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Study of Problem of Some Fixed Point Result For Kannan and Chhatterjea Contraction on Generalized Complex Valued Metric Spaces

Surendra Kumar Tiwari[1,*] and Mridusmita Gauratra[2]

1 Department of Mathematics, Dr. C. V. Raman University, Kota, Bilaspur, Chhattisgarh, India.

Abstract: In this paper, we study and established some fixed point theorem for general Kannan and Chhattrjea type contraction in generalized complex valued metric spaces. The results extend and improve the common fixed point result of Elkouch & Marhrani [5], which is introduced by Issara et al. [7].

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1. Introduction

The concept of metric space is a very important tools in many scientific field and particularly in the fixed point theory. The conspicuous growth of metric space was fundamentally done by French Mathematician Freheninthe in the year 1906. Of course the Banach contraction principle [1] is the first important result on fixed points for contractive type mappings. So for, there have been a lot of fixed point results dealing with mappings satisfying different type of contractive inequalities. In particular, the concept of K-contraction and C-contraction were introduced by Kannan [2] and Chhatterjea [3] as follows:

Definition 1.1 ([2]). *The mapping $T : X \to X$ in metric space (X, d) is called K-contraction mapping if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{1}$$

Definition 1.2 ([3]). *The mapping $T : X \to X$ in metric space (X, d) is called C-contraction mapping if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:*

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] \tag{2}$$

* *E-mail: sk10tiwari.@gmail.com*

Recently, Jileli and Samet [4] introduced a very interesting concept of a generalized metric space which covers different well known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, Modular metric spaces and so on. In 2017, Elkouch and Marhrani [5] extend the Kannan contraction theorem defined by [2] in generalized metric space and they proved existence result for the Kannan contraction. Also, they introduced the Chhatterjea contraction with $\alpha = \frac{1}{2}$ in (X, d) such that

$$D(Tx, Ty) \leq \alpha(D(y, Tx) + D(x, Tx)), \text{ for all } x, y \in X. \tag{3}$$

Then they proved that a mapping T has a fixed point in X.

In the final work they introduced the Hardy-Rogers contraction [6] and prove the uniqueness of the fixed point in X. Subsequently, there are large number of generalization in this direction. Very recently, Issara and Uraint Depan [7] defined the generalized complex valued metric space and established a fixed point theorem for general Hardy-Rogers contraction which is extend the results of Elkouch and Marhrani [5]. In this paper, we establish and generalized some results on fixed points in generalized complex valued metric space. These results obtained the existing results of [5].

2. Preliminaries

In this section, we give some definitions and Lemma for this work. Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order relation $\leq$ on C as follows: $z_1 \leq z_2$ if and only if $Re(z_1) \leq Re(z_2)$. Thus $z_1 \leq z_2$ if one of the following holds:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \leq z_2$ and $z_1 \neq z_2$, i.e. one of $(C_2), (C_3)$ and (C_4) is satisfied and we will write $z_1 < z_2$ only (C_4) is satisfied.

Remark 2.1. *We can easily to check the following:*

(i). If $a, b \in R$, $0 \leq a \leq b$ and $z_1 \leq z_2$ then $az_1 \leq bz_2$, for all $z_1, z_2 \in C$.

(ii). $0 \leq z_1 \leq z_2 \Rightarrow |z_1| < |z_2|$.

(iii). $z_1 \leq z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Azam et al. [8] defined the complex valued metric space in the following way:

Definition 2.2 ([8])**.** *Let X be a non empty set. Suppose that the mapping $d : X \times X \to C$ satisfies the following conditions*

1. *$0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$.*
2. *$d(x, y) = d(y, x)$, for all $x, y \in X$;*
3. *$d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$.*

Then d is called a complex valued metric on X and the pair (X, d) is called a complex valued metric space. In 2017, Elkouch and Marhrani [5] define a new class of metric space. Let X be a non empty set and $D : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, defined the set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\}$$

Definition 2.3 ([4]). *A mapping D is called a generalized metric if it satisfies the following conditions*

1. *for every $(x, y) \in X \times X$, we have $D(x, y) = 0 \Leftrightarrow x = y$*
2. *for every $(x, y) \in X \times X$, we have $D(x, y) = D(y, x)$*
3. *There exists a real constant $C > 0$ such that for all $(x, y) \in X \times X$ and $x_n \in C(D, X, x)$, we have $D(x, y) \leq C\ \limsup D(x_n, y)$.*

The pair (X, D) is called a generalized metric space.

