Int. J. Math. And Appl., 9(1)(2021), 27–33
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

On Permutation Labeling of Graphs

Mitesh J. Patel[1,*] and G. V. Ghodasara[2]

1 Department of Mathematics, Tolani College of Arts abd Science, Adipur-Kachchh, Gujarat, India.

2 Department of Mathematics, H. & H. B. Kotak Institute of Science, Rajkot, Gujarat,India.

Abstract: An injective function $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ is said to be permutation labeling if each edge uv is assigned with label ${}^{f(u)}P_{f(v)} = \frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$ are all distinct. A graph which admits permutation labeling is called permutation graph. In this paper we prove that wheel graph, restricted square and degree splitting graph of bistar graph are permutation graphs. We also proved that arbitrary super subdivision of path graph, star graph and cycle graph are permutation graphs.

MSC: 05C78.

Keywords: Permutation labeling, Wheel graph, Bistar graph, Arbitrary super subdivision.

1. Introduction

A labeling of a graph $G = (V, E)$ is a mapping that carries vertices, edges or both to the set of labels (usually to the positive or non-negative integers). For a summary on various graph labeling techniques one can go through *A Dynamic Survey of Graph Labeling* published by Joseph A. Gallian [6] in The Electronic Journal of Combinatorics.

A permutation labeling was defined by Hegde and Shetty [12].

Definition 1.1 ([12]). *An injective function $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ is said to be permutation labeling if the induced edge function $f^* : E(G) \to \mathbb{N}$ defined as $f^*(uv) = {}^{f(u)}P_{f(v)} = \frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$ is injective.*

A graph which admits permutation labeling is called permutation graph.

Baskar Babujee and Vishnupriya [7] derived that star graph $K_{1,n}$, path P_n, cycle C_n and complete binary tree with at least 3 vertices are permutation graphs. Seoud and Salim [9] obtained all permutation graph of order at most 9, they proved that every bipartite graph of order at most 50 is permutation graph. Shiama [8] derived some permutation graph related to shadow and splitting graph. Sonchhatra and Ghodasara [11] proved that $P_2 + \overline{K_n}$, book graph, cycle with one chord, cycle with twin chords, tadpole and lotus inside a circle are permutation graphs. Hegde and Shetty [12] proved that complete graph K_n is permutation graph if and only if $n \leq 5$, they also verified that all trees with at most fifteen vertices are permutation graphs and they strongly believed that all trees are permutation graphs. In [2], We proved that all trees admit permutation labeling. We also proved that complete bipartite graph $K_{3,n} (n \geq 3)$ for $n + 3$ prime, wheel graph $W_n (n \geq 3)$ for $n + 1$ is prime, dumbbell graph $D_{n,k,2} (n, k \geq 3)$, crown graph $C_n \odot K_1 (n \geq 3)$, one point union $C_n^{(k)} (k \geq 2, n \geq 3)$ of k

* *E-mail: miteshmaths1984@gmail.com*

copies of cycle C_n, middle graph of cycle $C_n(n \geq 3)$, t^*-ply $P_t^*(u, v)$, Petersen graph $P(5, 2)$ are permutation graphs.
In this paper we consider nonempty, simple, finite, undirected and connected graph. We refer to Bondy and Murty [5] for the standard terminology and notations related to graph theory and David M. Burton [1] for the terms related to number theory.

Definition 1.2 ([5]). *The wheel graph $W_n(n \geq 3)$ is the graph obtained by joining the graphs C_n and K_1. i.e. $W_n = C_n + K_1$. Here the vertices corresponding to C_n is called rim vertices and C_n is called rim of W_n while the vertex corresponding to K_1 is called apex vertex.*

Definition 1.3 ([6]). *The bistar $B_{n,n}$ is graph obtained by joining the apex vertices of two copies of star graph $K_{1,n}$ by an edge.*

Definition 1.4 ([6]). *The restricted square of $B_{n,n}$ is the graph G with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i | 1 \leq i \leq n\}$.*

Definition 1.5 ([6]). *Let $G = (V, E)$ be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \ldots S_t \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V - \bigcup_{i=1}^{t} S_i$.*
The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3 \ldots w_t$ and joining to each vertex of S_i, for $1 \leq i \leq t$.

