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A New Approach to Solve Simultaneous Linear Diophantine Equations in Two Variables

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Abstract: In this paper, we describe a new approach for solving simultaneous linear Diophantine equations in two variables. We also introduce a new theorem that computes the solution of simultaneous linear Diophantine equations in two variables and proved theorem with an elaborated example. The theorem also helps to decide the existence of integer solutions to the simultaneous linear Diophantine equations in two variables.

Keywords: Diophantine equation, a system of linear Diophantine equations, greatest common divisor(gcd), and integer solution.

1. Introduction

A Diophantine equation is an algebraic equation with integer coefficients of which the integer solutions are required. In mathematics, a Diophantine equation is a polynomial equation, usually in two or more unknowns, such that only the integer solutions are sought or studied [3].

Let $f(x_1, x_2, \dots, x_n)$ be a given polynomial in the variables $x_i, i = 1, 2, \dots, n$ with integral coefficients and form the equations $f(x_1, x_2, \dots, x_n) = 0$. This is called **a Diophantine equation** when we consider it from the point of view of determining the integer numbers $x_1, x_2, x_3, \dots, x_n$ which satisfy it.

Systems of linear Diophantine equations are systems of linear equations in which the solutions are required to be integers. These systems can be tackled initially using similar techniques to those found in linear equations over the real numbers, using elementary methods such as elimination and substitution or more advanced methods from linear algebra. One major difference is that a single linear Diophantine equation does not always have integer solutions, even though it always has real solutions: for example, the linear Diophantine equation $4x - 2y = 5$ has no integer solutions (the left side is even and the right side is odd), but it has infinitely many solutions over the real numbers, of the form $x = 1.25 + t, \quad y = 2t$ [2]. Finding the set of all solutions is called solving the system of linear Diophantine equations.

The question of finding solutions of a Diophantine equation with more than one variable is quite complex. In this paper, we discussed a method that helps to determine the solution of the system of Linear Diophantine equations in two variables.

2. Mathematical Preliminaries

Definition 2.1. *Suppose that $f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$(ring of integers). Then the equation $f(x_1, x_2, \dots, x_n) = 0$ in $\mathbb{Z}$ is called **a Diophantine equation**. If $deg f = 1$, it is called a **linear Diophantine equation**. If $deg f \geq 2$, it is called*

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*a **Diophantine equation of higher order**.*

Example 2.2. $3x + 4y + 5z = 8$ *and* $x^2 - 2y^2 - 3z^2 = 0$ *are respectively linear and higher order Diophantine equations.*

Definition 2.3. *An integer b is said to be **divisible** by an integer $a \neq 0$, in symbols $a|b$, if there exists some integer c such that $b = ac$. We write $a \nmid b$ to indicate that b is **not divisible** by a [1].*

Definition 2.4 ([1])**.** *Let a and b be given integers, with at least one of them different from zero. The **greatest common divisor** of a and b, denoted by $gcd(a, b)$, is the positive integer d satisfying the following:*

(i) $d|a$ and $d|b$.

(ii) If $c|a$ and $c|b$, then $c \leq d$.

3. Linear Diophantine Equation

A linear Diophantine equation in two unknown is an equation of the form

$$ax + by = c$$

where a, b, c are given integers and a, b are not both zero. A solution of this equation is a pair of integers x_0, y_0 that, when substituted into the equation, satisfy it; that is, we ask that $ax_0 + by_0 = c$. The question of existence of solutions is satisfactorily answered by the following theorem.

Theorem 3.1. *The linear Diophantine equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ is solvable(has a solution) if and only if $gcd(a_1, a_2, \ldots, a_n)|c$ [4].*

Proof. Suppose that $d = gcd(a_1, a_2, \ldots, a_n)$ and $d|c$. Then $c = dq$ for some integer q. This imply, there exist integers $y_1, y_2, \ldots, y_n$ such that $d = a_1y_1 + a_2y_2 + \ldots + a_ny_n$. Then

$$c = dq = (a_1y_1 + a_2y_2 + \ldots + a_ny_n)q = a_1(y_1q) + a_2(y_2q) + \ldots + a_n(y_nq).$$

Hence setting $x_i = y_iq, i = 1, 2, \ldots, n$, we notice that the n-tuple $(x_1, x_2, \ldots, x_n)$ is a solution of the Diophantine equation. To prove the converse, suppose $x_1, x_2, \ldots, x_n$ are integers such that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = c.$$

Since $d|a_i$ for each i, let $a_i = dq_i, q_i \in \mathbb{Z}$. Then $c = d(q_1x_1 + q_2x_2 + \ldots + q_nx_n)$. Thus d is a factor of c. This completes the proof of Theorem. □

Theorem 3.1 gives us easy criteria to check whether a linear Diophantine equation has a solution. For instance, the equation

$$155x + 35y + 45z = 15$$

has a solution while

$$155x + 35y + 45z = 26$$

has no solution. Indeed $gcd(115, 35, 45) = 5$ is a factor of 15 but not 26. The draw-back of Theorem 3.1, however, is it does not give a clue how to find even one solution.

