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Modeling Bridge Oscillations Induced by Marching Soldiers

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Abstract: This work simulates the effects of soldiers marching on a bridge, by first using Newton's second law to write a differential equation describing the bridge's motion, and then solving the equation to yield a simple harmonic oscillator. Each of the soldiers' steps is modeled as a constant impulse on the bridge. Computer simulations model the effects of between 5 and 200 soldiers marching both in and out of unison. We find that in both cases, the maximum amplitude of the oscillations of the bridge is proportional to the time the soldiers spent marching on the bridge. In the case where the soldiers march in unison, there is a linear relationship between the maximum amplitude of the oscillations and the number of marching soldiers. However, when the soldiers do not march in unison, the oscillations are proportional to the square root of the number of soldiers. Thus, it is preferable for soldiers to march out of unison on a bridge, since marching in unison could lead to a rapid increase in the amplitude of oscillations and the collapse of the bridge.

Keywords: Resonance, oscillation, bridge.

1. Introduction

The effects of resonant behavior on bridges are well-documented. One of the most famous examples is the Tacoma Narrows Bridge, which collapsed in 1940 due to periodic forces from wind and other aerodynamic effects [2]. This caused the bridge to dangerously swing back and forth at its resonance frequency, eventually causing the bridge to cave-in. Of course, resonance can be achieved in multiple ways: one notorious example is soldiers marching on a bridge in unison. In 1831, a group of soldiers marched in unison across England's Broughton Suspension Bridge, causing it to collapse [5]. This explains why soldiers are instructed to break stride when crossing a bridge to avoid risking such a collapse.

Scientists have established the basic principles underlying soldiers marching in unison, causing a bridge to collapse. In this work, we will investigate the physics underlying an infamous example of such resonance. We will treat the bridge as a simple harmonic oscillator, since it is reasonable to assume that the bridge is well-supported, and thus the size of the oscillations is small [3]. This simple harmonic oscillator is only allowed to move in the vertical direction, as shown in Figure 1.

To qualitatively describe the motion of a simple harmonic oscillator, let us consider the more familiar system of a single pendulum, such as a toddler on a swing in a park. In the presence of only gravity, if the pendulum was initially in motion, it will oscillate at a certain frequency that depends only on properties of the pendulum, such as its mass and length. This is known as its natural frequency, and it is this property of the system that we are most interested in. Of course, in the real world, the amplitude of the oscillations will decrease with time and the pendulum will eventually stop oscillating. This

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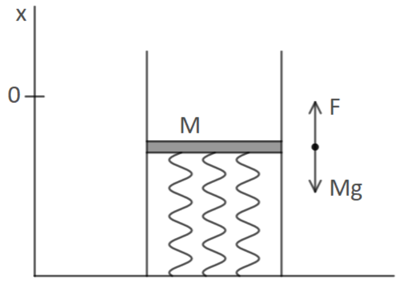

Figure 1. **A spring model (simple harmonic oscillator) of the bridge**

decrease in amplitude is caused by damping forces, such as friction and air resistance, which is why the toddler needs to constantly be pushed in order to keep swinging. The bridge behaves in a similar manner, but its oscillations are in the vertical direction, and the force needed to set it in motion is much greater than the force needed to swing a child. Nevertheless, this serves as an useful example to visualize the physics at play.

We can further extend this analogy by considering the case of a forced oscillator, such as a person pushing a swinging toddler. In that case, the person pushing the toddler provides the driving force. This applied force is periodic, and its frequency is the same as the natural frequency of the pendulum that consists of the child and swing. The amplitude of the oscillations of the pendulum increases because the force is applied at the natural frequency of the pendulum, a phenomenon known as resonance. Note that in reality, damping prevents the oscillations to continue increasing indefinitely.

The physics of the oscillations of a bridge caused by soldiers marching is analogous to the physics of the toddler being pushed on the swing. Just like a person pushing a toddler, soldiers marching on a bridge exert a periodic force that causes the bridge to oscillate. When this force is in resonance, the amplitude of the oscillations will increase over time. This increase is slowed down by damping, but sometimes, damping is not enough to prevent the bridge from collapsing.