Definition 1.6 ([6]). *The graph obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some positive integer m_i and $1 \leq i \leq q$ is called arbitrary super subdivision of G.*

1.1. Number theory results

We use the following number theory results for positive integers.

(1). If $n > r$ then ${}^{n(n+1)\ldots(n+r)}P_1 = {}^{(n+r)}P_{(n+1)}$.

(2). If $n > m$ and $r > k$ then ${}^nP_r > {}^mP_k$.

(3). If $n > k > r$ then ${}^nP_k > {}^kP_r$.

(4). If $n > r$ then ${}^nP_r < {}^{(n+1)}P_r < \ldots < {}^{(n+k)}P_r$.

(5). If $n > 4$ is even then ${}^{(\frac{n}{2}+2)}P_2 < {}^nP_2$.

2. Some New Permutation Graphs

The following are the results investigated in this paper.

Theorem 2.1. *Wheel $W_n(n \geq 3)$ is a permutation graph.*

Proof. Let $V(W_n) = \{u_0, u_1, u_2, \ldots, u_n\}$ and $E(W_n) = \{u_0u_i | 1 \leq i \leq n\} \cup \{u_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{u_1u_n\}$, where u_0 be apex and $u_1, u_2, \ldots, u_n$ be rim vertices of wheel graph W_n. Here $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.
We define a bijection $f : V(W_n) \to \{1, 2 \ldots, n + 1\}$ as follows.
Case 1: n is even.

$$f(u_0) = 1.$$

$$f(u_{2i-1}) = (i+1)\ ; 1 \le i \le \frac{n}{2}.$$
$$f(u_{2i}) = \frac{n}{2} + 1 + i\ ; 1 \le i \le \frac{n}{2}.$$

So from above defined function f, the following five possibilities for the produced edge labels can be considered.

(1). Labels in edge set $\{u_0u_{2i-1} | 1 \le i \le \frac{n}{2}\}$ are respectively $2, 3, \ldots, \frac{n}{2} + 1$.

(2). Labels in edge set $\{u_0u_{2i} | 1 \le i \le \frac{n}{2}\}$ are respectively $\frac{n}{2} + 2, \frac{n}{2} + 3, \ldots n + 1$.

(3). Label of the edge $\{u_2u_1\}$ is ${}^{(\frac{n}{2}+2)}P_2$.

(4). Label of the edge $\{u_nu_1\}$ is ${}^{n}P_2$.

(5). Labels in edge set $\{u_{2i}u_{2i+1}, u_{2i+2}u_{2i+1} | 1 \le i \le \frac{n}{2}\}$ are respectively ${}^{(\frac{n}{2}+2)}P_3, {}^{(\frac{n}{2}+3)}P_3, \ldots {}^{(n+1)}P_{(\frac{n}{2}+1)}$.

Using the number theory results described in Subsection 1.1, it is clear that edge labels from any of the above possibilities (1) to (5) are internally as well as externally in ascending order.

Case 2: n is odd.

$$f(u_0) = 1.$$
$$f(u_{2i-1}) = (i+1)\ ; 1 \le i \le \frac{n+1}{2}.$$
$$f(u_{2i}) = \frac{n+1}{2} + 1 + i\ ; 1 \le i \le \frac{n-1}{2}.$$

So from above defined function f, the following six possibilities for the produced edge labels can be considered.

(1). Labels in edge set $\{u_0u_{2i-1} | 1 \le i \le \frac{n+1}{2}\}$ are respectively $2, 3, \ldots, \frac{n+1}{2} + 1$.

(2). Labels in edge set $\{u_0u_{2i} | 1 \le i \le \frac{n-1}{2}\}$ are respectively $\frac{n+1}{2} + 2, \frac{n+1}{2} + 3, \ldots, n + 1$.

(3). Label of the edge $\{u_nu_1\}$ is ${}^{(\frac{n+1}{2}+1)}P_2$.

