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Oscillation of Bridge: Mathematical Modeling of the Amplitude Under Different Situations

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Abstract: We consider a function that shows the position of a bridge. This function includes the effect of the force applied by the soldier marching on it. We analyze the function and obtain how the impulse effect the position of the bridge in different cases, such as the resonance case-when the soldier applies a force with the same frequency as the frequency of the bridge oscillations-and the non-resonance case-when the soldier applies a force with a different frequency from the frequency of the bridge oscillations.

Keywords: Bridge, Oscillation, Modeling.

1. Introduction

By an amplitude we refer to the maximum extent of an oscillation. For example, we could be referring to an amplitude of a pendulum, wave, sine function, etc. The use of mathematical models to predict the how the amplitude of the bridge changes as a function of time is common practice and it is really helpful in real life. In this article we consider the amplitude of a bridge. This amplitude measures the distance between the center of the bridge and the position of the bridge at its equilibrium position. We assume that the only factors that affect the change in amplitude is the force that is applied on the bridge by the marching soldier. The center of the bridge can be viewed as a platform that is supported by a spring. The forces that is applied on it are its weight, the elastic force from the spring, and the force that is applied by the marching soldier. The free body diagram is shown below.

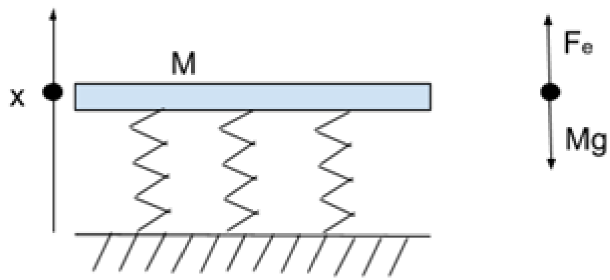

Figure 1. **Bridge as a platform. Free body diagram**

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This article is organized as follows. In Section 2, we describe the mathematical model of a bridge without pedestrians. In Section 3, we describe the effect of an impulse on the dynamics of the platform. In Section 4, we describe the mathematical model of a bridge pedestrians walking at unison. We conclude in Section 5 with a small discussion.

2. Mathematical Model Without Pedestrians

As described in the introduction, we view the center of the bridge as a platform on a spring that can only move vertically. We denote the mass of the bridge by M. The only forces that act on the platform are the weight, denoted by Mg, and the elastic force from the spring, denoted by F_e. We assume that the elastic force follows Hooke's law, and there is an equilibrium position at which the elastic force is zero. At any other height, this force becomes $-kx$, where k is the elastic constant, which is positive, and x represents the distance between the height of the spring and its equilibrium position. The negative sign exists because the force should be positive when x is negative and vice versa.

When we set the height of equilibrium position as zero, x also represents the position of the bridge. Thus, denoting by the gravitational constant, g, and a, the acceleration of the platform, Newton's second law, force is equal to mass times acceleration, implies

$$-kx - Mg = Ma \tag{1}$$

We will study the time evolution of the bridge. We use t to represent time and every function is a function of time t, such as $x = x(t)$, $v = v(t)$, and $a = a(t)$. We denote derivatives with primes, so $a = v' = x''$ since acceleration is the derivative of velocity, and it's the second derivative of position [2]. Substituting the a with x'', function (1) becomes

$$-kx - Mg = Mx'' \tag{2}$$

Both of the unknown function $x = x(t)$ and derivatives of the unknown function appear in the function (2). In this kind of functions, there are infinite number of solutions. In order to get the one solution that we want, we need to have the initial height and velocity of the bridge, which is $x(0)$ and $x'(0)$.

Without solving the function, it is helpful to realize that at the equilibrium position, where the acceleration is zero, $x(t) = -\frac{Mg}{k}$. Using this information, the functions becomes

$$z(t) = x(t) + \frac{Mg}{k} \tag{3}$$

This function shows the situation when the elastic force is equal to the weight of the bridge but in the opposite direction. If we define $w = \sqrt{k/M}$, the function can be simplified to

$$z'' + w^2 z = 0 \tag{4}$$

There are infinite solutions to function (4) as well [3]. The solutions are

$$z = A\sin(wt + \varphi) \tag{5}$$

In function (5), both of A and φ are constants, so each value of A and φ is a solution to function (5). The constant A shows the amplitude. Figure 2 shows a plot of the solution when $A = 1$, $w = 2\pi$, and $\varphi = -\pi/4$.

