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System Representation of the Lorentz Gas

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Abstract: We introduce a random environment system modeling a Lorentz Gas with semi-randomly positioned, periodic scatterers. Each system is determined through a probabilistic method producing a deterministic walk. We prove that the path within the generalized system must be periodic and calculate the expected number of loops within the path of a 1-door system. For n-door systems, where $n > 1$, we describe an algorithm to calculate the expected number of loops of its path by making use of its bijection to the Lorentz Gas.

Keywords: Lorentz Gas, Random Environment, Random walk, Algorithm.

1. Introduction

The Lorentz Gas model, proposed in 1905, has remained as one of the primary models for chaotic diffusion, utilizing either a random or periodic configuration of scatterers. Assuming that a free particle is moving within such environment, it changes direction through collisions with larger, immovable particles (scatterers). The model uses these scatterers to determine the path of the free particle similar to one in a billiard system, where it is specularly reflected on the surface of the larger particles.

Much work has been done on systems resembling the Lorentz Gas. Some have considered a Lorentz Gas model on the lattice with scatterers on its vertices in [1, 2], and [3]. The random walks and dynamic properties of the system has proven to be useful as the system itself can be modified easily for various applications. Research has also been done on the kinetic energy growth of such models [4].

Bunimovich has examined a larger set of Lorentz Gas models and their paths, in which he calls deterministic walks in random environments (DWRE), which he examined properties of both one-dimensional and multi-dimensional DWRE [5]. Furthermore, a Lorentz Gas with rotating scatterers was also introduced [6] to combat the lack of a local thermal equilibrium in the original model. While difficult to analyze, it has shown to model the more complex systems in statistical mechanics. In general, such models of diffusion can be partitioned into two subclasses, the first utilizing a stochastic process in which the direction of a particle is determined randomly and independently at each step. The second utilizes a system in which, while produced randomly, predetermines the path of the particle.

In this paper, we introduce a system resembling the latter, in which the status of a set of positioned scatterers is predetermined through a random process, determining a unique path. We explain several properties of the system both mathematically and through an algorithm (for the more complex systems) which utilizes the resemblance to the Lorentz Gas.

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2. The N-Door System

Suppose an ant was stuck in an infinitely tall building with circular floors in which each consecutive floor is connected by a single door. Each door, however, is preset to be either open or closed with a probability of $\frac{1}{2}$. Furthermore, the ant can only move along the circumference of the circular floors in a counter-clockwise direction, the only exception being when he moves up or down a floor through the door. Visually, one such system and its path labeled by numbers is shown in figure 1.

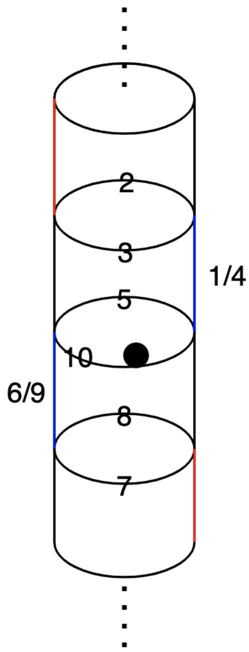

Figure 1. **Example of a 1-Door System**

The black circle represents the starting point of the ant, and each colored line represents the door, with blue being open and red being closed. The ant's path, then, is determined by moving along the circumference of the floor in a counter-clockwise direction, choosing only to move up or down a floor if the door it reaches is blue. Note while every building is infinitely long in both directions, we didn't need to show the other floors since the path was restricted in a periodic fashion. We call the system in figure 1 a 1-door system, since consecutive floors are connected by one door. Through the same logic, we can define an n-door system to be one with consecutive floors connected by n doors. However, we have to define the orientation of the doors. Suppose we were to project the system onto the ground, with each door's position labeled as a point. For example, the projection of a 1-door system is shown in figure 2.

