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Principal Topology on a Rees Matrix Semigroup using Green's Left Quasiorder

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Abstract: This paper introduces a principal topology on a Rees matrix semigroup using Green's left quasiorder. Since principal topologies are in one-one correspondance with quasiorder relations on a set, the relations are commonly used for constructing such topologies. The basis for the topology is the collection of minimal open neighbourhoods corresponding to each element in a given set. When semigroups are considered with Green's left quasiorder, minimal open neighbourhoods are the principal left ideals. Hence, the collection of principal left ideals will turn out to be a basis for the principal topology on a semigroup. As long as a Rees matrix semigroup is considered, it is observed that these ideals exhibit certain interesting properties. This paper analyses these ideals in the context of a Rees matrix semigroup. The properties thus observed actually determine the number of elements in the so formed principal topology. Further, the topology hence obtained is an example for a finite topology on an infinite set, provided the order of the Rees matrices is finite.

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1. Introduction

This paper is, in fact, a study about principal topology on a special class of semigroups, namely the Rees matrix semigroup, introduced by David Rees in 1940. Topology is actually a class of subsets of a set X satisfying certain axioms. An extension of this concept was made by Alexandroff in 1937, which contributed to the discovery of principal topology. It is a topology where in the arbitrary intersection of open sets is open. Quasiorder relation is the most common tool used for introducing principal topologies on a set.

The first section of this paper introduces some basic concepts of Rees matrix semigroups and principal topologies along with their basic properties and associated lemmas. After a detailed analysis of Rees matrix semigroup is done, the proceeding section discusses about the principal topology on the semigroup. As Alexandroff proved in [1], a quasiorder relation on a set corresponds to an equivalent principal topology. Suppose a set is given with a quasiorder relation, the minimal open neighbourhoods corresponding to each element in the set can easily be found out. These neighbourhoods are nothing but the basis elements of the principal topology. For semigroups with Green's left quasiorder, the principal left ideals are the corresponding minimal open neighbourhoods and the basis will then be the collection of these prinicipal left ideals generated by each and every element in the semigroup. Unlike to this general context, when a Rees matrix semigroup is considered in particular, it is observed that, the set of those principal left ideals generated by only a few elements in it, turns out to be the

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basis. These elements are significant as they determine the number of elements in the so formed topology. It is also noticed that these ideals have some significant properties. The properties are being explained through a collection of lemmas.
The first lemma provides the relation between the principal left ideals in two variations of Rees matrix semigroups. While the second lemma clearly specifies the principal left ideals generated by both the zero and nonzero elements in a Rees matrix semigroup. The third one compares the principal left ideals by two distinct nonzero elements. When a collection of subsets of a set with certain properties is given, the next lemma then provides a formula to calculate the number of elements in the topology generated by this collection of subsets. The following lemma, in fact, defines a basis for the principal topology on the two different variations of Rees matrix semigroups considered. Keeping in view of the properties thus discussed, finally the main result asserts that the principal topology on these variations of Rees matrix semigroups induced by Green's left quasiorder is, in fact, a finite topology. Also, it is found that the number of elements in the topology is determined by the order of the Rees matrices.

2. Preliminaries

As said above, this section explains the basic definitions and lemmas associated with the main topics of this paper. Most of the definitions and notations used in this sequel are from [2–4].
Generally, a semigroup is a non empty set equipped with an associative binary operation. Zero element of a semigroup $(S, \cdot)$ is an element z such that $zs = sz = z$, $\forall s \in S$. A subset I of S is a left ideal of S if $SI \subseteq I$, and a right ideal if $IS \subseteq I$. The principal left ideal on S generated by an element $x \in S$ is defined as $S^1x = Sx \cup \{x\}$ where $Sx = \{sx : s \in S\}$. A relation defined on a set X, that satisfies reflexivity and transitivity is known as a quasiorder relation on X. The symmetric, quasiorder relation is termed as an equivalence relation. A relation R on a semigroup S is called as a compatible relation on S if $s\ R\ t$ and $s^{'}\ R\ t^{'}$ implies $ss^{'}\ R\ tt^{'}$ for any $s, t, s^{'}, t^{'}$ in S^1. Also, congruence relation [4] on a semigroup S is an equivalence, compatible relation.

