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Laplacian Minimum Hub Energy of a Graph

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Abstract: In this paper, we introduce Laplacian minimum hub energy $LE_H(G)$ of a graph G, and compute Laplacian minimum hub energies of some standard graphs, also for a number of well-known families of graphs. Upper and lower bounds for $LE_H(G)$ are established.

MSC: 05C50, 05C99.

Keywords: Minimum hub set, Laplacian minimum hub matrix, Laplacian minimum hub eigenvalue, Laplacian minimum hub energy of a graph.

1. Introduction

Throughout this paper, all graphs we considered are simple and connected. Let $G = (V(G), E(G))$ be a simple connected graph with p vertices and q edges. For a vertex $v \in V(G)$, $d_G(v)$ denotes the degree of v. For graph theoretic terminology, we refer to [9].

M. Walsh [20] introduced the theory of hub in the year 2006. Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by $h(G)$ [20]. For more details on the hub number and hub parameters see [12, 15–18].

Eigenvalues and Eigenvectors provide insight into the geometry associated with the linear transformation. In 1978 Gutman [5], defined the energy of a graph G as the sum of absolute values of the eigenvalues of the adjacency matrix of graph G and denoted it by $E(G)$. Then, $E(G) = \sum_{i=1}^{p} |\lambda_i|$.

Theories on the mathematical concepts of graph energy can be seen in the articles [1, 2, 7] and the references cited there in. For various upper and lower bounds for energy of a graph can be found in [10, 14] and it was observed that graph energy has chemical applications in the molecular orbital theory of conjugated molecules [3, 6].

Let G be a graph with p vertices and q edges and let $A = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$

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of A, assumed in nonincreasing order, are the eigenvalues of the graph G. As A is real symmetric, let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_s$, respectively. The multiset

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}$$

of eigenvalues of $A(G)$ is called the adjacency spectrum of G, the eigenvalues of G are real with sum equal to zero. Gutman and Zhou [8] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with p vertices and q edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order p whose elements are defined as

$$L_{ij} = \begin{cases} -1, & \text{if } v_iv_j \in E; \\ 0, & \text{if } v_iv_j \notin E; \\ d_i, & \text{if } i = j. \end{cases}$$

Where d_i is the degree of the vertex v_i. Let $\vartheta_1, \vartheta_2, ..., \vartheta_p$ be the Laplacian eigenvalues of G. Laplacian energy $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^{p} |\vartheta_i - \frac{2q}{p}|$.

1.1. The minimum hub energy

Mathad and Mahde in [11] introduced the minimum hub distance energy of a graph G, and they in article [12] introduced the minimum hub energy of G as follows. Let G be a graph of order p with vertex set $V = \{v_1, v_2, ..., v_p\}$ and edge set E. Any hub set H of a graph G with minimum cardinality is called a minimum hub set. Let H be a minimum hub set of G. The minimum hub matrix of G is the $p \times p$ matrix $A_H(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in H; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_H(G)$ denoted by $f_p(G, \lambda)$ is defined as

$$f_p(G, \lambda) := det(\lambda I - A_H(G)).$$

The minimum hub eigenvalues of the graph G are the eigenvalues of $A_H(G)$. Since $A_H(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$. The minimum hub energy of G is defined as:

$$E_H(G) = \sum_{i=1}^{p} |\lambda_i|.$$

In this article, we introduce the concept of laplacian minimum hub matrix L_H.

2. The Laplacian Minimum Hub Energy of a Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then $L_H(G) = D(G) - A_H(G)$ is called the Laplacian minimum hub matrix of G. Consider $\vartheta_1, \vartheta_2, ..., \vartheta_p$ be the eigenvalues of $L_H(G)$, arranged in non-increasing order. These eigenvalues are called Laplacian minimum hub eigenvalues of G. The Laplacian minimum hub energy of the graph G is defined as

$$LE_H(G) = \sum_{i=1}^{p} |\vartheta_i - \frac{2q}{p}|.$$

Example 2.1. *The minimum hub sets for the following graph G shown in Figure* 1, *are* $\{v_1, v_2\}$ *and* $\{v_2, v_3\}$.

