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Set Domination Number of Jump Graph

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Abstract: Let $G = (V, E)$ be a connected graph. A Set $D \subseteq V(J(G))$ is a set dominating set (sd-set) of jump graph if for every set $T \subset V(J(G)) - D$, there exists a non empty set $S \subset D$ such that the subgraph $\langle S \cup T \rangle$ induced by $S \cup T$ is connected. The set domination number $\gamma_s(J(G))$ of $J(G)$ is the minimum cardinality of a sd-set of $J(G)$ .We also study the graph theoretic properties of $\gamma_s(J(G))$ and its exact values of some standard graphs. The relation between $\gamma_s(J(G))$ with other parameters is also investigated.

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1. Introduction

Let $G(p, q)$ be a graph with $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. All the graphs considered here are finite, non trivial, undirected and connected without loops or multiple edges.

Let $G(V, E)$ be a connected graph. A set $D \subset V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G was first introduced by E. J. Cockayne and etal in [1] and further the concept was developed by R. Laskar and etal in [5].

A set $D \subset V$ is a set dominating set (sd-set) of G if for every set $T \subset V - D$, there exists a non empty subset $S \subset D$ such that the subgraph $\langle S \cup T \rangle$ is connected. The set domination number $\gamma_s(G)$ of G is the minimum cardinality of a sd-set. This concept was first studied by E. Sampathkumar and etal in [7].

Thus in this paper we going to study the set domination number of jump graph for some standard graphs and also we discussed the graph theoretic properties of $\gamma_s(J(G))$ Any undefined term or notation in this paper can be found in F. Harary [3].

Throughout this paper, by a graph we mean a connected graph.

2. Preliminary Notes

Definition 2.1. *The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. We call the complement of line graph $L(G)$ as the jump graph $J(G)$ of G, found in*

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[2]. *The jump graph $J(G)$ of a graph is the graph defined on $E(G)$ and in which two vertices are adjacent in G. Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G.*

Remark 2.2. *The isolated vertices of G (if G has) play no role in line graph and jump graph transformation. We assume that the graph G under consideration is nonempty and has no isolated vertices* [2].

Definition 2.3. *A set $D \subseteq V(J(G))$ is said to be set dominating set (sd-set) of $J(G)$, is for every set $T \subset [V(J(G))] - D$, there exists a non empty subset $S \subset D$ such that the subgraph $\langle S \cup T \rangle$ is connected. The set domination number $\gamma_s(J(G))$ of $J(G)$ is the minimum cardinality of a sd-set.*

Remark 2.4. *For any graph G, with $p \leq 4$, the jump graph $J(G)$ of G is disconnected. Since we study only the connected jump graph, we choose $p > 4$* [4].

Remark 2.5. *Let G be graph without isolated vertices. If D is a minimal dominating set, then $V - D$ is a dominating set* [6].

3. Main Result

Theorem 3.1.

(1). For any path P_p with $p \geq 5$, $\gamma_s[J(P_p)] = 2$

(2). For any cycle C_p, with $p \geq 5$, $\gamma_s[J(C_p)] = 2$

(3). For any complete graph K_p, with $p \geq 5$, $\gamma_s[J(K_p)] = 3$

(4). For any complete bipartite graph $K_{m,n}$

$$\gamma_s[J(K_{m,n})] = \begin{cases} 2 & \text{for } K_{2,n} \text{ where } n \geq 2 \\ 3 & \text{for } K_{m,n} \text{ where } m, n \geq 3 \end{cases}$$

(5). For any wheel W_p, $\gamma_s[J(W_p)] = \begin{cases} 3 & \text{if } p = 5, 6 \\ 2 & \text{if } p \geq 7 \end{cases}$

Theorem 3.2. *For any connected graph G , $\gamma_s(J(G)) \geq 2$.*

Proof. Proof of the theorem is obvious. □

Theorem 3.3. *If D is any set dominating set (sd-set) of $J(G)$ such that $|D| = \gamma_s(J(G))$, then $V(J(G)) - D \leq \sum deg(v_i)$.*

Proof. Since every vertex in $V(J(G)) - D$ is adjacent to at least one vertex in D, there will be a contribution from each vertex of $V(J(G)) - D$ by one to the sum of degrees of vertices of D. Hence the proof of the theorem. □