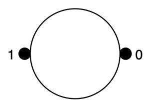

Figure 2. **Projection of a 1-Door System**

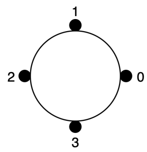

Figure 3. **Projection of a 2-Door System**

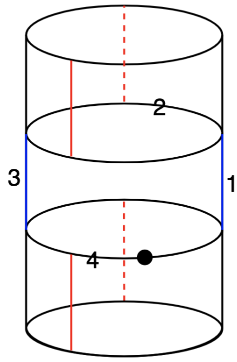

Figure 4. **Example of a 2-Door System**

Note that since the system is a 1-door system, there are two points on the circle, equidistant from each other. Every floor would either have the even or the odd indexed positioned door, switching between the two every level. In the case of a 2-door system, its projection would have 4 points, all equidistant on the circle. A 2-door system would have 2 doors on each floor all of which are located on either the even or the odd positions. An example of such a system and its path is shown in figure 4. Again, this 2-door system has a periodic path, and thus the rest of the system does not need to be shown. We can extend this definition to a n-door system, in which consecutive floors are connected by n doors, oriented in the even/odd indices.

3. Periodicity of the N-door System

Every example of an n-door system we have thus far seen have periodic paths, in that the path is closed and returns to the initial point. A natural question to ask is whether all n-door systems share this property. To prove periodicity we can prove that a path crosses over a door moving in the same direction more than once. This is because the ant's path is determined only by the status of its doors. This means that if it were to enter the same door under the same direction (up or down), the path has been repeated. In the case of the 1-door system, this is simple to show. Recall from the original definition that each door is preset to be open or closed with a probability of $\frac{1}{2}$. It is easy to see that if a door is closed in a 1-door system, the ant can no longer move past that level. Thus, if the ant is surrounded on top and bottom by closed doors at some point, than the path would have to be strictly within that region. If it were the case that all 1-door systems had a closed door at some point in both directions, it would conclude that the path would have to be periodic (a simple application of the pigeonhole principle would work here). Mathematically, this is easy to see as the probability of having only open doors in one direction is

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$$

Therefore, all 1-door systems can be generalized to have closed doors at some point in both directions, blocking off its path, indicating that it is periodic (see figure 5).

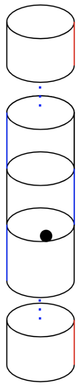

Figure 5. **General 1-Door System**

In the case of an n-door system, where $n > 1$, a floor of all closed doors isn't the only way for the ant switch directions. For example, in the system displayed on figure 4, the ant still returns back to its original path without all the doors on the floor being completely closed. However, it is true that a floor with all closed doors will switch the direction of the path, and so we can proceed with the similar logic as before. Similar to the 1-door system, all n-door systems must have closed boundaries. This can be seen since the probability of a given floor having all of its doors closed is $(\frac{1}{2})^n$, and so there is a probability less than 1 in which the floor is not all closed. This probability will be multiplied by itself infinitely many times, again leading towards a limit of 0. Thus, every path in a n-door system must be periodic.

4. Expected Number of Loops in a N-door System

We define a loop in an N-door system to be whenever the ant returns to its starting position projected down onto the floor. For example, in the 2-door system, with its projection diagram in Figure 3, the number of loops would be the number of times the ant reaches the position of index 0 before returning to its original position (by convention, we let the first door reached by the ant be index 0, increasing by 1 in the counterclockwise direction). Calculating the expected number of loops in a randomly generated n-door system is difficult, but can be done mathematically for a 1-door system. First, suppose that the 1-door system's periodic path consists of a path upwards of m levels and a path downwards n levels. The probability that the path upwards is of m levels is simply $(\frac{1}{2})^{m+1}$ (note that since the door on the mth level must be closed, a total of m+1 doors must be set). Because the path would consist of hemispheres of the circumference of each floor, going up and down the path would yield to reaching the index exactly m times (once every other floor going up and the other floors going down). This yields a total expected value of

$$\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n+1} = 1$$

Similarly, the total expected value of the path downwards is

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{2}\right)^{n+1} = 2$$

The only difference with this calculation is that it reaches one more loop in that we count when it returns to the original spot as the final loop, thus multiplying by $n+1$. Summing the two together yields the total expected number of loops in a 1-door system to be 3. However, finding this value for when $n > 1$ is much harder, as we can no longer characterize the path through an upward and downward sub-path since paths can switch directions multiple times before reaching periodicity. To find an accurate estimation, we make use of the system's similarities to a Lorentz Gas, creating an algorithm in which a computer code can run.