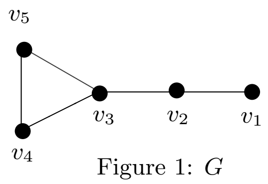

Figure 1: G

i). Let $H_1 = \{v_1, v_2\}$ is the minimum hub set of G. Then

$$A_{H_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, D_{H_1}(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L_{H_1}(G) = D_{H_1}(G) - A_{H_1}(G) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial of $L_{H_1}(G)$ is $\vartheta^5 - 8\vartheta^4 + 18\vartheta^3 - 5\vartheta^2 - 13\vartheta + 3$, and the Laplacian minimum hub eigenvalues are $\vartheta_1 = 4.0223, \vartheta_2 = -0.73914, \vartheta_3 = 1.4913, \vartheta_4 = 0.22554, \vartheta_5 = 3$, so

$$LE_{H_1}(G) = 9.478.$$

ii). If $H_2 = \{v_2, v_3\}$ is the minimum hub set of G, then

$$A_{H_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$L_{H_2}(G) = D_{H_2}(G) - A_{H_2}(G) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial of $L_{H_2}(G)$ is $\vartheta^5 - 8\vartheta^4 + 20\vartheta^3 - 13\vartheta^2 - 7\vartheta + 3$, and the Laplacian minimum hub eigenvalues are $\vartheta_1 = 3, \vartheta_2 = -0.50848, \vartheta_3 = 0.32036, \vartheta_4 = 3.3623, \vartheta_5 = 1.8228$, hence

$$LE_{H_2}(G) = 9.0169.$$

Then Laplacian minimum hub energy depends on the minimum hub set of the graph.

3. Laplacian Minimum Hub Energy of Some Standard Graphs

Theorem 3.1. *For the complete graph K_p, $p \geq 2$,*

$$LE_H(K_p) = p.$$

Proof. Let K_p be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_p\}$. Then the minimum hub number is $h(K_p) = 0$. Then

$$A_H(K_p) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{p\times p}$$

and

$$D(K_p) = \begin{pmatrix} p-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & p-1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & p-1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & p-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & p-1 \end{pmatrix}_{p\times p}$$

$$L_H(K_p) = D(K_p) - A_H(K_p)$$

$$= \begin{pmatrix} p-1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & p-1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & p-1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & p-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & p-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & p-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & p-1 \end{pmatrix}_{p\times p}.$$

The Characteristic equation is

$$\vartheta(\vartheta - p) = 0.$$

Laplacian minimum hub eigenvalues are $\vartheta = 0(one\ time)$ and $\vartheta = p(one\ time)$. Since $\frac{2q}{p} = p-1$, then Laplacian minimum hub energy,

$$\begin{aligned} LE_H(K_p) &= |0-(p-1)| + |p-(p-1)| \\ &= p-1+1 = p. \end{aligned}$$

□

Theorem 3.2. *For the complete bipartite graph $K_{n,n}$, $n \geq 3$, the Laplacian minimum hub energy is*

$$\sqrt{n^2-2n+5} + \sqrt{n^2+2n-3}.$$

Proof. Let $K_{n,n}$, $n \geq 3$ be the complete bipartite graph with vertex set $V = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The minimum hub set is $H = \{u_1, v_1\}$. Then

$$A_H(K_{n,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(2n)\times(2n)}$$

and

$$D(K_{n,n}) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}_{(2n)\times(2n)}$$

$$\begin{aligned} L_H(K_{n,n}) &= D(K_{n,n}) - A_H(K_{n,n}) \\ &= \begin{pmatrix} n-1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & n & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & n-1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & n \end{pmatrix}_{(2n)\times(2n)}. \end{aligned}$$

The Characteristic equation is

$$(\vartheta - n)(\vartheta^2 - (n-1)\vartheta - 1)(\vartheta^2 - (2n+n-1)\vartheta + n^2 + (n-1)^2 = 0.$$

Laplacian minimum hub eigenvalues are $\vartheta = n(one\ time)$, $\vartheta = \frac{n-1\pm\sqrt{n^2-2n+5}}{2}(one\ time\ each)$ and $\vartheta = \frac{3n+1\pm\sqrt{n^2+2n-3}}{2}(one\ time\ each)$. Since $\frac{2q}{p} = \frac{2n^2}{2n} = n$, then Laplacian minimum hub energy,

$$\begin{aligned} LE_H(K_p) &= |n-n| + |\frac{n-1+\sqrt{n^2-2n+5}}{2} - n| + |\frac{n-1-\sqrt{n^2-2n+5}}{2} - n| \\ &+ |\frac{3n+1+\sqrt{n^2+2n-3}}{2} - n| + |\frac{3n+1-\sqrt{n^2+2n-3}}{2} - n| \\ &= |0| + |\frac{-n-1+\sqrt{n^2-2n+5}}{2}| + |\frac{-n-1-\sqrt{n^2-2n+5}}{2}| \\ &+ |\frac{n+1+\sqrt{n^2+2n-3}}{2}| + |\frac{n+1-\sqrt{n^2+2n-3}}{2}| \\ &= \sqrt{n^2-2n+5} + \sqrt{n^2+2n-3}. \end{aligned}$$

□

Theorem 3.3. *For $p \geq 2$, the Laplacian minimum hub energy of a star graph $K_{1,p-1}$ is equal to* $\frac{p-2}{p} + \sqrt{p^2-2p+5}$.