Definition 2.1. *In order to completely study the properties of the elements in a semigroup, James Alexander Green introduced a class of four equivalence relations, namely, Green's relations, in 1951. As a prelude to the relations he defined four preorders (quasiorders) on a semigroup. Let $(S, \cdot)$ be a semigroup and $a, b \in S$. Then, the Green's left quasiorder on the semigroup S is defined by $a \leq_{\mathscr{L}} b$ if and only if $a = ub$ for some $u \in S^1$, and the Green's right quasiorder is defined as $a \leq_{\mathscr{R}} b$ if and only if $a = bv$ for some $v \in S^1$. The third quasiorder defined on S is $\leq_{\mathscr{H}}$ quasiorder where $a \leq_{\mathscr{H}} b$ if and only if $a = ub = bv$ for some u, v in S^1, and the fourth one is $\leq_{\mathscr{J}}$ where $a \leq_{\mathscr{J}} b$, if and only if $a = ubv$ for some u, v in S^1. These four quasiorders are collectively known as Green's quasiorders (preorders) [3] on a semigroup.*
If each one of these quasiorders is symmetric, then the corresponding equivalence relations, denoted by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{J}$ are collectively known as Green's relations.

Definition 2.2. *Let I be an ideal in a semigroup S, and ρ be a congruence on S defined by $x\rho y$ if and only if either $x = y$ or both x and y are in I. Then, the congruence ρ is called Rees congruence and S/I is called the Rees factor semigroup, where I is the zero element of S/I. Therefore, by identifying the zero element I as $0^{'}$, S/I can also be written as $(S \setminus I) \cup \{0^{'}\}$.*

Definition 2.3. *Consider a group G with identity element e, and I, Λ as non-empty index sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix whose entries are from the zero group G^0 where $G^0 = G \cup \{0\}$, 0 is the zero element of the group G. Also assume P is regular. Then, P will have no rows or columns that consists entirely of zeros. Formally,*

$$(\forall i \in I)\ (\exists \lambda \in \Lambda)\ p_{\lambda i} \neq 0,$$

$$(\forall \lambda \in \Lambda)\ (\exists i \in I)\ p_{\lambda i} \neq 0$$

Let $Q = I \times G^0 \times \Lambda$. *A multiplication on* Q *is defined by* $(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$. *Under this multiplication,* Q *will be a semigroup. Further,* $T = I \times \{0\} \times \Lambda$ *is an ideal of* Q. *Let* $M(G, I, \Lambda, P) = I \times G \times \Lambda$ *and consider* $M^{0'}(G, I, \Lambda, P)$ *as the Rees factor semigroup* Q/T. *Then,* $M^{0'}(G, I, \Lambda, P)$ *can be considered as the set* $M(G, I, \Lambda, P) \cup \{0'\}$ *under the multiplication*

$$(i, a, \lambda) *' (j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu), & if\ p_{\lambda j} \neq 0 \\ 0', & if\ p_{\lambda j} = 0 \end{cases} \tag{1}$$

$$0'(i, a, \lambda) = (i, a, \lambda)0' = 0'0' = 0'$$

where $0'$ *is the zero element of* $M^{0'}(G, I, \Lambda, P)$, *that is also denoted by* $(i, 0, \lambda)$ *for any* i *in* I *and* λ *in* Λ.

The semigroup $M^{0'}(G, I, \Lambda, P)$, *as defined above is known as* $I \times \Lambda$ *Rees matrix semigroup with zero [2] over the group* G^0, *with regular sandwich matrix* P. *If the matrix* P *consists of all nonzero entries, then the set* $M(G, I, \Lambda, P)$ *with the operation* $(i, a, \lambda) * (j, b, \mu) = (i, ap_{\lambda j}b, \mu)$ *is the Rees matrix semigroup without zero.*

By Rees $I \times \Lambda$ *matrix over* G^0 *we mean, an* $I \times \Lambda$ *matrix over* G^0 *which has at most one nonzero entry. If* $a \in G, i \in I$ *and* $\lambda \in \Lambda$, *then* $(a)_{i\lambda}$ *will denote the Rees* $I \times \Lambda$ *matrix over* G^0, *having* a *in the* $i-$*th row and* $j-$*th column, the remaining entries being* 0.