The phenomena of soldiers marching on a bridge, causing the bridge to collapse, is the subject of study of this article, which is organized as follows. In Section A, we describe the mathematical model of a bridge without marching soldiers. In Section B, we describe the effect of an impulse on the dynamics of the platform. In Section C, we describe the mathematical model of a bridge when soldiers march on it in unison. In Section D, we model the soldiers stepping on the bridge at random times, where throughout this paper, we define marching at "random times" as the soldiers marching out of unison. Section E presents the statistical results obtained by applying the model in Section D, and we conclude with a small discussion in Section F.

2. Main Results

A. **Mathematical Model:** As mentioned in the previous section, suppose the bridge can be modeled as a Hookean spring. That is, suppose there is some equilibrium position where the bridge experiences no force, and the bridge experiences a force, F, linearly proportional to the displacement, x, from this equilibrium position given by $F = -kx$. If we take the convention that upward motion corresponds to positive changes in the x-coordinate of the bridge, as suggested by Figure 1, then the net force on the bridge is

$$F_{net} = Ma = -kx - Mg, \tag{1}$$

where M is the mass of the bridge, $a = x''$ is its acceleration (here, x'' denotes the second derivative with respect to time of x), k is the spring constant, and g is the gravitational constant. For convenience, let us make the substitution

$z = x + Mg/k$ and $\omega^2 = k/M$. Then equation (1) becomes

$$z'' + \omega^2 z = 0. \tag{2}$$

The solutions to this equation are precisely the equations that govern the motion of a simple harmonic oscillator. They are well-known [1] and are in the form

$$z = A\sin(\omega t + \phi). \tag{3}$$

B. **Modeling Soldier Steps:** Now, we will consider what happens to this simple harmonic oscillator when we subject it to a force F for a short period of time Δt. The impulse delivered on the object by the force is defined as

$$J = F\Delta t. \tag{4}$$

Also, the momentum of an object, denoted p, is defined as $p = mv$. Newton's second law tells us that the change in momentum is equal to the impulse. In other words,

$$p_2 - p_1 = J. \tag{5}$$

Now, suppose we treat each soldier step on the bridge as a single, constant impulse $-J$ onto the platform at time $t = t_1$. This impulse will cause a change in momentum, and thus in velocity. This translates to having a change in the constants A and ϕ. More precisely, if $A = A_1$ and $\phi = \phi_1$ before the impulse, and $A = A_2$ and $\phi = \phi_2$ after the impulse, we have

$$z = \begin{cases} A_1\sin(\omega t + \phi_1) & t < t_1 \\ A_2\sin(\omega t + \phi_2) & t \geq t_1. \end{cases}$$

Also, we know that the momentum of the bridge is equal to the mass multiplied by the time derivative of z: $p = Mz' = M\omega\cos(\omega t + \phi)$. Now, the task is to solve for A_2 and ϕ_2. Using the fact that both the position z is continuous and the change in momentum equals $-J$, we obtain the two equations

$$A_1\sin(\omega t_1 + \phi_1) = A_2\sin(\omega t_1 + \phi_2) \tag{6}$$

$$M\omega A_1\cos(\omega t_1 + \phi_1) - J = M\omega A_2\cos(\omega t_1 + \phi_2). \tag{7}$$

For convenience, let us define $I = J/M\omega$. Dividing equation (7) by $M\omega$ gives the two equations

$$A_1\sin(\omega t_1 + \phi_1) = A_2\sin(\omega t_1 + \phi_2) \tag{8}$$

$$A_1\cos(\omega t_1 + \phi_1) - I = A_2\cos(\omega t_1 + \phi_2). \tag{9}$$

We can solve for A_2 by squaring both equations, adding them, and taking the square root:

$$A_2 = \sqrt{A_1^2 - 2A_1 I\cos(\omega t_1 + \phi_1) + I^2}. \tag{10}$$

To solve for ϕ_2, define $\theta = \arcsin(A_1\sin(\omega t_1 + \phi_1)/A_2)$. Notice that by definition of arcsin, we have $-\pi/2 \leq \theta \leq \pi/2$. Thus, if $A_1\cos(\omega t_1 + \phi_1) - I > 0$, then $\cos(\omega t_1 + \phi_2) \geq 0$, so we see that $\omega t_1 + \phi_2 = \theta$. On the other hand, if $\cos(\omega t_1 + \phi_2) < 0$, then $\pi/2 < \omega t_1 + \phi_2 < 3\pi/2$, so $\omega t_1 + \phi_2 = \pi - \theta$. Therefore, we have the following expression for ϕ_2:

