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Isolated Domination Polynomial of a Graph

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Abstract: A dominating set S of a graph G is called an isolated dominating set, if $\langle S\rangle$ contains at least one isolated vertex. The minimum cardinality of an isolated dominating set is called the isolated domination number, symbolized by $\gamma_o(G)$. In this paper, we initiate the study of a graph polynomial called isolated domination polynomial. We determine the isolated domination polynomial of some standard graphs and characterize some class of graphs through the isolated domination polynomial. Finally, we determine the isolated domination polynomial at some particular values.

MSC: 0569C.

Keywords: Dominating set, Isolated dominating set, Domination polynomial, isolated domination polynomial of a graph.

1. Introduction

The notion of isolated domination [4] is an interesting concept introduced by I. Sahul Hamid in 2015. Later, the concept was studied extensively and many research articles were published on that concept. Further, the concept of graph polynomial was introduced by Saeid alikhani [3] in 2009. In this article, we define and study isolated domination polynomial, which actually gives an algebraic connection for the parameter $\gamma_o(G)$. Prior to the definition, we assume that all graphs we considered here are undirected, simple and finite. Let $G=(V,E)$ be any graph of order n and size m. Degree of a vertex represents number of edges incident to that vertex. If there are no edges incident to a vertex, then it is called an isolated vertex. If only one edge is incident to a vertex then it is called a pendant vertex. The joining of graphs (G_1) and (G_2) denoted by $(G)_1\vee G_2$ is the graph such that $V(G_1)\vee V(G_2)=V(G_1)\cup(G_2)$ and$E(G)_1\vee V(G)_2)=E(G_1)\cup E(G_2)\cup uv: u\in V(G_1), v\in V(G_2)$. The star graph $K_{1,n}$ is a graph of order $n+1$ obtained by joining two graphs K_1 and $\overline{K}_n$. The bi-star graph $G=B(m,n)$ is obtained by considering a path P_2 and attaching m vertices to one vertex of P_2 and n vertices to another vertex of P_2. A friendship graph F_n is a planar undirected graph with $2n+1$ vertices and $3n$ edges.F_n can be constructed by joining n number of cycle graph $C_3$with a common vertex.

A subset S of $V(G)$ is called a dominating set, if each vertex in $V{-}S$ is adjacent to a vertex in S. The least cardinality of a dominating set is called the domination number of G and is usually symbolized by $\gamma(G)$. A dominating set S in G is called an isolated dominating set if $\langle S\rangle$ contains at least one isolated vertex. The minimum cardinality of an isolate dominating set is called the isolated domination number, symbolized by $\gamma_o(G)$. The domination polynomial of a graph G of order n is the polynomial $D(G,x)=\sum\limits_{i=\gamma(G)}^{n} d(G,i)x^i$ where $d(G,i)$ is the number of dominating sets of G of size i and $\gamma(G)$ denotes the domination number of G.

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2. The Isolated Domination Polynomial of a Graph

Definition 2.1. *The isolated domination polynomial of G is denoted by $D_o(G,x)$ and is defined as $D_o(G,x) = \sum_{i=\gamma_o(G)}^{n} d_o(G,i)x^i$ where $d_o(G,i)$ is the number of isolated dominating sets of size i. The isolated domination polynomial is defined for all non-trivial connected graphs.*

For any graph of order n, we have $D_o(G,x)$ a polynomial of degree n and hence by the consequence of fundamental theorem of algebra, it must posses n number of roots, counting their multiplicities. We shall denote the set of all roots of isolated domination polynomial of G by $Z_o(G,x)$.

Example 2.2. *Let $G \cong P_5$ be a path of order 5. Clearly, $\gamma_o(G) = 2$ and there are 3 isolated dominating sets of size 2, so $d_o(G,2) = 3$. Similarly there are 7 isolated dominating sets of size 3, and 2 isolated dominating sets of size 4. i.e, $d_o(G,3) = 7$ and $d_o(G,4) = 2$. Therefore, the isolated domination polynomial of P_5 is given by $D_o(P_5,x) = 2x^4+7x^3+3x^2 = x^2(2x+1)(x+3)$. The roots of the polynomial are $x = 0, -\frac{1}{2}$ and -3. That is $Z_o(P_5,x) = \{0, -\frac{1}{2}, -3\}$.*

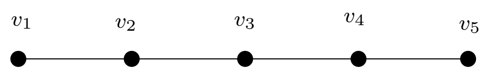

Figure 1: Path of order 5

From the definition of the isolated domination polynomial, we have following observations:

Observations 2.3.

(1). Let G be any connected graph of order n with $\Delta(G) = n-1$, then $\gamma(G) = \gamma_o(G) = 1$. Therefore, first term of the polynomial will have co-efficient 1.