Proof. Let $K_{1,p-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{p-1}\}$, such that v_0 is the center, and the minimum hub set is $H = \{v_0\}$. Then

$$A_H(K_{1,p-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{p\times p}$$

and

$$D(K_{1,p-1}) = \begin{pmatrix} p-1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{p\times p}$$

$$\begin{aligned} L_H(K_{1,p-1}) &= D(K_{1,p-1}) - A_H(K_{1,p-1}) \\ &= \begin{pmatrix} p-2 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{p\times p} \end{aligned}$$

The Characteristic equation is

$$(\vartheta - 1)(\vartheta^2 - (p-1)\vartheta - 1) = 0.$$

Laplacian minimum hub eigenvalues are $\vartheta = 1$(*one time*), and $\vartheta = \frac{p-1\pm\sqrt{p^2-2p+5}}{2}$ (one time each). Since $\frac{2q}{p} = \frac{2(p-1)}{p}$, then Laplacian minimum hub energy,

$$\begin{aligned} LE_H(K_{1,p-1}) &= |1 - \frac{2(p-1)}{p}| + |\frac{p-1+\sqrt{p^2-2p+5}}{2} - \frac{2(p-1)}{p}| + |\frac{p-1-\sqrt{p^2-2p+5}}{2} - \frac{2(p-1)}{p}| \\ &= |\frac{-p+2}{p}| + |\frac{p^2-p+p\sqrt{p^2-2p+5}}{2p}| + |\frac{p^2-p-p\sqrt{p^2-2p+5}}{2p}| \\ &= \frac{p-2}{p} + \sqrt{p^2-2p+5}. \end{aligned}$$

□

Definition 3.4 ([4]). *The double star graph $S_{n,m}$ (see Figure 2) is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. A vertex set $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{m-1}\}$ and edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j | 1 \le i \le n-1, 1 \le j \le m-1\}$.*

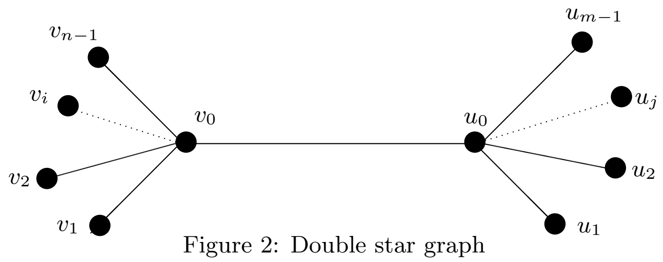

Figure 2: Double star graph

Theorem 3.5. *For $n \ge 3$, the minimum hub energy of the double star $S_{n,n}$ is equal to*

$$\frac{n}{n+1} + \frac{n\sqrt{n^2+4} + \sqrt{n^2+4}}{n+1} + \frac{n\sqrt{n^2+4n} + \sqrt{n^2+4n}}{n+1}.$$

Proof. For the double star graph $S_{n,n}$ with vertex set $V = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{n-1}\}$ the minimum hub set is $H = \{v_0, u_0\}$. Then

$$A_H(S_{n,n}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

and

$$D_H(S_{n,n}) = \begin{pmatrix} n+1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & n+1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{2n\times 2n}$$

$$L_H(S_{n,n}) = D(S_{n,n}) - A_H(S_{n,n})$$

$$= \begin{pmatrix} n & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & n & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 1 \end{pmatrix}_{2n\times 2n}.$$

The Characteristic equation is

$$(\vartheta - 1)(\vartheta^2 - n\vartheta - 1)(\vartheta^2 - (n+2)\vartheta + 1) = 0.$$