In this work, we consider a non empty set X, and $D : X \times X \to C$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n, x)| = 0\}.$$

Definition 2.4 ([7]). *Let X be a non empty set. Then a mapping $D : X \times X \to C$ is called a generalized complex valued metric if it satisfies the following conditions*

(i). for every $(x, y) \in X \times X$, we have $0 \leq D(x, y)$.

(ii). for every $(x, y) \in X \times X$, we have $D(x, y) = 0 \Leftrightarrow x = y$.

(iii). for every $(x, y) \in X \times X$, we have $D(x, y) = D(y, x)$.

(iv). There exists a complex constant $0 < r$ such that, for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, we have $D(x, y) \leq r \lim_{n\to\infty} sup|D(x, y)|$.

Then a pair (X, D) is called a general complex valued metric space.

Definition 2.5 ([8]).

1. *Let $X, D)$ be a generalized complex valued metric space and let $\{x_n\}$ be a sequence in X and $x \in X$. Then we say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D, X, x)$. It is denoted by $\lim_{n\to\infty} \{x_n\} = x$.*
2. *A sequence $\{x_n\}$ in X is called Cauchy sequence in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.*
3. *If every Cauchy sequence is convergent in X, then (X, D) is called generalized complete complex valued metric space.*

Lemma 2.6 ([8]). *Let λ is a real number such that $0 \leq \lambda < 1$, and let $\{x_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} b_n = 0$. Then for any sequence of positive numbers $\{a_n\}$ satisfying $a_{n+1} \leq a_n + b_n$, for all $n \in N$, we have $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

3.1. Problem of Fixed point for General Kannan contraction

In this section we prove some propositions for use in the main theorem and prove some fixed point theorem on generalized complex valued metric space.

Definition 3.1. *Let (X, D) be a generalized complex valued metric space. A self map f on X is called general Kannan contraction, if there exists $\lambda \in (0, \frac{1}{2})$ such that for all $x, y \in X$, satisfying*

$$D(fx, fy) \leq \lambda[D(x, fx) + D(y, fy)]. \tag{4}$$

Proposition 3.2. *Let (X, D) be a generalized complex valued metric space and let $f : X \to X$ be a Kannan contraction, for some $\lambda \leq (0, \frac{1}{2})$. Then any fixed point $u \in X$ of f satisfies $|D(u, u)| < \infty \Rightarrow D(u, u) = 0$.*

Proof. Let x_0 be a fixed point of such that $|D(u,u)| < \infty$ and $fu = u$. To show that $D(u,u) = 0$. Now

$$\begin{aligned} D(u,u) &= D(fu, fu) \\ &\leq \lambda[D(u, fu) + D(u, fu)] \\ &\leq 2\lambda D(u,u). \end{aligned}$$

By Remark 2.1(ii) we have, $|D(u,u)| = 0$. Hence $D(u,u) = 0$. □

Theorem 3.3. *Let (X, D) be a generalized complex valued metric space, and let f be a self mapping on X satisfying (4) for some constant $\lambda \in (0, \frac{1}{2})$ Such that $r\lambda < 1$, and there exists an element $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$. Then the sequence $\{f^k x_0\}$ converges to some $u \in X$ and u is a fixed point of f. Moreover, if v is a fixed point of f in X such that $|d(v,v)| < \infty$ and $|D(u,u)| < \infty$, then $u = v$.*

Proof. Let $K \in N(n \geq 1)$, for all $i, j \in N$, we have

$$D(f^{k+i}x_0, f^{k+j}x_0) = D(f(f^{k+i-1}x_0, f^{k+j-1}x_0)).$$

By (4), we have

$$D(f^{k+i}x_0, f^{k+j}x_0) \leq \lambda[D(f^{k+i-1}x_0, f(f^{k+i-1}x_0)) + D(f^{k+j-1}x_0, f(f^{k+j-1}x_0))].$$

By Remark 2.1(ii), we have

$$\begin{aligned} |D(f^{k+i}x_0, f^{k+j}x_0)| &\leq \lambda[|D(f^{k+i-1}x_0, f^{k+i}x_0) + D(f^{k+j-1}x_0, f^{k+j}x_0)|] \\ &\leq 2\lambda|\delta(D, f, f^{k-1}x_0)|. \end{aligned}$$