(2). Let G be any connected graph, Then $d_o(G,n) = \begin{cases} 1, & \delta(G) = 0 \ ; \\ 0, & \textit{otherwise.} \end{cases}$

(3). $D_o(G,x)$ is monotonically increasing function.

(4). $D_o(G,x)$ has no constant term. On the other hand, $0 \in Z_o(G,x)$ always.

(5). Zero is a root of $D_o(G,x)$ of multiplicity $\gamma_o(G)$.

(6). $d_o(G,x) = 0$ if and only if $i < \gamma_o(G)$ or $i > n$.

(7). Let G be any connected graph of order n, then $D_o(\overline{K}_n) = x^n$.

(8). Let G be a complete graph with n vertices. Then $D_o(G,x) = n$.

3. Main Results

Theorem 3.1. *Let G be any graph of order $n \geq 2$, then $D_o(G,x) = nx$ if and only if $G \cong K_n$.*

Proof. Let G be any graph of order $n \geq 2$ such that $D_o(G,x) = nx$. That is $d_o(G,1) = n$ and so $\gamma_o(G) = 1$. Hence G contains a vertex say v which is adjacent to all vertices in G. Since G contains n dominating sets of cardinality one, it is clear that every vertex in G leads to a dominating set in G. That is every vertex is adjacent to each vertex in G and hence G is complete. Converse is obvious. $\square$

Theorem 3.2. *Let G be any connected graph of order $n+1$. Then $D(G,x) = x + x^n$ if and only if $G \cong K_{1,n}$.*

Proof. Let G be any connected graph of order $n+1$ such that $D(G,x) = x + x^n$. From the definition of the domination polynomial it is clear that G contains exactly one dominating set of size 1 and n respectively. Further it contains a vertex v adjacent to remaining vertices in G. Let $S = v$ and let u and w be any two vertices in $V - S$. From the data it is clear that $V - S$ itself an isolated dominating set of G. For if, u and w are adjacent, then the set obtained by removing w from $V - S$ leads to an isolated dominating set, which is a contradiction. This contradiction establishes the result and Converse is obvious. □

Theorem 3.3. *Let G be a connected graph of order $m+n+2$ with $m, n \geq 2$. Then $D_o(G,x) = x^{m+1} + x^{n+1} + x^{m+n}$ If and only if $G \cong B(m,n)$.*

Proof. Let G be any connected graph order $m+n+2$ with $m, n \geq 2$. We shall call vertex set of G as $V(G) = v_1, v_2, \ldots v_n, v_{n+1}, u_1, u_2, \ldots, u_m, u_{m+1}$. Assume that $D_o(G,x) = x^{m+1} + x^{n+1} + x^{m+n}$. Then G contains exactly one dominating set of size $m+1, n+1$ and $m+n$ respectively. Let $V_m = u_1, u_2, \ldots, u_{m+1}$ and $V_n = v_1, v_2, \ldots v_{n+1}$ be a partition of $V(G)$. Since G is connected there exists at least one vertex say u_{m+1} in V_m adjacent to a vertex v_{n+1} in V_n. On contrary, assume there exists one more pair (u,v) of adjacent vertices from V_m and V_n, then $V(G) - u$ leads to an isolated dominating set of size $m+n+1$, which is not possible. Thus, there is exactly one vertex in V_m adjacent to a vertex inV_n. Next, let (u,v) be any pair of vertices in V_m, which are adjacent to each other. Then again, $S = V_m - u \cup V_n$will be an isolated dominating set of cardinality $m+n+1$, which is not possible. Hence, none of the vertices from V_m are adjacent to each other. This establishes the result. Converse is obvious. □

Lemma 3.4. *Let G_1 and G_2 be any two graphs, then $\gamma_o(G_1 \vee G_2) = \min\{\gamma_o(G_1), \gamma_o(G_2)\}$.*

Proof. Consider two graphs G_1 and G_2 of order n_1, n_2 respectively. Let $G \cong G_1 \vee G_2$, then G will be a graph order $n_1 + n_2$. It is obvious that $\gamma_o(G) \leq \gamma_o(G_1) + \gamma_o(G_2)$. On the other hand, Let S be any isolated dominating set of say G_1, then S dominates both G_1 and G_2 and induced subgraph contains isolated vertex. Hence, $\gamma_o(G_1 \vee G_2) = \min\{\gamma_o(G_1), \gamma_o(G_2)\}$, proving the result. □

Theorem 3.5. *Let G_1 and G_2 be any two connected graphs of order at least 2. Then $D_o(G_1 \vee G_2, x) = D_o(G_1, x) + D_o(G_2, x)$.*

Proof. Let G_1 and G_2 are any two graphs of order at least two. Let $G = G_1 \vee G_2$. Since every pair of vertices from $V(G_1)$ and $V(G_2)$ are adjacent, any isolated dominating set of G_1 or G_2 will be an isolated dominating set of G. Further, let S be any isolated dominating set of G, then S dominates both G_1 and G_2. Further, vertices in S are contained in $V(G_1)$ or $V(G_2)$ and the induced subgraph contains an isolated vertex. Therefore, S is either an isolated dominating set of G_1 or G_2, proving the result. □