Basically there are different techniques that could be used to find specific solution of a linear Diophantine equation. In this paper, we use Euclidean Algorithm only.

Theorem 3.2. *The linear Diophantine equation $ax+by=c$ has a solution if and only if $d|c$, where $d=gcd(a,b)$. If x_o, y_o is any particular solution of this equation, then all other solutions are given by*

$$x = x_o + \left(\frac{b}{d}\right)t \quad y = y_o - \left(\frac{a}{d}\right)t$$

where t is an arbitrary integer [1].

3.1. Euclidean Algorithm Method

It follows from the theorem that once a specific solution of $ax+by=c$ is known, then every solution is obtainable. One of the methods used to find a specific solution of the Diophantine equation $ax+by=c$ is the Euclidean Algorithm.

Example 3.3. *Find the complete solution of* $60x+25y=10$.

Solution. Since $gcd(60,25)=5$ and 5 is factor of 10, Theorem 3.1 guarantees that the Diophantine equation $60x+25y=10$ has a solution. First we express $5=gcd(60,25)$ as a linear combination of 60 and 25. Using Division Algorithm, we get:

$$60 = 2\times 25 + 10$$
$$25 = 2\times 10 + 5$$

Thus

$$5 = 25 - 2\times 10 = 25 - 2(60 - 2\times 25) = (-2\times 60) + (5\times 25).$$

Hence (-2, 5) is a particular solution of the Diophantine equation $60x+25y=5$. It follows that $(-4,10)$ is a particular solution of $60x+25y=10$. Therefore the complete solution of the Diophantine equation under consideration is given by $(x,y)=(-4+5t, 10-12t), t\in\mathbb{Z}$.

4. Simultaneous linear Diophantine Equations in Two Variables

Many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such a problem, you need to find the solutions of a system of linear Diophantine equations. The following is an example of a system of two linear Diophantine equations in x and y.

$$\begin{cases} 2x+5y=10, \\ 7x-2y=-4, \end{cases}$$

Definition 4.1. *A Diophantine equation of the form*

$$\begin{cases} ax+by=m, \\ cx+dy=n, \end{cases}$$

is called simultaneous linear Diophatine equations in two variables,

Definition 4.2. *The simultaneous linear Diophantine equations in two variables*

$$\begin{cases} ax+by=m \\ cx+dy=n \end{cases}$$

has integer solutions if the following conditions holds true.

(i). each linear Diophantine equation has an integer solutions.

(ii). the intersection of the solution set of each linear Diophantine equation is non empty i.e let S_1 is an integer solution set of $ax+by=m$ and S_2 is an integer solution set of $cx+dy=n$, then the system has an integer solution if $S_1 \cap S_2 \neq \emptyset$

Theorem 4.3. *The simultaneous linear Diophantine equations in two variables*

$$\begin{cases} ax+by=m \\ cx+dy=n \end{cases}$$

has integer solutions if and only if $t = \dfrac{d_1(n-cx_0-dy_0)}{r} \in \mathbb{Z}$ *where* $d_1 = gcd(a,b), r = cb-da \neq 0$ *and* (x_0,y_0) *are particular integer solution of* $ax+by=m$.