Definition 2.4. *Let* X *be a set. A collection* τ *of subsets of* X *is said to be a topology [5] on* X *if* τ *contains the whole set* X *and the empty set* ϕ *and* τ *is closed under arbitrary union and finite intersection.*

Definition 2.5. *Let* X *be a set. A collection* $\mathbb{B}$ *of subsets of* X *is said to be a basis [6] for a topology* τ *on* X *if, corresponding to each element* x *in* X *it must be possible to find at least one set in* $\mathbb{B}$ *containing* x, *and if* x *is in the intersection of two sets in* $\mathbb{B}$ *then there must exists another set in* $\mathbb{B}$ *containing* x *and contained in the intersection. The elements in the basis are called basis elements for the topology.*

Definition 2.6. *A topology defined on a set* X *with the property that the arbitrary intersection of open sets is open, is considered as a principal topology on the set* X. *Finite topologies are the most familiar examples for such topologies.*

Definition 2.7. *If* X *is a set with a principal topology* $\mathscr{T}$, *then the minimal open neighbourhood of an element* x *in* X *is defined as the intersection of all open sets containing* x. *That is,* $M_x = \cap\{U : x \in U, U \text{ is open}\}$. *The collection of* M_x *for each* x *in* X *is a basis for the topology* $\mathscr{T}$.

Lemma 2.8 ([7])**.** *Let* X *be a nonempty set and let* $\{B_i : i \in I\}$ *be a collection of nonempty subsets of* X. *Then* $\mathbb{B}$ *is a basis consisting of all the minimal open neighbourhoods for some principal topology on* X *if and only if*

(1). $\cup\mathbb{B} = X$

(2). For any subcollection $\mathcal{C} \subseteq \mathbb{B}$, *and* $x \in \cap\, \mathcal{C}$, *there exists* $B \in \mathbb{B}$ *with* $x \in B \subseteq \cap\, \mathcal{C}$.

(3). For any $B \in \mathbb{B}$, $\mathbb{B}$ *and* $\mathbb{B} \setminus \{B\}$ *are not equivalent bases. That is,* $\mathbb{B} \setminus \{B\}$ *is either not a basis for any topology on* X *or does not generate the same topology as* $\mathbb{B}$.

Alexandroff [1] gave the connection between quasiorders on any set X and the principal topology that can be defined on the same set. If X is a set with a quasiorder relation $\leq$, then corresponding to each x in X, the minimal open neighbourhood, that is, the set M_x can be found out. The collection $\{M_x : x \in X\}$ will form a basis for a topology, namely, the specialization topology on X, which is actually the principal topology on X. Conversely, if X is given with a principal topology, corresponding quasiorder is defined by $y \leq x$ whenever y lies in every open set containing x, for arbitrary elements x, y in X.

Lemma 2.9. *Let $(S,\cdot)$ be a semigroup and $\leq_{\mathscr{L}}$ be the Green's left quasiorder on S. Then, the minimal open neighbourhood corresponding to $\leq_{\mathscr{L}}$ is the set S^1x, where S^1x is the principal left ideal generated by an element x in S.*

Proof. Given a quasiorder $\leq$ on a set X, the minimal open neighbourhood for an element x in X is the set M_x, where $M_x = \{y \in X : y \leq x\}$. For semigroups with Green's left quasiorder $\leq_{\mathscr{L}}$, M_x is the collection of the elements y in S such that $y \leq_{\mathscr{L}} x$. That is, $M_x = \{y \in S : y \leq_{\mathscr{L}} x\}$, which is same as the set $\{ux : u \in S^1\}$. This collection is actually the principal left ideal generated by the element x. Therefore, $M_x = S^1x$ for a semigroup with $\leq_{\mathscr{L}}$. □

3. Principal topology on a Rees matrix semigroup

This section begins with an evaluation on the properties of these minimal open neighbourhoods for a Rees matrix semigroups without zero and then the observed properties are generalised to the same semigroup containing the zero element. Throughout this section, the semigroups $M(G,I,\Lambda,P)$ and $M^{0'}(G,I,\Lambda,P)$ are respectively denoted by L and S. To begin with, the relation between the principal left ideals in the semigroups L and S is evaluated through the following lemma.