Corollary 3.6. *Let $G \cong K_{m,n}$ be a complete bi-partite graph of order $m+n$. Then $D_o(G,x) = x^m + x^n$.*

Proof. Let $G \cong K_{m,n}$ be a complete bipartite graph of order $m+n$, then we have $G = \overline{K}_m \vee \overline{K}_n$. From the above theorem, we have $D_o(K_{m,n}) = D_o(\overline{K}_m) + D_o(\overline{K}_n)$. Further, from observation 7, it is evident that $D_o(\overline{K}_m) = x^m$ and $D_o(\overline{K}_n) = x^n$, proving the result.

Corollary 3.7. *Let $G \cong K_{n_1, n_2, \ldots, n_k}$ be a complete multipartite graph of order N, then $D_o(G,x) = \sum\limits_{i=1}^{k} x^i$.*

Proof. Let $G \cong K_{n_1, n_2, \ldots, n_k}$ be a multipartite graph, then we have G will be isomorphic to join of $\overline{K}_{n_i}$ as i varies. Therefore, from the above corollary, we have $D_o(G,x) = \sum\limits_{i=1}^{k} x^i$. □

Corollary 3.8. *Let $G \cong F_n$ be a friendship graph. Then $\gamma_o(G) = 1$.*

Theorem 3.9. *Let G be a friendship graph of order $n = 2m+1$. Then $D_o(G,x) = x + 2^m x^m + m(2^{m-1})x^{m+1} + m(2^{m-2})x^{m+2} + m(2^{m-3})x^{m+3} + ... + 2mx^{2m-1}$.*

Proof. Let G be a friendship graph of order $n = 2m+1$ and let $V(G) = v_1, v_2, \ldots, v_n$. Clearly, $d_o(G,1) = 1$, $d_o(G,n) = 0$ and any other isolated dominating set is obtained by choosing vertices of degree 2. Since G contains m cycles and there are exactly two possibility ways to select a vertex from each cycles, it follows that $d_o(G,m) = 2^m$. Next, choose an integer k, $1 \leq k \leq m-1$. An isolated dominating set of size $m+k$ is obtained by fixing a k-pair of vertices from the cycles and selecting $m-2k$ vertices from remaining cycles. Again, the selection of pair of edges can be repeated by considering one cycle in each step and there are exactly two possibilities to choose vertices from remaining cycle. Hence, $d_o(G, m+k) = m2^{m+k}$. Therefore, we get $D_o(G,x) = x + 2^m x^m + m(2^{m-1})x^{m+1} + m(2^{m-2})x^{m+2} + m(2^{m-3})x^{m+3} + ... + 2mx^{2m-1}$. □

The value of any graph's polynomial sometime giving some results in the applications of graph theory, in the following we calculate $D_o(G,-1)$ for some standard graph. □

Observations 3.10.

(1). Let G be a complete graph with n vertices, then $D_o(G,-1) = -n$.

(2). Let G be a totally disconnected graph of order n, then $D_o(G,-1) = (-1)^n$.

(3). Let G be a star on n vertices, then $D_o(G,-1) = -1 + (-1)^n$.

(4). Let G be a star graph of order $m+n+2$, Then $D_o(G,-1) = \begin{cases} 3, & \textit{if both m,n odd ;} \\ -1, & \textit{otherwise.} \end{cases}$

When we consider standard graphs to compute isolated domination polynomial, cycle graphs and path graphs are very crucial in the sense that no direct polynomial can be obtained easily. We must use recurrence relation that connects terms of polynomials of path or cycle graphs of different order. In this section, we shall try to have some observations on isolated domination polynomial and co-efficients of isolated domination polynomials of path and cycle graphs.

Observations 3.11.

(1). Let P_n be a path of order $n = 3k$, then $d_o(P_n, k) = 1$.

(2). Let P_n be a path of order $n = 3k-1$, then $d_o(G, \gamma_o) = k+1$.

(3). Let C_n be a cycle of order $n = 3k$, then $d_o(C_n, k) = 3$.

(4). Let C_n be a cycle of order $n \geq 3$, then $d_o(C_n, n-2) = n$.

(5). Let G be a path or cycle of order n, then $d_o(G,n) = 0$ and $d_o(G, n-1) = n$.

4. Conclusion

The concept of domination was well studied concept in domination and the isolated domination is one of the fundamental domination parameter that was introduced recently. In this article, the isolated domination polynomial was introduced and polynomial was determined for some standard graphs. Also, we have determined the polynomial at some particular point. Finally some observations on co-efficient of isolated domination polynomial of cycle and path graphs are obtained.

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