from each other and other at 1, being of diameter 2. There are $(m-2)$ vertices of degree $(m-1)$,$(n-2)$ vertices of degree $(n-1)$ and two vertices of degree $(m+n-3)$. Hence the possible degree sums are $2(m-1)$, $2(n-1)(2m+n-4)(2n+m-4)$ and $2(m+n-3)$.
For distance 1, corresponding to edges of $K_m O_e K_n$, we have ${}^{(m-2)}C_2$ pairs with degree sum $2(m-1)$, ${}^{(n-2)}C_2$ pairs with degree sum $2(n-1)$,$(m-2)$ pairs with degree sum $(2m+n-4)$,$(n-2)$ with degree sum $(2n+m-4)$ and one pair with degree sum $2(m+n-3)$. Total contribution for degree sum exponent distance will be, $2(m-1) \times {}^{(m-2)}C_2 + 2(n-1) \times (n-2)C_2 + (m-2)(2m+n-4) + (n-2)(2n+m-4) + 2(m+n-3)$.
For distance 2, there are $(m-2)(n-2)$ only possible vertex pairs with degree sum $(m+n-2)$ giving contribution of $(m-2)(n-2)(m+n-2)$. Adding we get the desired result. □

Definition 3.10. *The Corona of two graphs G_1 and G_2 was defined by Frucht and Harary. The Corona $G_1 0_e G_2$ where G_1 and G_2 have order p_1 and p_2 respectively, is the graph obtained by taking p_1 copies of G_2 and then joining i^{th} point of G_1 to every vertex of i^{th} copy of G_2. $G_1 0_e G_2$ has, by definition $p_1(1+p_2)$ vertices and $q_1 + p_1(p_2+q_2)$ edges. Similarly, the Corona $G_2 0_e G_1$ can be defined.*

We discuss the degree sum exponent distance of Corona of complete graphs. Let K_m and K_n denote complete graphs of order m and n respectively. Then Corona $K_m 0_e K_n$ it has $m(m+n)$ vertices and ${}^{m}C_2 + m(n + {}^{n}C_2)$ edges.

Theorem 3.11. *The degree sum exponent distance index of $K_m 0 K_n$ is given by, $\chi_{dist}(K_n O K_m)$=$m(n-1)^2(n-2) + \frac{m(m-1)}{2}(m+n-2) + m(n-1)(m+2n-3) + m(m-1)(n-1)(m+2n-3)^2 + (n-1)^2(m-1)^2(m-2)$.*

Proof. The Corona graph $K_m 0 K_n$ is of diameter 3. The three distinct degrees acquired by the vertices are $(m-1)$, $(n-1)$ and $(m+n-2)$ respectively. Hence the possible degree sums are $(m+n-2)$, $(m+2n-3)$ and $(n+2m-3)$ respectively. For distance 1, we have $m\ {}^{(n-1)}C_2$ vertex pairs with degree sum $2(n-1)$, ${}^{m}C_2$ pairs with degree sum $(m+n-2)$ and $m(n-1)$ pairs with degree sum $(m+2n-3)$. Hence, the total contribution is $m[\ {}^{(n-1)}C_2] \times 2(n-1) + {}^{m}C_2 \times (m+n-2) + m(n-1) \times (m+2n-3)$.

For distance 2, we have $m(m-1)(n-1)$ pairs of vertices with degree sum $(m+2n-3)$ giving the contribution $m(m-1)(n-1)(m+2n-3)^2$.

Finally for distance 3, we have $(n-1)^2 \times \sum_{i=1}^{m-1} i = 2(n-1)^2(m-2)$ pairs with degree sum $(m+2n-3)$ giving the contribution $2(n-1)^2(m-2)(m+2n-3)$. Adding all and simplifying we get the conclusion. □

Definition 3.12 (Lollipop graph). *The Lollipop graph $L_(n,m)$ is the graph obtained by merging a vertex of complete graph K_n with one pendant vertex of path graph P_m.*

Theorem 3.13. *The degree sum exponent distance index of the lollipop graphs $L_{n,m}$ is given by,*

$$\chi_{dist}(L_{n,m}) = (n-1)[n^m+n^2-n+1+\frac{(n+1)}{n}((n+1)^{m-1}n-2)]+(n+1)^{m-1}+\frac{3}{2}(3^{m-2}-1)+\frac{(n+2)}{(n+1)}((n+2)^{m-2}-1)+\sum_{i=m-3}^{1}\sum_{j=1}^{m-3} i.4^j$$

Proof. The $L_{n,m}$ lollipop graph is the graph obtained by joining a complete graph K_n to a path graph P_m with a bridge. It is a coalescence of a complete graph and a path graph with pendant vertex, where the K_n having $n-1$ vertices of degree $n-1$ and one vertex with degree n, for P_m having one vertex with degree 1 and remaining vertices with degree 2.