5. Lorentz Gas Bijection

As mentioned before, we define the index of a door to be the number assigned to its position on the projection onto the floor. We also define the level of the floor to be the number assigned to each floor, increasing by height starting from 0. Suppose we have the 2-door system shown in Figure 6 (note that the levels and indices are labeled). We can formulate a bijection of this system onto a matrix in which it has the levels by column and the door indices by row. The elements of the matrix are either 0, 1, 2, which 1 represents an open door, 2 a closed door, and 0 no door. The 0 is more so of a filler since not every position has a door each floor. We repeat the matrix of the system underneath one another to better visualize the path.

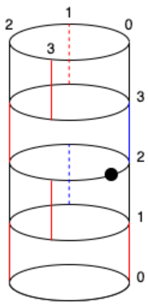

Figure 6. **2-Door System**

Index \ Level	0	1	2	3
0	2	0	1	0
1	0	1	0	2
2	2	0	2	0
3	0	2	0	2
	2	0	1	0
	0	1	0	2
	2	0	2	0
	0	2	0	2
	2	0	1	0
	0	1	0	2
	2	0	2	0
	0	2	0	2
	2	0	1	0

Figure 7. **Matrix Representation of Figure 6**

As shown in Figure 7, the path of the ant can be visualized by following the colored elements of the matrix. When it is on a blue (open) door, it continues the direction it was in, moving one down and over on the matrix. However, when it reaches a red (closed) door, it switches directions as it continues to move downwards. We end the path when it reaches the same element and direction as it started on.

We can also see its bijection to a Lorentz Gas (Figure 8). Taking only the closed doors and its position on the matrix, we can see that it represents the Lorentz Gas model with semi-randomly positioned, periodic scatterers. We say semi-randomly due to the fixed possible positions of the scatterers (doors) and periodic due to the configuration of scatterers repeating by row. This makes sense, as the path of the ant is uniquely determined by the doors (scatterers). The Lorentz Gas would have each door's position as an immovable molecule, with the ant (free molecule) colliding with the scatterers. The molecule is reflected specularly on the surface of the scatterers, but do so in a way that recreates the diagonal movement in our matrix.

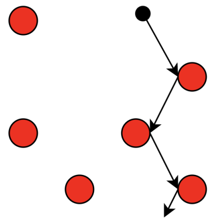

Figure 8. **Lorentz Gas Visualization of Figure 6**

6. The Algorithm

First, we would need to generate the system, which would be the equivalence of generating a matrix in which the doors in the left-most and right-most columns are closed. We then keep track of three variables, the floor, door, and direction. The direction variable will either be 1 or -1, with 1 being the initial southeast direction. To count the number of loops, all we would need to do is keep track of the number of times the path crosses over a door index of 0, which is done through a counting variable. The outputs of the algorithm, averaged over 1000 runs, produces the expected values indicated by table 1 (note our mathematically proven result for $n = 1$ is modeled correctly). An interesting observation would be that $E[n]$ does not seem to increase as rapidly as one would think. This can be explained due to the fact that while it does get increasingly harder to return to the original position due to the number of doors (probability of getting a floor with all closed doors is small), it also makes it harder to finish a loop.

Algorithm 1 Algorithm for expected number of loops in an N-Door System

```
Require: direction = 1, floor = START, door = 0, count = 0
  while not in original position and direction do
    if door is open then
      floor = floor + direction
      door = (door + 1) mod 2n
    else if door is closed then
      floor = floor - direction
      door = (door + 1) mod 2n
      direction = - direction
    end if
    if floor = 0 then
      count++
    end if
  end while
```

n	$E[n]$
1	2.957
2	4.559
3	6.062
4	7.208
5	8.525
6	9.895
7	10.852
8	12.2728
9	13.3431

Table 1. **Expected Number of Loops in an N-Door System**

7. Discussion

The N-Door System shares many of the properties with other random environments. The path, uniquely determined through the status of the doors, of any N-door System is proven to be periodic. It can be shown mathematically that the expected number of loops in a 1-Door System is 3, and through the system's similarities to a Lorentz Gas, an algorithm can be created and run to show that the expected number of loops in a N-Door System doesn't increase rapidly. A mathematical proof for an explicit formula has yet to be found, but we can expect by the results of the algorithm that the expected number of loops is dependent on the number of doors resembling a linear trend.

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