Int. J. Math. And Appl., 8(3)(2020), 77–84
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Fibonacci and Lucas Numbers-Continued Fraction Expansion

S. A. Bhanotar[1,*]

1 Department of Mathematics, L J Institute of Engineering and Technology, Ahmedabad, Gujarat, India.

Abstract: In modern science, there are lots of applications of Fibonacci and Lucas numbers have been used in number theory, computer science, applied mathematics and biology. In this paper, I have discussed Fibonacci number, Lucas number, the golden ratio, and their relations to them. I study numerous new properties of these sequences and focused to the generalized the formula for the Fibonacci number, Lucas number and some important identity; some observation of sunflower the head is carried out with the angular momentum velocity and represented continued fraction expansion of such number based on the golden ratio.

MSC: 11B39, 11J70.

Keywords: Fibonacci number, Lucas number, Golden Ratio, Continued fraction expansion.

1. Introduction

Fibonacci numbers were initially found by a person whose name is Leonardo Pisano. He was known by his nickname, Fibonacci. As we are Familiar that Fibonacci number has been published in the year 1202, it was considered on famous rabbit puzzle: a person put a different gender-wise pair of newly born rabbits in the field. It is crucial that Rabbits take a month to mature to adult. One month after mating, females give to one male-female pair and then mate again (see [5]). It is going to be an infinite sequence, and there is no Rabbits die. Infect, we can calculate the Rabbit pair at the end of the year. We can analysis from Table 1 Fibonacci's rabbit growth that, at the starting of each month, the Teenage pairs, adult pair and total pairs are shown. It is quite easy to understand from the bellow table 1 that each teenage will become an adult after each month and able to give a new-born. So, it's representation of total number, $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$. This great number is called Fibonacci's number (see [5]). We can note from the below table 1 that at the end of one-year total Rabbits becomes, $F_{13} = 233$, where F_n, denotes Fibonacci's number corresponding to n belong to N. Furthermore, it is clear to write the recurrence relation, $F_{n+1} = F_n + F_{n-1}$. To elaborate next number, it is necessary to have first two initial numbers, $F_1 = 1$, $F_2 = 1$ based on Rabbit's Problem gives you better understanding of this number.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec	Jan
Teenage	1	0	1	1	2	3	5	8	13	21	34	55	89
Adult	0	1	1	2	3	5	8	13	21	34	55	89	144
Total	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1: Fibonacci's rabbit growth

* *E-mail: sbhanotar@gmail.com*

Roughly speaking, Fibonacci can be found in nature, not only based upon the rabbit's problem but also on the head of a sunflower, the seeds are packed in the pattern of the Fibonacci numbers. The petals of flowers may also be related to the Fibonacci numbers. This number pattern can be extended to non-positive indices using the relation,

$$F_n = F_{n+2} - F_{n+1}.$$

And it can be expressed as, $F_{-n} = (-1)^{n+1} F_n$, and proved by mathematical induction.

In 2008, Steven [1] studied about application based on Fibonacci, Lucas and golden ration. Then Omar Yayenie [2] studies the new properties of Fibonacci sequence and investigated a sequence the generalized of Lucas's first kind of form in 2011. In 2014 Edson and Yayenie [4] and in 2016 Emrah Polatl ı [6] has derived relation between Fibonacci and Lucas numbers the generalization of the Fibonacci and Lucas quaternion.

In 2013 Daniel Panario, Murat S, and Wang [3] and in 2016 Jeffrey [1] has represented the matrix form and extended the Binet's formula for generalized continuant. Moreover, book Author Hubert [7] has represented analytic approach of continued fraction based on Fibonacci number in 2016. In this paper, I have proved some identity which gives enormous representation of Fibonacci number, Lucas's number and golden ratio. Moreover, one can observe the way of continued fraction, which you can observed details in following sub sequential parts. Now we follow the following some important theorem.

Theorem 1.1. *Prove that $f_{n+2} = F_{n+1}Q + F_nP$, using the generalized Fibonacci sequence satisfies,*

$$f_{n+1} = f_n + f_{n-1},$$

With starting values $f_1 = P$, and $f_2 = Q$.