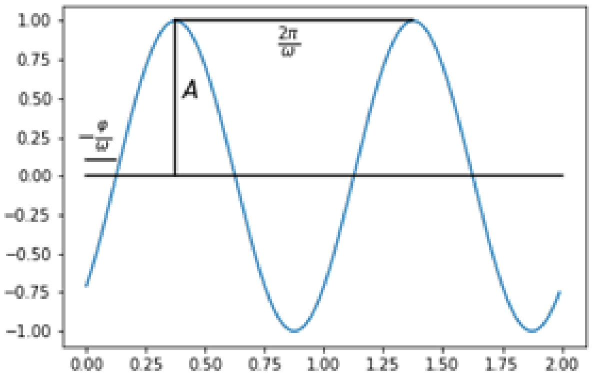

Figure 2. **Plot of $z = A\sin(wt + \varphi)$ with when $A = 1$, $w = 2\pi$, and $\varphi = -\pi/4$**

As it is shown in figure 2, the bridge moves at a constant rate w, and $w = \sqrt{k/M}$, with a constant amplitude A. The frequency w depends on the properties of the system, but the amplitude A is always the initial amplitude.

3. Effect of an Impulse on the Dynamics of the Platform

When there is a soldier marching on the bridge, there is a force F that is applied on the bridge in a period of time Δt. The impulse J created by this force on the bridge is

$$J = F\Delta t \tag{6}$$

The momentum p of an object is usually the product of velocity v and mass M

$$p = Mv \tag{7}$$

And the impulse is the change in momentum

$$p_2 - p_1 = J \tag{8}$$

When a force is applied on the bridge at $t = t_1$, there is an impulse of $-J$. It is negative because the velocity would decrease since the force is applied downwards and p_2 would be smaller than p_1. Also, function (5) would be divided into two parts at $t = t_1$. Because of the impulse, the value of A and φ would change. To differentiate the constants, we denote them by A_1 and φ_1 before $t = t_1$, and we denote them by A_2 and φ_2 after $t = t_1$. Then function (5) becomes

$$z = \begin{cases} A_1 \sin(wt + \varphi_1) & \text{if } t < t_1 \\ A_2 \sin(wt + \varphi_2) & \text{if } t > t_1 \end{cases} \tag{9}$$

The constants A_2 and φ_2 can be shown in terms of A_1 and φ_1, using the facts that the position z is continuous and that the change in momentum $-J$ at time $t = t_1$. These two facts lead to the following two equations respectively

$$\begin{aligned} A_2 \sin(wt_1 + \varphi_2) &= A_1 \sin(wt_1 + \varphi_1) \\ MwA_2 \cos(wt_1 + \varphi_2) &= MwA_1 \cos(wt_1 + \varphi_1) - J \end{aligned} \tag{10}$$

For convenience, we define the constant

$$I = \frac{J}{Mw}$$

After dividing the second equation of equation (10) by Mw, equation (10) becomes

$$\begin{aligned} A_2 \sin(wt_1 + \varphi_2) &= A_1 \sin(wt_1 + \varphi_1) \\ A_2 \cos(wt_1 + \varphi_2) &= A_1 \cos(wt_1 + \varphi_1) - J \end{aligned} \tag{11}$$

We square both equations, add them, and take the square root to get

$$A_2 = \sqrt{(A_1)^2 - 2A_1 I \cos(wt_1 + \varphi_1) + I^2} \tag{12}$$

In order to find φ_2, we need to go back to function (11). When $A_2 = 0$, φ_2 can be any value. We arbitrarily select $\varphi_2 = 0$. If $A_2 \neq 0$, we define $\theta = \sin^{-1}(A_1 \sin(wt_1 + \varphi_1)/A_2)$. From the first equation we have that either $wt_1 + \varphi_2 = \theta$ or $wt_1 + \varphi_2 = \pi - \theta$. We use the second equation to decide. Simple arguments lead to the following:

$$\varphi_2 = \begin{cases} \sin^{-1}(A_1 \sin(wt_1 + \varphi_1)/A_2) - wt_1 & \text{if } A_1 \cos(wt_1 + \varphi_1) - I > 0 \\ \pi - \sin^{-1}(A_1 \sin(wt_1 + \varphi_1)/A_2) - wt_1 & \text{if } A_1 \cos(wt_1 + \varphi_1) - I < 0 \end{cases} \tag{13}$$