$$\phi_2 = \begin{cases} \arcsin(A_1\sin(\omega t_1 + \phi_1)/A_2) - \omega t_1 & A_1\cos(\omega t_1 + \phi_1) - I > 0 \\ \pi - \arcsin(A_1\sin(\omega t_1 + \phi_1)/A_2) - \omega t_1 & A_1\cos(\omega t_1 + \phi_1) - I < 0. \end{cases}$$

C. **Multiple Soldiers Marching at the Same Time:** We will now extend these results to the case of multiple soldiers marching at the same time. If we take equation (8) and multiply it by i, and then we add it to equation (9), we obtain

$$A_1 e^{i(\omega t_1 + \phi_1)} - I = A_2 e^{i(\omega t_1 + \phi_2)} \tag{11}$$

where we use the fact that $\cos(x) + i\sin(x) = e^{ix}$. Equivalently, multiplying equation (11) by $e^{-i\omega t_1}$ yields

$$A_1 e^{i\phi_1} - I e^{-i\omega t_1} = A_2 e^{i\phi_2}. \tag{12}$$

Now, suppose all the soldiers march at the same time. Let T be a positive number, N be the number of soldiers, and let the nth soldier take a step at time $t_n = nT$. Note that

$$\begin{aligned} A_N e^{i\phi_N} &= (A_N e^{i\phi_N} - A_{N-1} e^{i\phi_{N-1}}) + (A_{N-1} e^{i\phi_{N-1}} - A_{N-2} e^{i\phi_{N-2}}) + \cdots + (A_1 e^{i\phi_1} - A_0 e^{i\phi_0}) \\ &= -I e^{i\omega(N-1)T} - I e^{i\omega(N-2)T} - \cdots - I. \end{aligned}$$

If $\omega T = 2k\pi$, where k is some positive integer, then we can see that $A_N e^{i\phi_N} = NI$, and this case is known as **resonance**. Otherwise, by the geometric series formula, we have

$$A_N e^{i\phi_N} = \frac{1 - e^{-iN\omega T}}{1 - e^{i\omega T}}. \tag{13}$$

We modeled the effect of multiple soldiers marching on the bridge at the same time, both in resonance and when there is no resonance. In each case, we took the original impulse, J, delivered by each soldier, and multiplied it by the number of soldiers to determine the net impulse delivered. In Figure 2, we see the resonance case–the amplitude of the bridge oscillations grows linearly with time.

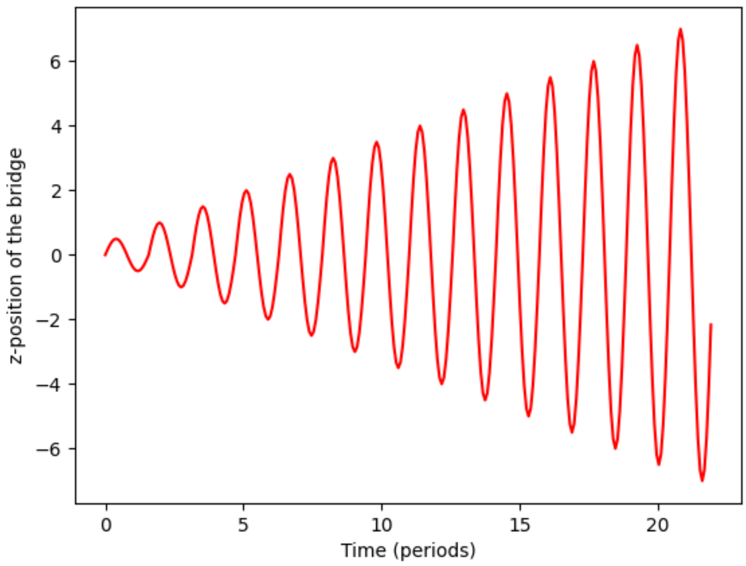

Figure 2. **A plot of resonant bridge oscillations with $N = 100$ soldiers propagated out to 20 periods**

However, as Figure 3, shows, the maximum amplitude of oscillations also grows linearly with the number of soldiers.