Theorem 3.4. *For any connected (p, q) graph G, $\gamma_s(J(G)) \leq q - \beta_1(G) + 1$.*

Proof. Let $V_1 = \{v_1, v_2, v_3, \ldots v_n\}$ be the set of vertices in $J(G)$ corresponding to the set of independent edges $\{e_1, e_2, e_3, \ldots e_n\}$ of G. By the definition of $J(G)$, the elements of V_1 form an induced subgraph $\langle K_n \rangle$ in $J(G)$. Further, let $S \cup \{v_1\}$, where $S \subset [V(J(G) - (v_1)]$ be a set dominating set in $J(G)$. It follows that $|S \cup \{v_1\}| \leq |V(J(G)) - (v_1)| + |v_1|$ therefore $\gamma_s[J(G)] \leq q - \beta_1(G) + 1$. □

The following theorem gives the relationship between set domination numbers of a graph with its set domination number of a jump graph.

Theorem 3.5. *For any connected (p,q) graph G, $\gamma_s(G) + \gamma_s(J(G)) < \left(\frac{p+1}{2}\right)^2$.*

Proof. For any (p,q) graph G, we have $\gamma_s(G) \leq min\{|D|, |V-D|\} \leq \frac{p}{2}$ by virtue of remarks. Further, by the definition of jump graph, we have $V(J(G)) = q$. Hence, $\gamma_s(J(G)) \leq \frac{q}{2}$. But for any simple graph G, $q \leq \frac{p(p-1)}{2}$ and therefore, we get $\gamma_s(J(G)) \leq \frac{p(p-1)}{4}$. From the above equations, we get

$$\begin{aligned}\gamma_s(G) + \gamma_s(J(G)) &\leq \frac{p}{2} + \frac{p(p-1)}{4} \\ &\leq \frac{p(p+1)}{4} \\ &< \left(\frac{p+1}{2}\right)^2.\end{aligned}$$

□

Theorem 3.6. *For any connected (p,q) graph G with diameter, $diam(G) \geq 2$, $\gamma_s(J(G)) \geq 2$.*

Proof. Let uv be a path of maximum distance in G. Then $d(u,v) = diam(G)$. We can prove the theorem with the following cases.

Case 1: For $diam(G) = 2$, choose a vertex v_1 of eccentricity 2 with maximum degree among others. Let $V_1 = \{v_1', v_2' \ldots\}$ corresponding to the elements of $\{e_1, e_2 \ldots\}$ forming a set dominating set in jump graph $J(G)$. Every vertex $u \notin V_1$ Hence V_1 is a minimum set dominating set. So set domination number of the jump graph will be equal to the degree of v_1, also by known remarks, we say $\gamma_s(J(G)) > 2$.

Case 2: For $diam(G) > 2$, let e_1 be any edge adjacent to u and e_2 be any edge adjacent to v. Let $\{e_1, e_2\} \subseteq E(G)$ form a corresponding vertex set $\{v_1', v_2'\} \subseteq V(J(G))$. These two vertices form a set dominating set in jump graph. Since these vertices $\{v_1', v_2'\}$ are adjacent to all other vertices of $V(J(G))$, it follows that $\{v_1, v_2\}$ becomes a minimum set dominating set. Hence $\gamma_s(J(G)) = 2$. In view of above cases, we can conclude that for any connected graph G, $\gamma_s(J(G)) \geq 2$. □

Theorem 3.7. *For any connected (p,q) graph G with circumference, $circum(G) \geq 4$, $\gamma_s(J(G)) \geq 2$.*

Proof. Let $u - v - u$ be a longest cycle in G. Then $C(G) = Circum(G)$. We can prove the theorem with the following cases.

Case 1: For $Circum(G) = 4$, choose a vertex v_1 that have longest cycle among others. Let $V_1 = \{v_1', v_2', \ldots\}$ corresponding to the elements of $\{e_1, e_2, \ldots\}$ forming a set dominating set in jump graph $J(G)$. Every vertex $u \notin V_1$ is adjacent to a vertex in V_1. Hence V_1 is a minimum set dominating set. So the set domination number of the jump graph will be equal to the longest cycle. we say $\gamma_s(J(G)) > 2$.