Two examples shown in the plot below. Both of them are graphs of $z(t)$ with different parameters, and on each plot, there is a dot on the line, which indicates where the impulse is applied. The parameters of the graph on the left are $A_1 = 1$, $w = 2\pi$, $\varphi_1 = -\frac{\pi}{4}$, $I = 0.3$, and $t_1 = 3.1$. The parameters of the graph on the right are $A_1 = 1$, $w = 2\pi$, $\varphi_1 = -\frac{\pi}{4}$, $I = 0.3$, and $t_1 = 3.6$. As we can observe, the amplitude in the left graph decreased after the impulse was applied. So we can get the conclusion that if an impulse is applied when the bridge is moving upwards, the amplitude will decrease. However, the amplitude in the other graph increased after the impulse was applied, which leads to the conclusion that if an impulse is applied when the bridge is moving downwards, the amplitude will increase.

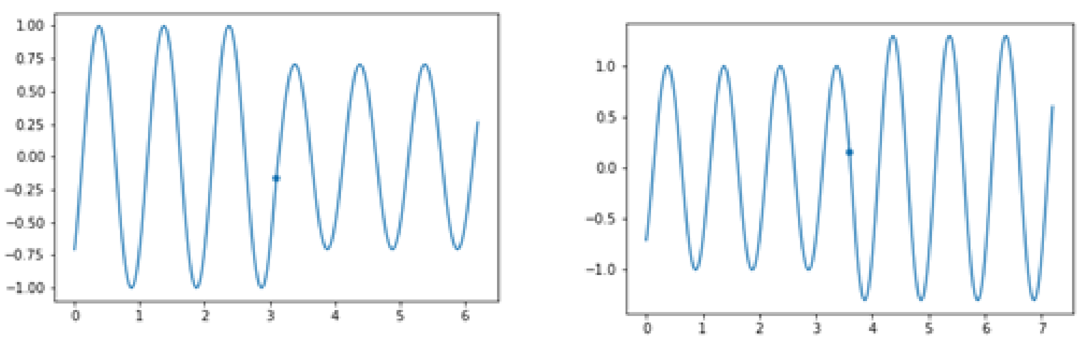

Figure 3. **Illustration of the dynamics of the platform when subject to an impulse at the time of the solid dot**

4. The Mathematical Model of a Bridge Pedestrians Walking at Unison

We model the effect of pedestrians walking at unison as the pedestrians are applying an impulse of $-J$ onto the platform. Again, it has a negative sign because the impulse is applied downwards. We assume the pedestrians are having each step at a certain frequency that never changes. Let a positive constant T represent the time difference between each step, and $t_n = nT$. The impulse of $-J$ is applied to the platform for each time that $t = t_n$. Extending from the last section, we get

$$z = A_n \sin(wt + \varphi_n) \quad if\ t_{n-1} < t < t_n \tag{14}$$

As before, we define $I = \frac{J}{Mw}$. With a similar analysis as before, leads to the equation for the amplitudes

$$A_{n+1} = \sqrt{A_n^2 - 2A_n I \cos(wt_n + \varphi_n) + I^2} \quad and \quad A_0 = 0$$

Note that $A_0 = 0$ corresponds to the platform not moving before the first step of the pedestrians. We also have the equations $\varphi_{n+1} = 0$ if $A_{n+1} = 0$, otherwise

$$\varphi_{n+1} = \begin{cases} \sin^{-1}(A_n \sin(wt_n + \varphi_n)/A_{n+1}) - wt_n & \text{if } A_n \cos(wt_n + \varphi_n) - I > 0 \\ \pi - \sin^{-1}(A_n \sin(wt_n + \varphi_n)/A_{n+1}) - wt_n & \text{if } A_1 \cos(wt_1 + \varphi_1) - I < 0 \end{cases} \tag{15}$$

Figure 4 is a graph of $z(t)$. In this example, $w = 2\pi$, $I = 0.2$, and $T = 1.2$. Note that $z(t)$ is a periodic function, so the amplitude is bounded. In fact, we have $|z(t)| < 0.16$ for all time. The period is 6. In other words, $z(t+6) = z(t)$. This is a consequence of a more general fact that we state as an observation.