Proof. We use mathematical induction to prove this. The result is true for $n = 1$. We write,

$$\begin{aligned} f_3 &= F_2Q + F_1P \\ &= Q + P. \end{aligned}$$

To prove that the result is true for $n = 2$, we have,

$$\begin{aligned} f_4 &= F_3Q + F_2P \\ &= (Q + P) + P \\ &= Q + 2P \\ &= F_2Q + F_3P. \end{aligned}$$

Suppose that the result is true for, $n = k$,

$$f_{k+2} = F_{k+1}Q + F_kP,$$

And $n = k + 1$,

$$f_{k+3} = F_{k+2}Q + F_{k+1}P \tag{1}$$

Then we have claim to prove that the result is true for, $n = k + 2$, using recurrence relation of Fibonacci numbers.

$$f_{k+4} = f_{k+3} + f_{k+2}$$

$$= F_{k+2}Q + F_{k+1}P + F_{k+1}Q + F_kP$$
$$= (F_{k+2} + F_{k+1})\, Q + (F_{k+1} + F_k)P$$
$$= F_{k+3}Q + F_{k+2}P$$

Therefore, the principle of induction the result, $f_{n+2} = F_{n+1}Q + F_nP$, is true for all positive integers. □

2. Lucas Number

The Lucas numbers are an integer sequence named after the mathematics scientist Francois Edouard Anatole Lucas (1842-91), studied about Fibonacci numbers and derived Lucas sequences it is closely related to Fibonacci numbers. The first few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$. It also follows the same recurrence relation of Fibonacci numbers, $L_{n+1} = L_n + L_{n-1}$, n belong to N. To extend these number systems it is required to have first two initial numbers, $L_0 = 2$, $L_1 = 1$. See [1–6]. From the Fibonacci number and Lucas number we can observe that and can be proved by reader,

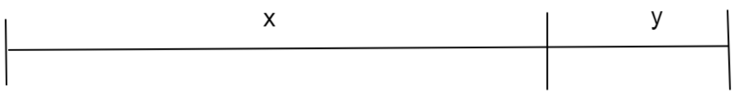
$$L_n = F_{n-1} + F_{n+1},$$

and

$$F_n = \frac{1}{5}\left(L_{n-1} + L_{n+1}\right). \tag{2}$$

3. The Golden Ratio

The Golden ratio was derived from the Fibonacci sequence. We now write the classical definition of golden ratio.



From the figure, two numbers x and y, with $x > y > 0$ are said to be in the golden ratio between their sum and the larger number, that is

$$\psi = \frac{x}{y} = \frac{x+y}{x} \tag{3}$$
$$\psi = 1 + \frac{y}{x}$$
$$\psi = 1 + \frac{1}{\psi} \tag{4}$$
$$\psi^2 - \psi - 1 = 0 \tag{5}$$

Solving we get,

$$\psi = \frac{1 \pm \sqrt{5}}{2}$$

But ψ is the positive number, so ψ cannot be negative number. Therefore, we have

$$\psi = \frac{\sqrt{5}+1}{2} \approx 1.618$$

The non-positive root of the quadratic equation (5) is called golden ratio conjugate

$$\emptyset = \frac{\sqrt{5}-1}{2} \approx 0.618 \tag{6}$$

Theorem 3.1. *If ψ and $\emptyset$ are the golden ratio and golden conjugate ratio respectively, then prove that,*

(a). $\emptyset = \frac{1}{\psi}$,

(b). $\psi^{n+1} = \psi^n + \psi^{n-1}$.

Proof. The relationship between $\emptyset$ and ψ is given by, $\emptyset = \psi - 1$ and by (4), we have

$$\psi - 1 = \frac{1}{\psi}$$
$$\therefore \emptyset = \frac{1}{\psi}$$

From equation (5), we have

$$\psi^2 = \psi + 1.$$

Multiply both sides by ψ^{n-1} yields the outcome,

$$\psi^{n+1} = \psi^n + \psi^{n-1}$$

□

Theorem 3.2. *If* $\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \psi$*, prove that* $\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \psi^k$*.*

Proof. Write,

$$\frac{F_{n+k}}{F_n} = \frac{F_{n+k}}{F_{n+k-1}} \times \frac{F_{n+k-1}}{F_{n+k-2}} \times \ldots . \times \frac{F_{n+1}}{F_n} \tag{7}$$