Laplacian minimum hub eigenvalues are $\vartheta = 1$ (one time), and $\vartheta = \frac{n\pm\sqrt{n^2+4}}{2}$ (one time each) and $\vartheta = \frac{n+2\pm\sqrt{n^2+4n}}{2}$ (one time each). Since $\frac{2q}{p} = \frac{2n+1}{n+1}$, then Laplacian minimum hub energy,

$$\begin{aligned} LE_H(S_{n,n}) &= |1 - \frac{2n+1}{n+1}| + |\frac{n+\sqrt{n^2+4}}{2} - \frac{2n+1}{n+1}| + |\frac{n-\sqrt{n^2+4}}{2} - \frac{2n+1}{n+1}| \\ &+ |\frac{n+2+\sqrt{n^2+4n}}{2} - \frac{2n+1}{n+1}| + |\frac{n+2-\sqrt{n^2+4n}}{2} - \frac{2n+1}{n+1}| \\ &= |\frac{n}{n+1}| + |\frac{n^2-3n-2+n\sqrt{n^2+4}+\sqrt{n^2+4}}{2(n+1)}| + |\frac{n^2-3n-2-n\sqrt{n^2+4}-\sqrt{n^2+4}}{2(n+1)}| \\ &+ |\frac{n^2-n+n\sqrt{n^2+4n}+\sqrt{n^2+4n}}{2(n+1)}| + |\frac{n^2-n-n\sqrt{n^2+4n}-\sqrt{n^2+4n}}{2(n+1)}| \\ &= \frac{n}{n+1} + \frac{n\sqrt{n^2+4}+\sqrt{n^2+4}}{n+1} + \frac{n\sqrt{n^2+4n}+\sqrt{n^2+4n}}{n+1}. \end{aligned}$$

□

4. Some Properties of Laplacian Minimum Hub Energy of Graphs

Theorem 4.1. *If H is a minimum hub set of a graph G and $\vartheta_1, \vartheta_2, ..., \vartheta_p$ be the eigenvalues of $L_H(G)$. Then*

(1). $\sum_{i=1}^{p} \vartheta_i = 2q - |H|$.

(2). $\sum_{i=1}^{p} \vartheta_i^2 = 2q + \sum_{i=1}^{p}(deg(v_i) - b_i)^2$, *where* $b_i = \begin{cases} 1, & \text{if } v_i \in H; \\ 0, & \text{if } v_i \notin H. \end{cases}$

Proof.

(1). The sum of the principal diagonal elements of $L_H(G)$ is equal to $\sum_{i=1}^{p} deg(v_i) - |H| = 2|E| - H$, and the sum of eigenvalues of $L_H(G)$ is trace of $L_H(G)$. Then

$$\sum_{i=1}^{p} \vartheta_i = 2|E| - H.$$

(2). The sum of squares of the eigenvalues of $L_H(G)$ is the trace of $(L_H(G))^2$. Then

$$\sum_{i=1}^{p} \vartheta_i^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} L_{ij}L_{ji}$$

$$= \sum_{i=1}^{p} L_{ii}^2 + \sum_{i\neq j}^{p} L_{ij}L_{ji}$$
$$= \sum_{i=1}^{p} L_{ii}^2 + 2\sum_{i<j}^{p} L_{ij}^2$$
$$= 2q + \sum_{i=1}^{p}(d_i - b_i)^2,$$

where

$$b_i = \begin{cases} 1, & \text{if } v_i \in H; \\ 0, & \text{if } v_i \notin H. \end{cases}$$

□

5. Bounds on Laplacian Minimum Hub Energy of Graphs

In this section, we shall investigate some bounds for Laplacian minimum hub energy of graphs.

Theorem 5.1. *Let G be a graph with p vertices, q edges and H be a minimum hub set of G. Then*

$$LE_H(G) \leq \sqrt{2Zp} + 2q.$$

Where, $Z = |E| + \frac{1}{2}\sum_{i=1}^{p}(deg(v_i) - H)^2$.

Proof. By using the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right).$$

Let us take $a_i = 1$ and $b_i = |\vartheta_i|$, then

$$\left(\sum_{i=1}^{p} |\vartheta_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} \vartheta_i^2\right) = 2pZ.$$

So, $\left(\sum_{i=1}^{p} |\vartheta_i|\right)^2 \leq 2pZ$, implies that $\sum_{i=1}^{p} |\vartheta_i| \leq \sqrt{2pZ}$. By triangle inequality, $|\vartheta_i - \frac{2q}{p}| \leq |\vartheta_i| + |\frac{2q}{p}|$, $\forall i = 1, 2, ..., p$. This means that $|\vartheta_i - \frac{2q}{p}| \leq |\vartheta_i| + \frac{2q}{p}$, $\forall i = 1, 2, ..., p$. Therefore,

$$\sum_{i=1}^{p} |\vartheta_i - \frac{2q}{p}| \leq \sum_{i=1}^{p} |\vartheta_i| + \sum_{i=1}^{p} \frac{2q}{p}$$
$$\leq \sqrt{2pZ} + 2q.$$