Lemma 3.1. *Consider the semigroups $(L,*)$ and $(S,*')$. Then, $*'|L \times L = *$ if the corresponding entry in P is nonzero. Also, for any nonzero element (i,a,λ) in S, $L^1(i,a,\lambda) \cup \{0'\} = S^1(i,a,\lambda)$ and $*'|L^1(i,a,\lambda) \times L^1(i,a,\lambda) = *$ whenever the corresponding entry in P is nonzero.*

Proof. To compare the semigroup operations on both L and S, consider (i,a,λ) and (j,b,μ) be two nonzero elements in S. Then, $(i,a,\lambda) *' (j,b,\mu) = (i, ap_{\lambda j}b, \mu)$. Assume $p_{\lambda j} \neq 0$, then this product will be in L itself and $(i,a,\lambda) *' (j,b,\mu) = (i, ap_{\lambda j}b, \mu) = (i,a,\lambda) * (j,b,\mu)$ since a, b, $p_{\lambda j}$ are elements in the group G. If $p_{\lambda j} = 0$, then $(i, ap_{\lambda j}b, \mu) = (i,0,\mu)$. Since L doesn't contain zero elements, clearly this element lies in S itself. Hence, $*'|L = *$ only if $p_{\lambda j} \neq 0$. Now, consider a nonzero element (i,a,λ) in S. By the definition of the principal left ideal, $L^1(i,a,\lambda)$ consists of all the left multiples of (i,a,λ). Since L doesn't contain the zero element, $L^1(i,a,\lambda)$ consists of all the nonzero left multiples of (i,a,λ). On the other hand, $S^1(i,a,\lambda)$ is the collection of all the left multiples of (i,a,λ) including the zero element. Therefore, $L^1(i,a,\lambda) \subset S^1(i,a,\lambda)$ and $L^1(i,a,\lambda) \cup \{0'\} = S^1(i,a,\lambda)$. For the semigroup operations, let x and y be two nonzero elements in $S^1(i,a,\lambda)$. Then, $x = (j,b,\mu)(i,a,\lambda) = (j, bp_{\mu i}a, \lambda)$ for some $b, p_{\mu i}, a$ in G and $y = (k,c,\nu)(i,a,\lambda) = (k, cp_{\nu i}a, \lambda)$ for some $c, p_{\nu i}, a$ in G. Since x and y are nonzero elements, $bp_{\mu i}a \neq 0$ and $cp_{\nu i}a \neq 0$, which implies that both $p_{\mu i}$ and $p_{\nu i}$ are nonzero elements in G. Now, $xy = (j, bp_{\mu i}a, \lambda)(k, cp_{\nu i}a, \lambda) = (j, (bp_{\mu i}a)p_{\lambda k}(cp_{\nu i}a), \lambda)$. Clearly, xy will be zero only if $p_{\lambda k} = 0$. Therefore, if $p_{\lambda k} \neq 0$, then $xy \neq 0$ and it lies in $L^1(i,a,\lambda)$. Hence $x * y = x *' y$ as long as $p_{\lambda k} \neq 0$. □

Lemma 3.2. *Consider the semigroups L and S. Then the principal left ideal $L^1(i,a,\lambda) = I \times G \times \{\lambda\}$ for any $(i,a,\lambda) \in L$ and the principal left ideal for an element $(i,a,\lambda) \in S$ is given by,*

$$S^1(i,a,\lambda) = \begin{cases} I \times G^0 \times \{\lambda\}, & if\ a \neq 0 \\ \{0'\}, & if\ a = 0 \end{cases} \tag{2}$$

Proof. Let $(i,a,\lambda) \in L$, then the principal left ideal generated by (i,a,λ) consists of all the left multiples of (i,a,λ). Consider the principal left ideal $L^1(i,a,\lambda)$. For any $x \in L^1(i,a,\lambda)$, $x = (j,b,\mu)(i,a,\lambda) = (j, bp_{\mu i}a, \lambda)$ for some j in I, b in G and $p_{\mu i}$ in G. Then, clearly, x lies in $I \times G \times \{\lambda\}$. Therefore, $L^1(i,a,\lambda) \subseteq I \times G \times \{\lambda\}$. For the other part, consider $y = (k,g,\lambda)$ as an arbitrary element in $I \times G \times \{\lambda\}$. In particular, if $g = e$, the identity element of the group G, take $p_{\mu i} = e$. Since a is a nonzero element in G^0, a^{-1} exists in G. Now, consider the element $z = (k, a^{-1}, \mu)$. Then, $y = (k,g,\lambda) = (k,e,\lambda) = (k,a^{-1},\mu)(i,a,\lambda)$, which implies y is a left multiple of (i,a,λ). Hence, $y \in L^1(i,a,\lambda)$. If g is a non identity element in G, then from the group table of G, there exists an element b in G such that $g = b(p_{\mu i}a)$, for some fixed elements $p_{\mu i}, a$ in G. Then, $y = (k, b, \mu)(i, a, \lambda)$ for some k in I and μ in Λ. Therefore, $y \in L^1(i, a, \lambda)$. Thus, $I \times G \times \{\lambda\} = L^1(i, a, \lambda)$.