We have

$$|D(f^{k+i}x_0, f^{k+j}x_0)| \leq 2\lambda\delta|(D, f, f^{k-1}x_0)|. \tag{5}$$

By (5), we see that $2\lambda\delta(D, f, f^{k-1}x_0)$ is upper bound of the set $\{|D(f^{k+i}x_0, f^{k+j}x_0)|\}$. Since $\delta(D, f, f^k x_0)$ is least upper bound of $\{|D(f^{k+i}x_0, f^{k+j}x_0)|\}$, it follows that $\delta(D, f, f^k x_0) < 2\lambda\delta(D, f, f^{k-1}x_0)$. Similarly, we induction to

$$\begin{aligned} \delta(D, f, f^k x_0) &\leq 2\lambda\delta(D, f, f^{k-1}x_0) \\ &\leq \ (2\lambda)^2\delta(D, f^{k-2}x_0) \\ &\cdots\cdots\cdots\cdots \\ &\leq (2\lambda)^k\delta(D, f, x_0). \end{aligned}$$

By

$$|D(f^k x_0, f^m x_0)| = |D(f(f^{k-1}x_0, f(f^{m-1}x_0))|,$$

we have

$$|D(f^k x_0, f^m x_0)| \leq 2\lambda\delta(D, f, f^{k-1}x_0) \\ \leq (2\lambda)^{k-1}\delta(D, f, x_0), \tag{6}$$

for all integer m such that $m > k$. Since $\delta(D, f, x_0) < \infty$ and $2\lambda \in [0, 1]$. We have

$$\lim_{k\to\infty} \lambda^k \delta(D, f, x_0) = 0.$$

Thus

$$\lim_{m,k\to\infty} |D(f^k x_0, f^m x_0)| = 0. \tag{7}$$

Then, we have $\{f^k x_0\}$ is a Cauchy sequence. Since X be a complete generalised complex valud metric space, so there exists $u \in X$ such that

$$\lim_{k\to\infty} |D(f^k x_0, u)| = 0.$$

By Definition 2.4(iv)

$$D(fu, u) \leq r \lim_{k\to\infty} sup|D(fu, f^{k+1}x_0)|. \tag{8}$$

By Remark 2.1(ii), we have

$$|D(fu, u)| \leq |r| \lim_{k\to\infty} sup|D(fu, f^{k+1}x_0)|. \tag{9}$$

By (4), we have

$$D(f^{k+1}x_0, fu) \leq \lambda[D(f^{k+1}x_0, f^k x_0) + D(u, fu)]. \tag{10}$$

By Remark 2.1(ii), we have

$$|D(f^{k+1}x_0, fu)| \leq \lambda[|D(f^{k+1}x_0, f^k x_0)| + |D(u, fu)|]. \tag{11}$$

Now from (6) & (11), we obtain

$$\lim_{k\to\infty} supD(fu, f^{k+1}x_0) = \lambda D(fu, u).$$

But by Remark 2.1(ii)

$$\lim_{k\to\infty} sup|D(fu, f^{k+1}x_0)| = |\lambda||D(fu, u)|.$$

Using (8)

$$D(fu, u) \leq r\lambda D(fu, u).$$

By Remark 2.1(ii), we get

$$|D(fu,u)| \leq |r|\lambda |D(fu,u)|.$$

Since $r\lambda < 1$, we have $|r|\lambda < 1$, then $|D(fu,u)| = 0 \Rightarrow D(fu,u) = 0 \Rightarrow fu = u$. Therefore, u is fixed point of f in X. If v is another fixed point of f such that $D(u,v) < \infty$, we obtain

$$\begin{aligned} D(u,v) &= D(fu,fv) \\ &\leq \lambda(D(fu,u) + D(fv,v)) \\ &\leq \lambda(D(u,u) + D(v,v)) \\ &\leq 0. \end{aligned}$$

By Remark 2.1(ii), we have, $D(u,v) \leq 0$. Since $D(u,v) < \infty$ and $\lambda \in [0, \frac{1}{2}]$ so, we have, $|D(u,v)| = 0 \Rightarrow D(u,v) = 0 \Rightarrow u = v$. Therefore, u is the unique fixed point of f in X. □

3.2. Problem of fixed point for general chatterjea contraction

Definition 3.4. *Let (X,D) be a generalized complex valued metric space. A set mapping f on X is called general Chaterjea contraction, if there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$D(Tx,Ty) \leq \lambda[D(y,Tx) + D(x.Ty)], for\ all x, y \in X. \tag{12}$$

Theorem 3.5. *Let (X,D) be a generalized complex valued metric space and let T be a self mapping on X satisfying the condition (12) for some constant $\lambda \in [0, \frac{1}{2}]$ such that $r\lambda < 1$, for all $x, y \in X$. If there exists a point $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$, the sequence $\{f^n x_0\}$ converges to some $u \in X$. Moreover, if $D(x_0, Tu) < \infty$, then u is fixed point of f, and for any fixed point v of T such that $D(u,v) < \infty$, we have $u = v$.*