Observation: If there are positive integers m and k such that $wmT = 2k\pi$ and m does not divide k, then $z(t)$ is periodic with period mT, i.e. $z(t + mT) = z(t)$ for all t.

In our example of Figure 4, we have that $m = 5$ and $k = 6$ satisfy the conditions of the above observation because $wmT = 2\pi(5)(1.2) = 12\pi$, and $2k\pi = 2(6)\pi = 12\pi$. This example of figure 4 corresponds to a nonresonant case.

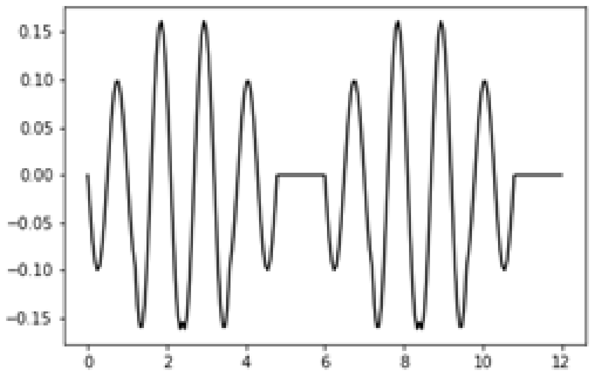

Figure 4. **Plot of $z(t)$ with $w = 2\pi$, $I = 0.2$, and $T = 1.2$**

Figure 5 shows a plot of $z(t)$. In this example, $w = 2\pi$, $I = 0.2$, and $T = 1$. Note that $z(t)$ is not a periodic function of t. In fact, its amplitude grows linearly with time. More precisely, we have that $z\left(n + \frac{3}{2}\right) = nI$ for all positive integers n. This is a consequence of a more general fact that we state as an observation.

Observation: If there are positive integer k such that $wT = 2k\pi$, then the amplitude of $z(t)$ grows linearly with time. More precisely,

$$z \approx -\frac{I}{T} t \sin(wt),$$

where the symbol $\approx$ means asymptotically as t becomes large. This means that the function above is an approximation, but it is an accurate approximation. The larger the t is, the better the approximation becomes. This example of Figure 5 corresponds to a resonant case [1]. This corresponds to the scenario of the marching soldiers collapsing the bridge. The bridge collapses because the soldiers march at unison and with a frequency equal to the natural frequency of the bridge.

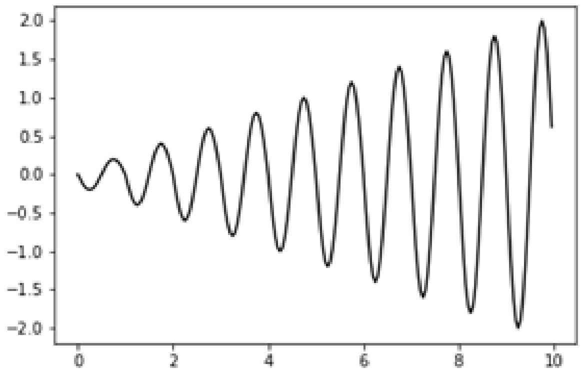

Figure 5. **Plot of $z(t)$ with $w = 2\pi$, $I = 0.2$, and $T = 1$**

5. Discussion

In this article we describe a simple mathematical model of the that pedestrians have on the dynamics of a bridge as they cross it in unison. This is an example of a phenomenon known as force oscillators. Our analysis shows the difference between the resonant and nonresonant case. The resonant case leads to unbounded increases in the amplitude of the oscillations, and in our case of the bridge, its eventual collapse. Naturally, this is the simplest model we could think of that captures the relevant physics and as a consequence, we have neglected several other physical effects that do not play a determinant role in our context. Our model could be considered a toy model.

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