Case 2: For $Cirum(G) > 4$. Let e_1 be any edge adjacent to u and e_2 be any edge adjacent to v. Let $\{e_1, e_2\} \subseteq E(G)$ form a corresponding vertex set $\{v_1', v_2'\} \subseteq V(J(G))$. These two vertices form an set dominating set in jump graph. Since these vertices $\{v_1', v_2'\}$ are adjacent to all other vertices of $V(J(G))$, it follows that $\{v_1, v_2\}$ becomes a minimum set dominating set. Hence $\gamma_s(J(G)) = 2$.

In view of above cases, we can conclude that for any connected graph G, $\gamma_s(J(G)) \geq 2$. □

Theorem 3.8. *For any tree T with diameter greater than 3, $\gamma_s(J(T)) = 2$.*

Proof. If the diameter is less than or equal to 3, then the jump graph will be disconnected. Let uv be a path of maximum length in a tree T where diameter is greater than 3. Let e_i be the pendant edge adjacent to u and e_k be the pendant edge adjacent to v. The vertex set $\{v_1', v_2'\}$ of $J(T)$ corresponding to the edges of $\{e_i, e_k\}$ in T will form the set dominating set in $J(T)$. Since all the other vertices of $V(J(T))$ are adjacent with $\{v_1', v_2'\}$ it forms a minimum set dominating set. Hence $\gamma_s(J(T)) = 2$. □

Theorem 3.9. *For any connected (p, q) graph G, $\gamma_s(J(G)) \leq q - \Delta(G)$ where $\Delta(G)$ is the maximum degree of G.*

Proof. Let $V = \{v_1, v_2, v_3, ...v_n\}$ be the set of vertices in G and let $V_1 = V - v_1$ where v_1 is one of the vertices with maximum degree. By definition of jump graph, $E(G) = V(J(G))$. Consider $I = \{e_1, e_2, e_3 \dots e_k\}$ as the set of edges adjacent to v_1 in G. Let $H \subseteq V(J(G))$ be the set of vertices of $J(G)$. Such that, $H \subseteq E - I$. Then H itself forms a minimally set dominating set. Therefore $\gamma_s(J(G)) \leq |E| - |I|$. Hence $\gamma_s(J(G)) \leq q - \Delta(G)$. □

Theorem 3.10. *For any connected (p, q) graph G, $2 \leq \gamma_s(J(G)) \leq \left\lfloor \frac{q}{2} \right\rfloor$.*

Proof. An edge $\{e_i\}$ in any connected graph G is adjacent to at least one more edge in G. In Jump graph, the vertex $\{v_1'\}$ corresponding to $\{e_i\}$ is non-adjacent to $\{v_i^k, v_i^j\}$ of $\{e_k, e_i\}$ in $J(G)$. Therefore by definition of set domination number of graph $\gamma_s(G)$, the set dominating set contains at least two elements. Hence

$$\gamma_s(J(G)) \geq 2. \tag{1}$$

Let E be the set of edges in G. Then $E = V(J(G))$. Suppose $D = \{v_1, v_2, v_3, \dots v_n\}$ be the set dominating set. Then $V - D$ is also a set dominating set. One among these two sets will form a minimal set dominating set. So by the definition of set domination number of graph, we can say set domination number, $\gamma_s(J(G))$ of jump graph is given by

$$\begin{aligned} \gamma_s(J(G)) &\leq min\{|D|, |V - D|\} \\ &\leq \left\lfloor \frac{q}{2} \right\rfloor \end{aligned} \tag{2}$$

From (1) and (2), we get

$$2 \leq \gamma_s(J(G)) \leq \left\lfloor \frac{q}{2} \right\rfloor$$

□

Theorem 3.11. *For any connected (p, q) graph G without pendent vertex, $\gamma_s(J(G))) \leq \delta(G)$.*

Proof. Let $V = \{v_1, v_2, v_3, \dots v_n\}$ be the set of vertices in G and v_1 be one among the vertices with minimum degree. Let $I = \{e_1, e_2, e_3, \dots e_k\}$ be the set of edges adjacent to v_1 in G. Then $E_1 \subseteq V(J(G))$ will form the set dominating set in $J(G)$. So $|E_1| = \delta(G)$. Obviously it becomes the minimum set dominating set. Therefore $\gamma_s(J(G)) \leq \delta(G)$. □

4. Conclusion

We showed that the exact set domination number for standard graph like path, cycle, complete graph, complete bipartite graph, wheel etc. Also we showed that the graph theoretic properties and relation between the other parameters is also investigated.

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