(4). Label of the edge $\{u_2u_1\}$ is ${}^{(\frac{n+1}{2}+2)}P_2$.

(5). Labels in edge set $\{u_{2i}u_{2i+1}, u_{2i+2}u_{2i+1} | 1 \le i \le \frac{n-3}{2}\}$ are respectively ${}^{(\frac{n+1}{2}+2)}P_3, {}^{(\frac{n+1}{2}+3)}P_3, \ldots, {}^{(n+1)}P_{(\frac{n+1}{2})}$.

(6). Label of the edge $\{u_{n-1}u_n\}$ is ${}^{(n+1)}P_{(\frac{n+1}{2}+1)}$.

Using the number theory results described in subsection 1.1, it is clear that edge labels of above possibilities (1) to (6) are internally as well as externally in ascending order.

So above defined function f in both the cases, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct. Hence wheel $W_n (n \ge 3)$ is a permutation graph. □

Corollary 2.2. *Gear $G_n (n \ge 3)$ is a permutation graph.*

Corollary 2.3. *Shell $S_n (n \ge 3)$ is a permutation graph.*

Theorem 2.4. *Restricted square of bistar $B_{n,n}$ is a permutation graph.*

Proof. Let G be the restricted square of bistar $B_{n,n}$ with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i | 1 \leq i \leq n\}$. Here $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. We define a bijection $f : V(G) \to \{1, 2, \ldots, 2n+2\}$ as follows.

$$\begin{aligned} f(u) &= 2n+2. \\ f(u_i) &= n+1+i\ ; 1 \leq i \leq\ n. \\ f(v) &= 1. \\ f(v_i) &= i+1; 1 \leq i \leq\ n. \end{aligned}$$

So from above defined function f, the following five possibilities for the produced edge labels can be considered.

(1). Labels in edge set $\{vv_i | 1 \leq i \leq\ n\ \}$ are respectively $2, 3, \ldots, n+1$.

(2). Labels in edge set $\{vu_i | 1 \leq i \leq\ n\ \}$ are respectively $n+2, n+3, \ldots 2n+1$.

(3). Label of the edge $\{uv\}$ is $2n+2$.

(4). Labels in edge set $\{uv_i | 1 \leq i \leq\ n\ \}$ are respectively of the form ${}^{(2n+2)}P_k, 2 \leq k \leq n+1$.

(5). Labels in edge set $\{uu_i | 1 \leq i \leq\ n\ \}$ are respectively of the form ${}^{(2n+2)}P_k, n+2 \leq k \leq 2n+1$.

Using the number theory results described in subsection 1.1, it is clear that edge labels of above possibilities (1) to (5) are internally as well as externally in ascending order.
So above defined function f, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct.
Hence the restricted square of bistar $B_{n,n}$ is a permutation graph. □

Theorem 2.5. *The degree splitting graph of $B_{n,n}$ is a permutation graph.*

Proof. Let $G = DS(B_{n,n})$ be the degree splitting graph of $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup \{w_1, w_2, u_i, v_i | 1 \leq i \leq n\}$ and edge set $E(G) = \{w_1u_i, w_1v_i, uu_i, vv_i, uv, uw_2, vw_2 | 1 \leq i \leq n\}$. Here $|V(G)| = 2n+4$ and $|E(G)| = 4n+3$.
We define a bijection $f : V(G) \to \{1, 2 \ldots, 2n+4\}$ as follows.

$$\begin{aligned} f(u) &= 2n+2. \\ f(u_i) &= 1+i\ ; 1 \leq i \leq\ n. \\ f(v) &= 2n+3. \\ f(v_i) &= n+i+1; 1 \leq i \leq\ n. \\ f(w_1) &= 1. \\ f(w_2) &= 2n+4. \end{aligned}$$

So from above defined function f, the following seven possibilities for the produced edge labels can be considered.

(1). Labels in edge set $\{u_iw_1 | 1 \leq i \leq\ n\ \}$ are respectively $2, 3, \ldots, n+1$.