Proof. Let $n \in N(n \geq 1)$. For all integer i, j, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) = D[T(T^{n+i-1}x_0, T^{n+j-1}x_0)].$$

By (12), we have

$$\begin{aligned} D(T^{n+i}x_0, T^{n+j}x_0) &\leq \lambda[D(T^{n+j-1}x_0, T(T^{n+j-1}x_0) + D(T^{n+i-1}x_0, T(T^{n+j-1}x_0)] \\ &\leq \lambda[D(T^{n+j-1}x_0, T^{n+i}x_0) + D(T^{n+i-1}x_0, T^{n+j}x_0)] \\ &\leq 2\lambda\delta(D,T,T^{n-1}x_0). \end{aligned}$$

By Remark 2.1(ii), we have

$$|D(T^{n+i}x_0, T^{n+j}x_0)| \leq 2\lambda\delta|(D,T,T^{n-1}x_0)|. \tag{13}$$

Hence $\delta(D,T,T^n x_0) \leq 2\lambda\delta(D,T,T^{n-1}x_0)$, and consequently $\delta(D,T,T^n x_0) \leq (2\lambda)^2\delta(D,T,x_0)$. By

$$|D(T^n x_0, T^m x_0)| = |D(T(T^{n-1}x_0, T(T^{m-1}x_0))|,$$

we have

$$|D(T^n x_0, T^m x_0)| \leq 2\lambda\delta(D, T, T^n x_0)$$
$$\leq (2\lambda)^n \delta(D, T, x_0), \quad (14)$$

for all integer n, m such that $m > n$. Since $\delta(D, T, x_0) < \infty$ and $2\lambda \in [0, 1)$, we obtain

$$\lim_{m,n\to\infty} (2\lambda)^n \delta(D, T, x_0).$$

Thus

$$\lim_{n,m\to\infty} |D(T^n x_0, T^m x_0)| = 0. \quad (15)$$

Then, it follows that$\{x_n\}$ is a Cauchy sequence, Since X be a complete generalized complex valued metric space, so there exists $u \in X$ such that

$$\lim_{n\to\infty} |D(T^n x_0, u)| = 0.$$

By Definition 2.4(ii), we have

$$D(T^n x_0, u) \leq r \lim_{n\to\infty} sup D(T^n x_0, T^m x_0). \quad (16)$$

By Remark 2.1(ii), we have

$$|D(T^n x_0, u)| \leq |r| \lim_{n\to\infty} sup|D(T^n x_0, T^m x_0)|$$
$$(2\lambda)^n |r||\delta(D, T, x_0)|$$
$$\leq |r||\delta(D, T, x_0)|, \text{ for all } n \in N.$$

By (12), we have

$$D(T^{n+1} x_0, Tu) \leq \lambda[D(T^{n+1} x_0, u) + D(T^n x_0, Tu)].$$

By Remark 2.1(ii),we have

$$|D(T^{n+1} x_0, Tu)| \leq \lambda[|D(T^{n+1} x_0, u)| + |D(T^n x_0, Tu)|]. \quad (17)$$

Since $D(x_0, Tu) < \infty$, we have, $D(T^n x_0, Tu) < \infty$, for all $n \in N$. By Lemma 2.6, we obtain

$$\lim_{n\to\infty} D(T^n x_0, Tu) = 0.$$

By Remark 2.1(ii), we have

$$\lim_{n\to\infty} |D(T^n x_0, Tu)| = 0. \quad (18)$$

It follows by Definition 2.4(iv), we get

$$D(Tu, u) \leq r \lim_{n\to\infty} sup|D(Tu, T^n x_0)|.$$

So, by Remark 2.1(ii), we have

$$|D(Tu, u)| \leq |r||\lambda||D(Tu, u)|. \tag{19}$$

Since $r\lambda < 1$. Then $|D(Tu, u)| = 0$. Thus $D(Tu, u) = 0 \Rightarrow Tu = u$. Let v be any fixed point of T such that $D(u, v) < \infty$ and $|D(v, v)| < \infty$. Then by (12), we have

$$\begin{aligned} D(u, v) &= D(Tu, Tv) \\ &\leq \lambda[D(v, Tu) + D(u, Tv)] \\ &\leq \lambda[D(v, u) + D(u, v)] \\ &\leq 2\lambda D(u, v). \end{aligned}$$

By Remark 2.1(ii), we have

$$|D(u, v)| \leq 2\lambda|d(u, v)|$$

Since $r\lambda < 1$ and $D(u, v) < \infty$. So $|D(U, V)| = 0 \Rightarrow D(u, v) = 0 \Rightarrow u = v$. Therefore, u is the unique fixed point of T in X. This completes the proof □

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