Now, consider semigroup S. Clearly, the principal left ideal generated by the zero element is the set containing the zero matrix only and hence the result holds trivially. Consider the principal left ideal generated by a nonzero element (i, a, λ) in S, that is, $S^1(i, a, \lambda)$. But, from lemma 3.1, $S^1(i, a, \lambda) = L^1(i, a, \lambda) \cup \{0^{'}\}$. Also, using the above part, $L^1(i, a, \lambda) = I \times G \times \{\lambda\}$, $S^1(i, a, \lambda) = I \times G \times \{\lambda\} \cup \{0^{'}\}$, where $0^{'}$ is the zero element of the Rees matrix semigroup. Therefore, $S^1(i, a, \lambda) = I \times G^0 \times \{\lambda\}$. □

Lemma 3.3. *Consider the semigroups L and S. Then, for any two distinct nonzero elements (i, a, λ), (j, b, μ) in S, the principal left ideals satisfies*

- $L^1(i, a, \lambda) = L^1(j, b, \mu)$ *if* $\lambda = \mu$
- $L^1(i, a, \lambda) \cap L^1(j, b, \mu) = \phi$ *if* $\lambda \neq \mu$. *and*
- $S^1(i, a, \lambda) = S^1(j, b, \mu)$ *if* $\lambda = \mu$.
- $S^1(i, a, \lambda) \cap S^1(j, b, \mu) = \{0^{'}\}$ *if* $\lambda \neq \mu$.

Proof. Let (i, a, λ) and (j, b, μ) be two distinct elements in L such that $\lambda = \mu$. Then by lemma 3.2, the principal left ideal generated by them coincides and $L^1(i, a, \lambda) = I \times G \times \{\lambda\} = L^1(j, b, \mu)$. To prove that their principal ideals will have empty intersection, assume, on the contrary, an element, say p lies in their intersection. Then $p \in L^1(i, a, \lambda)$ and $p \in L^1(j, b, \mu)$. Therefore, by lemma 3.2, p is of the form $p = (k, c, \lambda)$ for some k in I and c in G. Also, p has the form $p = (k^{'}, c^{'}, \mu)$ for some $k^{'}$ in I and $c^{'}$ in G. This happens simultaneously only if $k = k^{'}$, $c = c^{'}$ and $\lambda = \mu$. But, we have assumed $\lambda \neq \mu$. Hence, a contradiction arises. Therefore, the principal left ideals $L^1(i, a, \lambda) \cap L^1(i, a, \mu) = \phi$ for any two distinct elements λ and μ in Λ.

For the semigroup S, let (i, a, λ) and (j, b, μ) be two distinct nonzero elements in S such that $\lambda = \mu$. Since $L^1(i, a, \lambda) \cup \{0^{'}\} = S^1(i, a, \lambda)$ and using $L^1(i, a, \lambda) = L^1(j, b, \mu)$ just proved above, $S^1(i, a, \lambda) = S^1(i, a, \mu)$. (This can also be achieved from lemma 3.2, since $S^1(i, a, \lambda) = I \times G^0 \times \{\lambda\} = S^1(j, b, \mu)$.) Now, Consider two distinct nonzero elements (i, a, λ) and (j, b, μ) in S with $\lambda \neq \mu$, we have to prove that their principal left ideals intersects at the zero element. Obviously, $\{0^{'}\} \subseteq S^1(i, a, \lambda) \cap S^1(j, b, \mu)$, since both left ideals $S^1(i, a, \lambda)$ and $S^1(j, b, \mu)$ contains the zero element. Now, it remains to show that their intersection does not contain any nonzero elements. To prove this, it has already established that $L^1(i, a, \lambda) \cap L^1(j, b, \mu) = \phi$ for $\lambda \neq \mu$. Thus, $S^1(i, a, \lambda) \cap S^1(j, b, \mu) = \{0^{'}\}$ whenever $\lambda \neq \mu$. □