(2). Labels in edge set $\{v_iw_1 | 1 \leq i \leq\ n\ \}$ are respectively $n+2, n+3, \ldots 2n+1$.

(3). Labels in edge set $\{uu_i | 1 \leq i \leq\ n\}$ are respectively of the form ${}^{(2n+2)}P_k, 2 \leq k \leq n+1$.

(4). Labels in edge set $\{vv_i | 1 \leq i \leq \; n\}$ are respectively of the form ${}^{(2n+3)}P_k, n+2 \leq k \leq 2n+1$.

(5). Label of the edge $\{uv\}$ is ${}^{(2n+3)}P_{(2n+2)}$.

(6). Label of the edge $\{w_2 u\}$ is ${}^{(2n+4)}P_{(2n+2)}$.

(7). Label of the edge $\{w_2 v\}$ is ${}^{(2n+4)}P_{(2n+3)}$.

Using the number theory results described in subsection 1.1, it is clear that edge labels of above possibilities (1) to (7) are internally as well as externally in ascending order.

So above defined function f, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct.

Hence degree splitting graph of $B_{n,n}$ is a permutation graph. □

Theorem 2.6. *Arbitrary super subdivision of path graph P_n is a permutation graph.*

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1} | 1 \leq i \leq n-1\}$. Let G be a graph obtained by arbitrary super subdivision of path graph P_n. That is, for $1 \leq i \leq n-1$ each edge e_i of the path P_n is replaced by a complete bipartite graph K_{2,m_i}, where m_i is positive integer. Let u_{ij} be the vertices of m_i vertex section, where $1 \leq i \leq n-1$ and $1 \leq j \leq m_i$. Let $M = \sum\limits_{i=1}^{n} m_i$ then $|V(G)| = n + M$ and $|E(G)| = 2M$. We define a bijection $f : V(G) \rightarrow \{1, 2 \ldots, n+M\}$ as follows.

$$\begin{aligned} f(v_1) &= 1. \\ f(u_{ij}) &= f(v_i) + j; 1 \leq j \leq m_i. \\ f(v_{i+1}) &= f(u_{im_i}) + 1; 1 \leq i \leq n-1. \end{aligned}$$

So above defined function f, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct.

Hence the graph obtained by arbitrary super subdivision of path graph is a permutation graph. □

Theorem 2.7. *Arbitrary supersubdivision of cycle C_n is a permutation graph.*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(C_n) = \{e_i = v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{e_n = v_n v_1\}$. Let G be a graph obtained by arbitrary super subdivision of C_n as each edge e_i of C_n is replaced by complete bipartite graph $K_{2,mi}$, where m_i is a positive integer. Let u_{ij} be the vertices of m_i vertex section, where $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Let $M = \sum_{i=1}^{n} \; m_i$ then $|V(G)| = n + M$, $|E(G)| = 2M$. We define a bijection $f : V(G) \rightarrow \{1, 2 \ldots, n+M\}$ as follows.

Case 1: n is even.

$$\begin{aligned} f(u_i) &= 1 \; ; \; i = 1. \\ f(u_i) &= f(u_{n+2-i, m_{n+2-i}}) + 1 \; ; \; 2 \leq i \leq \frac{n}{2} + 1. \\ f(u_i) &= f(u_{n+2-i}) + 1 \; ; \; \frac{n}{2} + 1 < i \leq \; n. \\ f(u_{ij}) &= f(u_{n+2-i}) + j \; ; \; 1 \leq j \leq m_i, 2 \leq i \leq \frac{n}{2}. \\ f(u_{ij}) &= f(u_{n+2-i, m_{n+2-i}}) + 1 \; ; \; 1 \leq j \leq m_i, \frac{n}{2} < i \leq \; n. \end{aligned}$$

Case 2: n is odd.