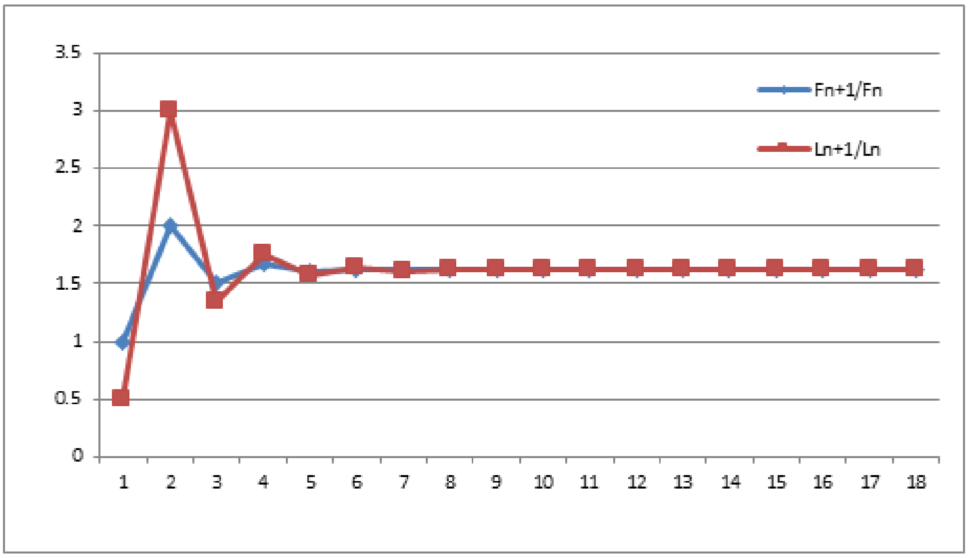
Letting limit n tends to ∞, and using the fact, $\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \psi$, implies, $\lim_{k\to\infty} \frac{F_k}{F_{k-1}} = \psi$. Directly, we can conclude that,

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \lim_{n\to\infty} \left(\frac{F_{n+k}}{F_{n+k-1}} \times \frac{F_{n+k-1}}{F_{n+k-2}} \times \ldots . \times \frac{F_{n+1}}{F_n} \right) = \psi^k.$$

□

Now looking at the ratios of Fibonacci and Lucas numbers it tends to golden ratio 1.618, which can be followed by the following tables and observed it converges to Golden ratio.

F_n	L_n	$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$
0	2	—	—
1	1	1	0.5
1	3	2	3
2	4	1.5	1.3333
3	7	1.666667	1.75
5	11	1.6	1.571428571
8	18	1.625	1.636363636
13	29	1.615385	1.611111111
21	47	1.619048	1.620689655
34	76	1.617647	1.617021277
55	123	1.618182	1.618421053
89	199	1.617978	1.617886179
144	322	1.618056	1.618090452
233	521	1.618026	1.618012422
377	843	1.618037	1.618042226
610	1364	—	—

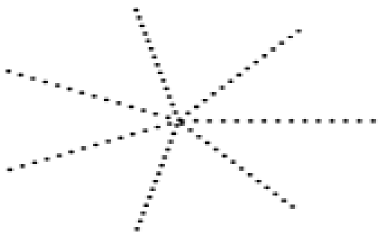


4. Model: The Growth of a Sunflower

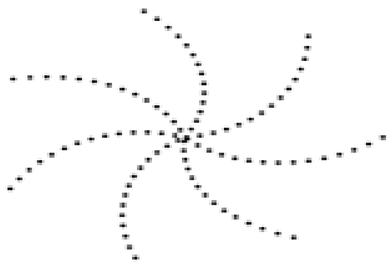
Look at the model for the growth of a head of a sunflower, and why the sequences of Fibonacci appear. Suppose that during creation, florets are created in the centre of the head and move forward to rapidly outward with the uniform speed as constant during the development of it. Also, it moves radially when a constant rotation angle observed.