Then, the bound holds. □

Theorem 5.2. *Let G be a graph of order and size p and q, respectively. If $H = |det(A_H(G))|$, then*

$$LE_H(G) \geq \sqrt{2M + p(p-1)\left[\prod_{i=1} \vartheta_i\right]^{2/p}}.$$

Proof. Suppose that
$$\left(\sum_{i=1}^{p}|\vartheta_i|\right)^2=\left(\sum_{i=1}^{p}|\vartheta_i|\right)\left(\sum_{i=1}^{p}|\vartheta_i|\right)=\sum_{i=1}^{p}|\vartheta_i|^2+\sum_{i\neq j}|\vartheta_i||\vartheta_j|.$$
So, $\sum\limits_{i\neq j}|\vartheta_i||\vartheta_j|=\left(\sum\limits_{i=1}^{p}|\vartheta_i|\right)^2-\sum\limits_{i=1}^{p}|\vartheta_i|^2$. Applying inequality between the arithmetic and geometric means, we have

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\vartheta_i||\vartheta_j|\geq\left(\prod_{i\neq j}|\vartheta_i||\vartheta_j|\right)^{1/[p(p-1)]}.$$

Then

$$\begin{aligned}\left(\sum_{i=1}^{p}|\vartheta_i|\right)^2-\sum_{i=1}^{p}|\vartheta_i|^2&\geq p(p-1)\left(\prod_{i\neq j}|\vartheta_i||\vartheta_j|\right)^{1/[p(p-1)]}\\&\geq p(p-1)\left(\prod_{i=1}^{p}|\vartheta_i|^{2(p-1)}\right)^{1/[p(p-1)]}\\\left(\sum_{i=1}^{p}|\vartheta_i|\right)^2-2Z&\geq p(p-1)\left|\prod_{i=1}^{p}\vartheta_i\right|^{2/p}\\\left(\sum_{i=1}^{p}|\vartheta_i|\right)^2&\geq 2Z+p(p-1)\left|\prod_{i=1}^{p}\vartheta_i\right|^{2/p}\\\sum_{i=1}^{p}|\vartheta_i|&\geq\sqrt{2Z+p(p-1)\left|\prod_{i=1}^{p}\vartheta_i\right|^{2/p}}.\end{aligned}$$

We know that $|\vartheta_i|-|\frac{2q}{p}|\leq|\vartheta_i-\frac{2q}{p}|$, $\forall i=1,2,...,p$. i.e., $|\vartheta_i|-\frac{2q}{p}\leq|\vartheta_i-\frac{2q}{p}|$, $\forall i=1,2,...,p$.

$$\begin{aligned}\sum_{i=1}^{p}|\vartheta_i|-\sum_{i=1}^{p}\frac{2q}{p}&\leq\sum_{i=1}^{p}|\vartheta_i-\frac{2q}{p}|\\\sum_{i=1}^{p}|\vartheta_i|-2q&\leq LE_H(G).\end{aligned}$$

Then

$$\begin{aligned}LE_H(G)&\geq\sum_{i=1}^{p}|\vartheta_i|-2q\\&\geq\sqrt{2Z+p(p-1)\left|\prod_{i=1}^{p}\vartheta_i\right|^{2/p}}-2q.\end{aligned}$$

□

Theorem 5.3. *Let G be a graph with a minimum hub set H. If the Laplacian minimum hub energy $LE_H(G)$ of G is a rational number, then*

$$LE_H(G)\equiv|H|(mod\ 2).$$

Proof. Let $\vartheta_1,\vartheta_2,...,\vartheta_p$ be Laplacian minimum hub eigenvalues of a graph G of which $\vartheta_1,\vartheta_2,...,\vartheta_p$ are positive and the remaining are non-positive, then

$$\begin{aligned}\sum_{i=1}^{p}|\vartheta_i|&=(\vartheta_1+\vartheta_2+...+\vartheta_s)-(\vartheta_{s+1}+...+\vartheta_p)\\&=2(\vartheta_1+\vartheta_2+...+\vartheta_s)-(\vartheta_1+\vartheta_2+...+\vartheta_p)\end{aligned}$$

$$= 2(\vartheta_1 + \vartheta_2 + ... + \vartheta_s) - \sum_{i=1}^{p} \vartheta_i$$

$$= 2(\vartheta_1 + \vartheta_2 + ... + \vartheta_s) - (2q - |H|)$$

$$= 2(\vartheta_1 + \vartheta_2 + ... + \vartheta_s - q) + |H|.$$

Then $\sum_{i=1}^{p} |\vartheta_i| \equiv |H| \pmod 2$. □

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