Lemma 3.4. *Let X be a set and choose $\mathbb{B} = \{B_1, B_2, ..., B_n\} \cup \{C\}$ as a collection of non-empty proper subsets of the set X such that $C \subset B_i$ for every i where $i = 1, 2, .., n$ and $\cup_{i=1}^{n} B_i = X$. Also assume $\mathbb{B}$ is a basis for a particular topology τ on X. Then, τ is a finite topology with $2 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ elements.*

Proof. Let X be a set and $\mathbb{B} = \{B_1, B_2, ..., B_n\} \cup \{C\}$ and let $\mathbb{B}$ be a basis for a topology τ on X. That is, $\mathbb{B}$ generates τ. An evaluation of the set τ is done for calculating the number of elements in τ. Then C, as an element in $\mathbb{B}$ and $\mathbb{B} \subseteq \tau$ by definition, definitely belongs to τ. Trivially, $\phi \in \tau$. Since the union of the elements in a basis is a topology, the topology τ consists of the union of two element sets, that is the sets of the form $B_1 \cup B_2, B_2 \cup B_3 \ldots$ and there are $\binom{n}{2}$ such unions. Similarly, τ contains the union of three element sets, that is the sets of the form $B_1 \cup B_2 \cup B_3, B_2 \cup B_3 \cup B_4 \ldots$, there are $\binom{n}{3}$ such unions. Further, continuing the process until union of these B_i' s equals X. Then, τ has $2 + \binom{n}{1} + \binom{n}{2} + \cdots +$

$\binom{n}{n}$ elements. Also, by the construction of τ, τ is a topology and since the number of elements in τ depends only on the number of the sets B_i' s and the collection $\{B_i\}$ is finite, τ is a finite topology. ☐

Lemma 3.5. *Consider the semigroups L and S with $\leq_{\mathscr{L}}$ as the Green's left quasiorder on them. For a fixed i in I and a in G, consider the set $X = \{i\} \times \{a\} \times \Lambda$, then $\mathbb{B} = \left\{L^1x : \ x \in \ X\right\}$ is a basis for the principal topology $\mathscr{T}$ on L induced by $\leq_{\mathscr{L}}$ and $\mathbb{D} = \left\{S^1x : \ x \in \ X\right\} \cup \{0^{'}\}$ is a basis for the principal topology $\mathscr{U}$ on S induced by $\leq_{\mathscr{L}}$.*

Proof. In order to show that the sets $\mathbb{B}$ and $\mathbb{D}$ are the basis for the principal topology induced by $\leq_{\mathscr{L}}$ on L and S respectively, a verification of lemma 2.8 is required. The verification is done simultaneously for both L and S.

For proving the union of elements in $\mathbb{B}$ is L, let (j, b, μ) be an arbitrary element in L, then definitely there exists an element x in X, that is, $x = (i, a, \mu)$ in X such that L^1x contains (j, b, μ), for $L^1x = I \times G \times \{\mu\}$ by lemma 3.2. That is, $(j, b, \mu) \in L^1x \subseteq \cup_{x\in X} L^1x \subseteq \cup\mathbb{B}$. On the other hand, since each principal left ideal L^1x for $x \in X$ are proper subsets of S, $\cup\mathbb{B} \subseteq S$. Therefore, $L = \cup\mathbb{B}$. Similarly, to prove $\cup\mathbb{D} = S$, take an arbitrary element $y = (i, a, \lambda)$ in S. If $y = 0$, then, obviously $y \in \{0^{'}\}$ which is contained in $\cup_{x\in X} S^1x \cup \{0^{'}\}$. If y is nonzero, then similar to the above case, y lies in S^1x for some $x \in X$. Hence, $y \in \cup_{x\in X} S^1x \cup \{0^{'}\}$. On the other hand, since each principal left ideal S^1x for $x \in X$ and the set containing the zero element are proper subsets of S, $\cup\mathbb{D} \subseteq S$. Therefore, $S = \cup\mathbb{D}$. Now, for the semigroup L, the second condition of lemma 2.8 is trivial because of lemma 3.3. For the semigroup S, consider $\mathcal{C}$ as a subcollection of $\mathbb{D}$. Then, by lemma 3.3, $\cap\, \mathcal{C}$ contains the zero element, $0^{'}$. But, since $0^{'}$ is itself an element in $\mathbb{D}$, and $0^{'} \in \{0^{'}\} \subseteq \cap\, \mathcal{C}$, $\mathbb{D}$ satisfies the second condition of lemma 2.8.