$$\begin{aligned} f(u_i) &= 1 \; ; \; i = 1. \\ f(u_i) &= f(u_{n+2-i, m_{n+2-i}}) + 1 \; ; \; 2 \leq i \leq \frac{n+1}{2}. \end{aligned}$$

$$f(u_i) = f(u_{n+2-i}) + 1 \ ; \ \frac{n+1}{2} < i \leq \ n.$$
$$f(u_{ij}) = f(u_{n+2-i}) + j \ ; 1 \leq j \leq m_i, 2 \leq i \leq \frac{n+1}{2}.$$
$$f(u_{ij}) = f(u_{n+2-i,m_{n+2-i}}) + 1 \ ; \ 1 \leq j \leq m_i, \frac{n+1}{2} < i \leq \ n.$$

So above defined function f, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct. Hence the graph obtained by arbitrary super subdivision of cycle C_n is a permutation graph. □

Theorem 2.8. *Arbitrary supersubdivision of star $K_{1,n}$ is a permutation graph.*

Proof. Let $V(K_{1,n}) = \{v_0, v_1, v_2, \ldots, v_n\}$ and $E(K_{1,n}) = \{v_0v_i | 1 \leq i \leq n\}$, where v_0 be apex vertex and $v_1, v_2, \ldots, v_n$ be pendent vertices of star $K_{1,n}$. Let G be a graph obtained by arbitrary super subdivision of $K_{1,n}$ as each edge e_i of $K_{1,n}$ is replaced by complete bipartite graph $K_{2,mi}$, where m_i is positive integer. Let u_{ij} be the vertices of m_i vertex section, where $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Let $M = \sum_{i=1}^{n} m_i$ then $|V(G)| = n + M + 1, |E(G)| = 2M$. We define a bijection $f : V(G) \to \{1, 2 \ldots, n + M + 1\}$ as follows.

$$f(v_i) = 1 \ ; \ i = 0.$$
$$f(v_i) = cm_n + i \ ; \ 1 \leq i \leq \ n.$$
$$f(u_{ij}) = cm_{i-1} + j \ ; \ 1 \leq j \leq m_i, 1 \leq i \leq n$$

where $m_0 = 0$ and $cm_i =$ cumulative values of m_i. So above defined function f, each edge uv is identified the label $\frac{(f(u))!}{|f(u)-f(v)|!}$ $(f(u) > f(v))$, which are all distinct.
Hence the graph obtained by arbitrary super subdivision of star $K_{1,n}$ is a permutation graph. □

3. Conclusion

Permutation labeling is a connection between number theory and graph theory. Here we discuss some graphs satisfying the conditions of permutation labeling. To investigate equivalent results for different graph families is an open area of research.

Acknowledgment

We would like to thank the editor and the referees for their constructive suggestions and comments.

References

[1] D. M. Burton, *Elementary number theory* (Sixth edition), Tata McGraw-Hill, (2006).

[2] G. V. Ghodasara and Mitesh J. Patel, *More on permutation labeling of graphs*, International Journal of Applied Graph Theory, 1(2017), 30-42.

[3] G. V. Ghodasara and Mitesh J. Patel, *Some new combination graphs*, International Journal of Mathematics And its Applications, 5(2017), 153-161.

[4] G. V. Ghodasara and Mitesh J. Patel, *Some bistar related square sum graphs*, International Journal of Mathematics Trend and Technology, 47(2017), 172-177.

[5] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Elsevier Science Publication.

[6] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronics Journal of Combinatorics, (2019), 1-535.

[7] J. Baskar Babujee and V. Vishnupriya, *Permutation labeling for some trees*, International Journal of Mathematical and Computer Science, 3(2008), 31-38.

[8] J. Shiama, *Permutation labeling of some shadow graphs*, International Journal of Computer Applications, 40(2012), 31-35.

[9] M. A. Seoud and M. A. Salim, *On permutation labeling*, Journal of Egyptian Mathematical society, 19(2011), 134-136.

[10] M. A. Seoud and A. E. A. Mahram, *On permutation graphs*, Journal of Egyptian Mathmetical society, 20(2012), 57-63.

[11] S. G. Sonchhatra and G. V. Ghodasara, *Some new permutation graphs*, International Journal of Mathematical Archive, 7(2016), 209-215.

[12] S. M. Hegde and S. Shetty, *Combinatorial labeling of graphs*, Applied Mathematics E-Notes, 6(2006), 251-258.