Proof. Suppose the system has a solution. Now, (x_0,y_0) is a particular solution of $ax+by=m$ and $d_1 = gcd(a,b)$. Hence $x = x_0 + \frac{b}{d_1}t, y = y_0 - \frac{a}{d_1}t$ is the a general solution of $ax+by=m$. Now, substitute $x = x_0 + \frac{b}{d_1}t, y = y_0 - \frac{a}{d_1}t$ in the second Diophantine equation $cx+dy=n$ and determine t.

$$\begin{aligned} c\left(x_0 + \frac{b}{d_1}t\right) + d\left(y_0 - \frac{a}{d_1}t\right) &= n \\ cx_0 + \frac{cb}{d_1}t + dy_0 - \frac{da}{d_1}t &= n \\ cx_0 + dy_0 + \frac{cb}{d_1}t - \frac{da}{d_1}t &= n \\ \left(\frac{cb}{d_1} - \frac{da}{d_1}\right)t &= n - cx_0 - dy_0 \\ t &= \frac{n-cx_0-dy_0}{\frac{cb}{d_1} - \frac{da}{d_1}} \\ t &= \frac{n-cx_0-dy_0}{\frac{cb-da}{d_1}} \\ t &= \frac{d_1(n-cx_0-dy_0)}{cb-da} \end{aligned}$$

Since the system has a solution, then $t = \dfrac{d_1(n-cx_0-dy_0)}{cb-da} \in \mathbb{Z}$.

Conversely, suppose $t = \dfrac{d_1(n-cx_0-dy_0)}{cb-da} \in \mathbb{Z}$ where $d_1 = gcd(a,b), r = cb-da$ and (x_0,y_0) are particular integer solution of $ax+by=m$. This implies $x = x_0 + \frac{b}{d_1}t, y = y_0 - \frac{a}{d_1}t$ is a general solution of $ax+by=m$. Next to determine the solution of the second Diophantine equation substitute the above general solution of the first Diophantine equation to the second Diophantine equation.

$$\begin{aligned} cx+dy &= c\left(x_0 + \frac{b}{d_1}t\right) + d\left(y_0 - \frac{a}{d_1}t\right) \\ &= cx_0 + \frac{(cb)}{d_1}t + dy_0 - \frac{(da)}{d_1}t \\ &= cx_0 + dy_0 + \frac{(cb)}{d_1}t - \frac{(da)}{d_1}t \\ &= cx_0 + dy_0 + \frac{(cb-da)}{d_1}t \\ &= cx_0 + dy_0 + \frac{(cb-da)}{d_1}\frac{d_1(n-cx_0-dy_0)}{cb-da} \\ &= cx_0 + dy_0 + n - cx_0 - dy_0 \\ &= n \end{aligned}$$

Therefore, the simultaneous linear Diophantine equations has a solution of the form

$$x = x_0 + \frac{b}{d_1}t, \quad y = y_0 - \frac{a}{d_1}t \qquad \text{where} \quad t = \frac{d_1(n - cx_0 - dy_0)}{cb - da} \in \mathbb{Z}$$

. □

Example 4.4. *Find all the integer solutions to the following simultaneous linear Diophantine equations.*

$$\begin{cases} 3x + 9y = 15 \\ 6x - 8y = 4 \end{cases}$$

Solution. To determine the solution of the system first we need to check the existence of the solution of each linear Diophantine equation. The linear Diophantine equation $3x + 9y = 15$ has a solution because $d_1 = gcd(3, 9) = 3|15$ and $6x - 8y = 4$ has a solution because $d_2 = gcd(6, -8) = 2|4$. Next we evaluate the general solution of the first Linear Diophantine equation. To obtain the integer 3 as a linear combination of 3 and 9.

$$3 = 3(1) + 9(0)$$

Upon multiplying this relation by 5, we arrive at $15 = 5 \times 3 = 5[3(1) + 9(0)] = 3(5) + 9(0)$, so that $x_0 = 5, y_0 = 0$ provide one solution to the first linear Diophantine equation in question. All other solutions are expressed by

$$x = 5 + \frac{9}{3}t = 5 + 3t, \quad y = 0 - \frac{3}{3}t = -t, \quad t \in Z$$

Now to determine the solution of the simultaneous linear Diophantine equation we need to determine the value of t. So,

$$t = \frac{d_1(n - cx_0 - dy_0)}{cb - da} = \frac{3(4 - 6(5) - (-8)0)}{6(9) - (-8)3} = \frac{3(4 - 30)}{54 + 24} = \frac{3(-26)}{78} = -1 \in \mathbb{Z}.$$

Thus the simultaneous linear Diophantine equation has a solution. Therefore, the solution of a simultaneous linear Diophantine equations is $x = 5 + 3t = 5 + 3(-1) = 2, \quad y = -t = -(-1) = 1$.

5. Conclusion

We conclude that to solve a system of linear Diophantine equations in two variables first we need to check the existence of integer solutions for each linear Diophantine equation. And next, checking the existence of solution of a system of linear Diophantine equations. Finally, find a solution to the system of linear Diophantine equations using the proposed method.

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