For distance 1, corresponding to edges of K_n, we have $(n-1)$ pairs with degree sum $(2n-1)$, ${}^{n-1}C_2$ pairs with degree sum $(2n-2)$.

For distance 2, corresponding to edges of K_n and P_m we have $n-1$ vertices with degree sum $(n+2)$.

For distance 3, corresponding to edges of K_n and P_m we have $n-1$ vertices with degree sum $(n+2)$ and so on till distance $m-2$. Also one pair of distance $m-1$ has degree $n+1$.

Now for remaining vertices of P_m only two distinct degree i.e, 1 and 2, corresponding to pendant and non pendant vertices respectively. There are,

(i). $(m-1)$ pair at distance 1, out of which one vertex have degree 3, one vertex with degree n which is coalescence of a K_n with a P_m and remaining $(m-3)$ with degree sum 4.

(ii). $(m-2)$ pair at distance 2, out of which one pair with degree sum 3, one vertex with degree n which is coalescence of a K_n with a P_m and remaining $(m-3)$ with degree sum 4.

Continuing this way finally we are left with a pair of pendent vertices at distance $n-1$. Adding we get,

$$\begin{aligned}\chi_{dist}(L_{n,m}) &= [2n-1](n-1) + [n+2]^1 + [n+2]^2 + \ldots [n+2]^{m-2} + [n+1]^{m-1} + {}^{n-1}C_2(2n-2) \\ &+ (n-1)([n+1]^2 + [n+1]^3 + \ldots [n+1]^{m-1} + [n+1]^m) + 3^1 + 3^2 \\ &+ \ldots 3^{m-2} + 4^1(m-3) + 4^2(m-4) + \ldots 4^{m-4}.2 + 4^{m-3}.1\end{aligned}$$

On simplification we get the desired result. □

Theorem 3.14. *The degree sum exponent distance index of the windmill graph $Wd_{p,n}$ is given by,* $\chi_{dist}(Wd_{p,n}) = (n-1)^2 + [p(n-2)+p(p+1)] + 2p(p+1) + 2p(n-1)^4(p-1)$.

Proof. The $Wd_{p,n}$ windmill graph is obtained by joining n copies of the complete graph K_p at a shared universal vertex. It has $(p-1)n+1$ vertices and $\dfrac{np(p-1)}{2}$ edges.
For distance 1, $p(n-1)$ pairs of vertices have degree sum $(n-1)(p+1)$ and pairs of vertices have degree sum $n-1$.
For distance 2, there are $(n-1)^2$ pairs of vertices with degree sum $2(n-1)$.
Adding all contribution we get, $\chi_{dist}(Wd_{p,n}) = 2p(n-1)(\ ^{n-1}C_2) + p(p+1)(n-1)^2 + 2p(p-1)(n-1)^4$. □

Definition 3.15 (Splitting Graph)**.** *The splitting graph $S_p(G)$ of a graph G of order n, is obtained from G by adding one set of n new vertices say $(v_1, v_2, \ldots..v_n)$ corresponding to each vertex of G say $(u_1, u_2, \ldots, u_n)$ and joining each u_i to neighbor of v_i.*

Theorem 3.16. *The degree sum exponent distance of splitting graph of complete graph K_n is given by,* $\chi_{dist}(S_p(K_n)) = (n-1)^2(14n-9)$.

Proof. In splitting graph of K_n there are n vertices of degree $(2n-2)$ and n vertices of degree $(n-1)$. Hence possible degree sums are $(4n-4)$, $(3n-3)$ and $(2n-2)$.
For distance 1, corresponding to edges we have pairs of vertices of degree sum $(4n-4)$ and $n(n-1)$ pairs of vertices of degree sum $(3n-3)$. Hence the contribution is $(4n-4) + n(n-1)(3n-3)$.
For distance 2, there are exactly pairs of vertices with degree sum (3n-3) so the contribution is $\ ^{n}C_2 \times (3n-3)^2$. Adding and simplifying we get the result. □

Definition 3.17. *The parallel join of complete graphs is obtained by taking two copies of complete graph K_n with vertex sets (u_i) and (v_i) and then joining u_i to v_i, $i = 1, 2, \ldots, n$. We denote it by K_nPK_n.*

Theorem 3.18. *The degree sum exponent distance index of K_nPK_n is,* $\chi_{dist}(K_nPK_n) = 2n^3(2n-1)$.

Proof. The graph K_nPK_n is n regular.
For distance 1, $2 \times (\ ^{n}C_2)$ pairs of vertices with degree sum $2n$ and n pairs of vertices have degree sum $2n$, so the contribution is $n(n-1)2n + n \times 2n = 2n^2 + 2n^2(n-1) = 2n^3$.
For distance 2, $n(n-1)$ pairs of vertices have degree sum $2n$ giving the contribution $n(n-1)(2n)^2$.
On adding and simplifying we get the result. □

4. Conclusion

We discussed the degree sum exponent distance index for some graphs.

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