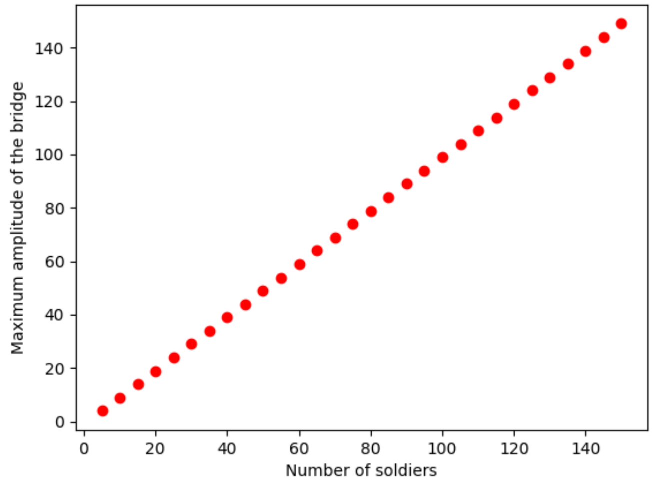

Figure 3. A plot of the maximum amplitude of the bridge oscillation as a function of the number of soldiers when $\omega = 4$ and $T = \pi/2$

When the soldiers are *not* marching in resonance with the bridge, then the growth of the amplitude of the oscillations is bounded, as seen in Figure 4. This is due to the following observation:

Observation: If there exist relatively prime positive integers m and k such that $\omega mT = 2k\pi$, then $z(t)$ is periodic with period mT; that is, $z(t + mT) = z(t)$ for all t.

In the example in Figure 5, the numbers $m = 5$ and $k = 6$ satisfy the conditions of the observation, since $\omega mT = 2\pi \cdot 5 \cdot 1.2 = 12\pi = 2\pi \cdot 6 = 2k\pi$.

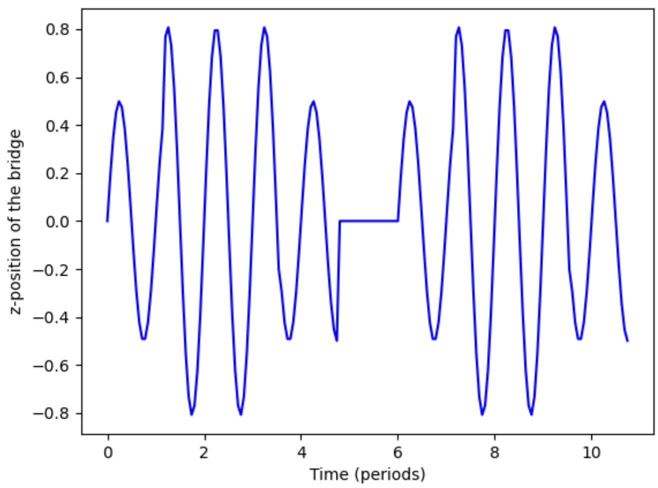

Figure 4. A plot of the z-position of the bridge under non-resonant conditions when $\omega = 2\pi$ and $T = 1.2$

D. **Multiple Soldiers Marching at Random Times:** Next, we will more explicitly consider the case where the soldiers do not march in unison. Again, let A equal the maximum amplitude of the bridge divided by the total number of periods $T = 2\pi/\omega$ that have elapsed, and let N equal the number of soldiers that are marching. In this case, each soldier's first step occurs at a random time between $t = 0$ and $t = T$, and soldiers take steps every $T = 2\pi/\omega$ seconds.

To analyze what happens when the soldiers do not match in unison, we varied the number of soldiers from $N = 5$ to $N = 200$ soldiers in increments of 5. For each soldier count, we increased the number of trials by 10 until successive runs gave results within 10% of each other.