In order to prove the last condition, if possible assume the collection $\mathbb{B} \setminus B$ is a basis generating the topology $\mathscr{T}$ on L, for some nonzero element B in $\mathbb{B}$. Then, $B = L^1x$ for some $x \in X$. Also, $B \subset L$. But, by lemma 3.2, $\cup(\mathbb{B} \setminus B)$ is only a proper subset of L, which implies $\cup(\mathbb{B} \setminus B) \neq L$. Thus, $\mathbb{B} \setminus B$ is not a basis for $\mathscr{T}$, and hence for any topology defined on L. Similarly, for S, assume, if possible, $\mathbb{D} \setminus \{D\}$ is a basis generating the topology $\mathscr{T}$ on S. If D is a nonzero basis element, then following the similar arguments, it will arrive at $\cup\mathbb{D} \setminus \{D\} \neq S$. If D is the zero element in the basis, then, $D = \{0^{'}\}$. But, then, clearly the element $0^{'} \in \cap\, \mathcal{C}$ for any subcollection $\mathcal{C}$ of $\mathbb{D}$. But, there is no element in $\mathbb{D} \setminus \{D\}$ containing $0^{'}$ and contained in $\cap\, \mathcal{C}$. This implies that $\mathbb{D} \setminus \{0^{'}\}$ is not a basis for $\mathscr{U}$. Hence, for any D in $\mathbb{D}$, $\mathbb{D} \setminus D$ will not be a basis for any topology on S. Therefore, $\mathbb{D}$ is a basis for the topology, more precisely the principal topology $\mathscr{U}$ on S. ☐

Theorem 3.6. *Consider the semigroups L and S with $|\Lambda| = m$, that is, m be the number of columns of the Rees matrices. Let $\mathscr{T}$ be the principal topology induced by $\leq_{\mathscr{L}}$ on L and $\mathscr{U}$ be the principal topology induced by $\leq_{\mathscr{L}}$ on S Then, $\mathscr{T}$ and $\mathscr{U}$ are finite topologies with $1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}$ elements and $2 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}$ elements respectively.*

Proof. Assume $|\Lambda| = m$. That is $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$. Then, by lemma 3.5, the basis for $\mathscr{T}$ on L induced by $\leq_{\mathscr{L}}$ is the set $\{L^1(i, a, \lambda_1), L^1(i, a, \lambda_2), ..., L^1(i, a, \lambda_m)\}$. By lemma 3.2, each of these basis elements are proper subsets of L. Thus, generating the topology $\mathscr{T}$ from this basis by taking all possible unions and finite intersections, the collection of $\mathscr{T}$ will be a collection of subsets of L, satisfying the axioms of the topology. Also since $\phi \in \mathscr{T}$ trivially, the number of elements in $\mathscr{T}$ is $1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}$. Similarly, $\{S^1(i, a, \lambda_1), S^1(i, a, \lambda_2), ..., S^1(i, a, \lambda_m)\} \cup \{0^{'}\}$ is the basis for the topology $\mathscr{U}$ on S by lemma 3.5. Using lemma 3.2, it is clear that each of these basis elements are proper subsets of S and $\{0^{'}\} \subset S^1(i, a, \lambda_i)$ for every $i = 1, 2, \ldots m$. Then by constructing $\mathscr{T}$ from the basis as in the proof of lemma 3.4, the set $\mathscr{U}$ will be a collection of subsets of S, satisfying the axioms of the topology. Again by lemma 3.4, the number of elements in $\mathscr{U}$ is $2 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}$. Then, it is immediate that the number of elements in $\mathscr{T}$ and $\mathscr{U}$ depends only on the cardinality of the index set Λ, that is, the number of columns of the Rees matrices. Further, only the finite ordered Rees matrices are considered, $|\Lambda|$ is finite always which in turn makes the sets $\mathscr{T}$ as well as $\mathscr{U}$ finite. ☐

4. Conclusion

It is quite clear from the work of P. Alexandroff that given a set with quasiorder relation, the relation immediately creates a principal topology on the set. For a semigroup with Green's left quasiorder, the basis elements will be the principal left ideals. This paper focuses on the study of these principal left ideals and it is seen that they have certain interesting properties that can be used in calculating the number of elements in the principal topology so formed. Further, the same topology is an example for a finite topology on an infinite set, provided the order of the Rees matrices is finite.

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