Famous the revolution point be through $2\pi\alpha$. Firstly, the possible worth of α is a rational number, verbalize n/m, wherever m and n are relatively prime number, with $n < m$. A florets rotation will return to the radial line which was initially commenced. The outcome results of sunflower head involve the florets lying along m not the curve shaped but straight lines. There is such a simulation of the head of the sunflower for $\alpha = 1/7$ is shown in the following fig. 1(a), where it is clear seven straights lines. Infect the value of α-rational value does not well-spaced florets. One noted that a number of rotations are not going to be returning the florets to their radial line for the case of irrational number α. For an instant, if $\alpha = \pi - 3$, then the outcomes of the result follow fig. 1(b) and it is on counterclockwise spirals.

Number say irrational that are going to elaborate the development of a sunflower head with aligned equally-mannered space of florets. On the contrary, it cannot be well-approximated by judicious an irrational number. Here we take the brilliant golden angle, taking $\alpha = 1 - \psi$. This convey to $1 - \psi$ is given by $\frac{F_n}{F_{n+2}}$, Therefore, the flourish number of spirals can be followed to the correspond to the Fibonacci numbers.



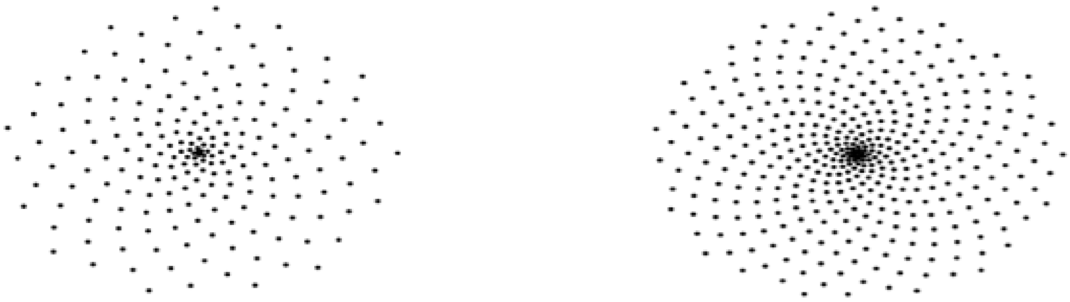
(a) Sunflower model for $\alpha = \frac{1}{7}$



(b) Sunflower model for $\alpha = \pi - 3$

Figure 1

At two simulations of the sunflower head with $\alpha = 1 - \psi$ are shown in the following fig. 2(a) and 2(b). It observes that differ by only the choice of velocity v_0. In the fig. 2(a), there are numbers that are observed 13 clockwise spirals and 21 counter-clockwise spirals; and in fig. 2(b), 21 clockwise spirals and 34 clockwise spirals.



(a) Model for $\alpha = 1 - \psi$ and clockwise direction $v_0 = \frac{1}{2}$

(b) Model for $\alpha = 1 - \psi$ and Clockwise direction $v_0 = \frac{1}{4}$

Figure 2

5. Continued Fraction

If ψ and $\emptyset$ are the golden ratio and golden conjugate ratio respectively satisfy

$$\psi = \frac{x}{y}, \quad \emptyset = \frac{y}{x}$$

Therefore, $\psi = \frac{1}{\emptyset}$, with $\psi = 1 + \emptyset$, we can obtain the golden angle,

$$\begin{aligned} \frac{g}{2\pi} &= \frac{y}{x+y} \\ &= \frac{\emptyset}{1+\emptyset} \\ &= \frac{\emptyset}{\psi} = \emptyset^2 \end{aligned}$$

Since $\emptyset^2 = 1 - \emptyset$, we get,

$$\frac{g}{2\pi} = 1 - \emptyset \Longrightarrow g = 2\pi\,(1 - \emptyset) \tag{8}$$

Or in degree $g \approx 137.5^\circ$. To detect the continued fraction for $\dfrac{g}{2\pi} = \dfrac{y}{x+y} = \dfrac{1}{1+\psi}$ and by equation (3),

$$\begin{aligned} \psi &= 1 + \frac{1}{\psi} \\ \frac{g}{2\pi} &= \frac{1}{1+1+\frac{1}{\psi}} \\ &= \frac{1}{1+1+\frac{1}{1+\frac{1}{1+\dots}}} \end{aligned}$$

Which belong to $[0; 2, 1]$. The training ones in the continued fraction conform that $\frac{g}{2\pi}$ is difficult to represent as a rational number? The successive approximation to $\frac{g}{2\pi}$ can be computed to be $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{8}, \dfrac{5}{13}, \dots$ which is looks like $\dfrac{F_n}{F_{n+2}}$.