Number of Soldiers	Maximum Amplitude	Number of Trials	Number of Periods
5	4.88009	10	10
5	4.96517	10	20
5	6.40421	20	10
5	5.97257	20	20
5	5.97362	30	10
5	6.27691	30	20

Table 1. **Arriving at the final maximum amplitude of 6.27691 for $N = 5$ soldiers**

Again, for each fixed number of trials, we increased the number of periods the soldiers marched for by 10 until successive runs gave results within 10% of each other. For each trial, the maximum amplitude was found by simply taking the maximum value obtained by z. A few sample values are shown in Table 1.

E. **Best-fit Model:** After applying the methodology detailed in the previous section, the results for the maximum amplitude over number of periods versus the number of soldiers marched is given in Figure 5.

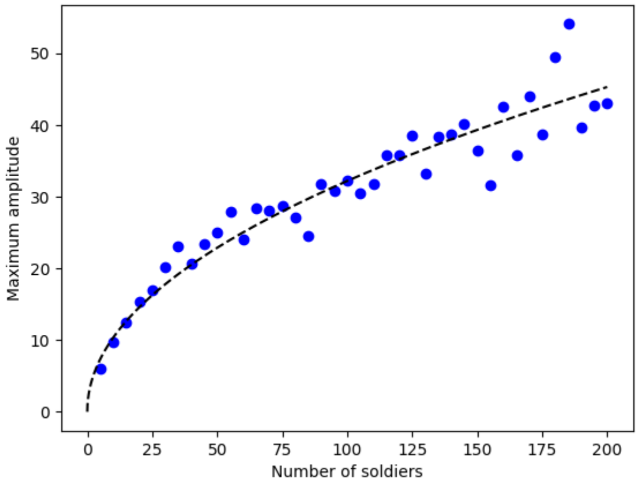

Figure 5. A plot of the maximum amplitude data when soldiers march randomly (blue dots) and the curve of best fit (black dashed line)

After applying log-log regression to the data [4], we obtained the following coefficients to the equation of best fit $\log(A) \sim b \log(aN)$:

$$a = 1.20 \pm 0.08$$

$$b = 0.49 \pm 0.02.$$

The value of b is the most pertinent here, since it gives us the asymptotic behavior of the amplitude A as a function of the number of soldiers N. We see that 0.5 is well within the margin of error for the b value, so it is natural to suggest that the maximum amplitude of the bridge is proportional to the square root of the number of soldiers when they march randomly. As seen in Figure 6, this is much more desirable than the linear behavior of the amplitude that is produced when the soldiers march in stride, thus explaining why soldiers are instructed to march out of stride when crossing a bridge.

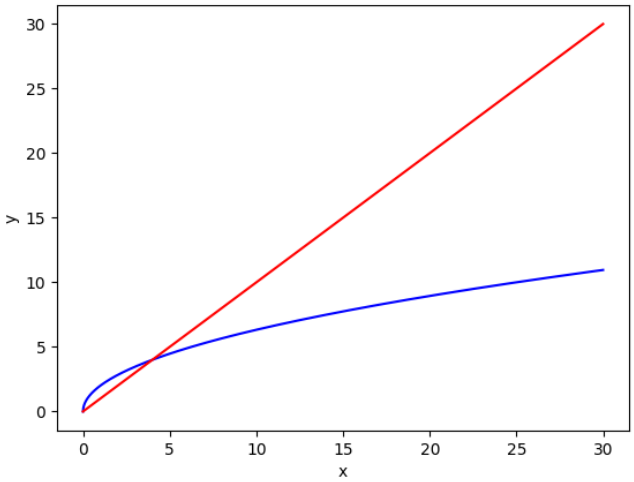

Figure 6. **Comparison of linear (red curve) versus square root (blue curve) behavior**

3. Conclusion

The potential for resonance to cause large-scale oscillatory effects is an important one to consider for all engineers, as seen by historical examples. More importantly, this work provides statistical evidence for the effect of soldiers marching in unison on the oscillations of bridges, showing that marching in unison can directly cause their collapse. However, we assumed the bridge could be modeled as a simple harmonic oscillators because the magnitude of oscillations was taken to be relatively small. Thus, future work is needed to determine if the resonant behavior detailed in this work is observed in non-linear systems.

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