Lemma 5.1. *Define ω_n is the n^{th} rational approximation to ω, which obtained from continued fraction.*

For example, $\omega_0 = [A_0;]$, $\omega_1 = [A_0; A_1]$, and $\omega_2 = [A_0; A_1, A_2]$, using $\frac{g}{2\pi} = [0; 2, 1]$, we can determine $\frac{g_0}{2\pi}$, $\frac{g_1}{2\pi}$ and $\frac{g_2}{2\pi}$.

Solution. We have

$$\begin{aligned}
\frac{g_0}{2\pi} &= [0;] = 0,\\
\frac{g_1}{2\pi} &= [0; 2] = \frac{1}{2} = \frac{F_1}{F_3},\\
\frac{g_2}{2\pi} &= [0; 2, 1] = \frac{1}{2 + \frac{1}{1}} = \frac{1}{3} = \frac{F_2}{F_4},
\end{aligned}$$

and so on. In general, in other words, we can follow in the following ways.

Lemma 5.2. *If the golden angle is given by*

$$g \;=\; 2\pi(1 \;-\psi).$$

The n^{th} rational approximation to $\frac{g}{2}\pi$ from its continued fraction is given by $\frac{g_n}{2\pi} = \frac{F_n}{F_{n+2}}$.

Proof. It is very easy to conclude that,

$$\begin{aligned}
\frac{g_n}{2\pi} &= \frac{1}{1 + \psi_n}\\
&= \frac{1}{1 + \frac{F_{n+1}}{F_n}}\\
&= \frac{F_n}{F_n + F_{n+1}}\\
&= \frac{F_n}{F_{n+2}}.
\end{aligned}$$

□

Moreover, one can find the continue fraction of the fourth approximation to $\sqrt{2}$ from its continued fraction is given by,

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = 1 + \frac{1}{2 + \frac{2}{5}} = 1 + \frac{5}{12} = \frac{17}{12}.$$

Example 5.3. *Show that $e = [2; 1, 2, 1, 1, 4, \ldots]$.*

Solution.

$$\begin{aligned}
e &= 2 + 0.718281\\
&= 2 + \frac{1}{1.392211\ldots}\\
&= 2 + \frac{1}{1 + 0.392211}\\
&= 2 + \frac{1}{1 + \frac{1}{2.549646..}}\\
&= 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1.819350\ldots}}}\\
&= 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1.220479\ldots}}}}\\
&= 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4.535573\ldots}}}}}
\end{aligned}$$

and so on. Thus, the expansion of $e = [1; 1, 2, 1, 1, 4, \ldots]$. Infect, this expansion can be carried out in a regular as $e = [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots]$. This is called Euler's continued fraction.

Acknowledgement

The author is exceedingly appreciation to the mysterious referees for their valuable comments and helpful suggestions on the primary draft of the paper.

References

[1] S. Vajda, Fibonacci and Lucas numbers and the golden section: theory and applications, Dover Publications Inc, Mineola, New York Corporation, (2008).

[2] O. Yayenie, *A note on generalized Fibonacci sequences*, Applied Mathematics and Computation, 217(12)(2011), 5603-5611.

[3] D. Panario, M. Sahin and Q. Wang, *A family of Fibonacci-like conditional sequences*, Integers, 13(2013).

[4] G. Bilgici, *Two generalizations of Lucas sequence*, Applied Mathematics and Computation, 245(2014), 526-538.

[5] Jeffrey R. Chasnov, *Fibonacci Numbers and the golden ratio*, Hongkong: book, (2016).

[6] E. Polatlı, *A generalization of Fibonacci and Lucas Quaternions*, Advances in Applied Clifford Algebras, 26(2)(2016), 719-730.

[7] Hubert Stanley Wall, *Analytic theory of continued fractions*, Courier Dover Publications, (2018).