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A Note on Multiplication Operators Between Orlicz Spaces

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Abstract: In this paper, we study the boundedness of multiplication operators between any two Orlicz spaces.

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1. Introduction

Let $X = (X, \Sigma, \mu)$ be a σ- finite complete measure space. A nondecreasing continuous convex function $\phi : [0, \infty) \to [0, \infty)$ for which $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$ is said to be an *Orlicz function*. For any $f \in L^0(X)$, we define the *modular*

$$I_\phi(f) = \int_X \phi(|f(x)|)d\mu(x)$$

and the *Orlicz space*

$$L^\phi(\mu) = \left\{f \in L^0(X) \mid I_\phi(\lambda f) < \infty \text{ for some } \lambda = \lambda(f) > 0\right\}.$$

This space is a Banach space with two norms: the *Luxemburg − Nakano norm*

$$\|f\|_\phi = \inf\{\lambda > 0 \mid I_\phi(f/\lambda) \leq 1\}$$

and the *Orlicz norm*

$$\|f\|_\phi^0 = \inf_{k>0}(1 + I_\phi(kf))/k.$$

For the study of Orlicz spaces, one can refer to [11, 13–15]. Multiplication operators on Orlicz spaces have also been studied in [5], [8] and references therein. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another (see, [13, Page 45] for details).

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2. Boundedness of Multiplication Operators

In this section, we study the boundedness of multiplication operators on weighted Orlicz spaces.

Lemma 2.1 ([13, Lemma 8.3])**.** *Let (X, Σ, μ) be a σ-finite nonatomic measure space, $\{\alpha_n\}$ a sequence of positive numbers and $\{s_n\}$ a sequence of measurable, finite, non-negative functions on X such that for $n = 1, 2, \ldots$*

$$\int_X s_n(x) d\mu(x) \geq 2^n \alpha_n.$$

Then there exist an increasing sequence $\{n_k\}$ of integers and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that for $k = 1, 2, \ldots$

$$\int_{A_k} s_{n_k}(x) d\mu(x) = \alpha_{n_k}.$$

Theorem 2.2. *Let (X, Σ, μ) be a σ-finite nonatomic measure space and $\theta : X \to \mathbb{C}$ be a measurable function. Then the multiplication operator $M_\theta : L^{\phi_1}(X) \longrightarrow L^{\phi_2}(X)$ is bounded if and only if there exist $a, b > 0$ and $0 \leq h \in L^1(X)$ such that $\phi_2(a|\theta(x)|u)) \leq b\phi_1(u) + h(x)$ for almost all $x \in X$ and for all $u \geq 0$.*

Proof. Suppose that the converse holds. Let $0 \neq f \in L^{\phi_1}(X)$ and choose any $M \geq 1$ such that $M(b + \|h\|_1) \geq 1$. Then

$$\begin{aligned}
I_{\Phi_2}\left(\frac{M_\theta f}{(M(b+\|h\|_1)\|f\|_{\Phi_1})/a}\right) &= \int_X \phi_2\left(\frac{a|\theta(x)f(x)|}{M(b+\|h\|_1)\|f\|_{\Phi_1}}\right) d\mu(x) \\
&\leq \frac{1}{M(b+\|h\|_1)} \int_X \phi_2\left(\frac{a|\theta(x)||f(x)|}{\|f\|_{\Phi_1}}\right) d\mu(x) \\
&\leq \frac{1}{M(b+\|h\|_1)} \int_X \left(b\ \phi_1\left(\frac{|f(x)|}{\|f\|_{\Phi_1}}\right) + h(x)\right) d\mu(x) \\
&\leq 1.
\end{aligned}$$

Thus $\|M_\theta f\|_{\Phi_2} \leq \frac{M}{a}(b + \|h\|_1)\|f\|_{\Phi_1}$ and hence M_θ is bounded. Consider the function

$$h_n(x) = \sup_{u \geq 0} \left(\phi_2(2^{-n}|\theta(x)|u) - 2^n \phi_1(u)\right).$$

Write $X = \bigcup_{i=1}^{\infty} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence of measurable subsets of X with $\mu(X_i) < \infty$ for every $i = 1, 2, \ldots$. For every $q \in \mathbb{Q}^+$,we put $f_{q,i}(x) = q\chi_{X_i}(x)$, where χ_{X_i} is the characteristic function of X_i. Then it can be shown that

$$h_n(x) = \sup_{\substack{r \in \mathbb{Q}^+ \\ i \in \mathbb{N}}} \left(\phi_2(2^{-n}|\theta(x)|f_{r,i}(x)) - 2^n \phi_1(f_{r,i}(x))\right).$$

Taking (f_k) to be a rearrangement of $(f_{q,i})$ with $f_1 = f_{0,i}$, the above equation can be rewritten as

$$h_n(x) = \sup_{k \in \mathbb{N}} \left(\phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x))\right)$$

It is clear that h_n are measurable and $h_n(x) \geq 0$ for each $x \in X$. To complete the proof, we need only to show that $\int_X h_n(x) d\mu(x) < \infty$ for some n. Suppose this is not true. Denote

$$b_{m,n}(x) = \max_{1 \leq k \leq m} \left(\phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x))\right).$$

Then $b_{m,n}$ are measurable, $b_{m,n}(x) \geq 0$ and $b_{m,n}(x)$ is a non-decreasing sequence tending to $h_n(x)$ as $m \to \infty$ for every $x \in X$. Thus for any n, there exists m_n such that $\int_X b_{m_n,n}(x) d\mu(x) \geq 2^n$. Writing $b_n = b_{m_n,n}$, we have $\int_X b_n(x) d\mu(x) \geq 2^n$ for $n = 1, 2, \ldots$. Let

$$E_{n,k} = \left\{x \in X \mid \phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x)) = b_n(x)\right\}$$

and

$$E_n = X \setminus (E_{n,1} \cup E_{n,2} \cup \cdots \cup E_{n,m_n}).$$

Then $\mu(E_n) = 0$. Let

$$\tilde{f}_n(x) = \begin{cases} 0 & \text{if } x \in E_{n,1} \cup B_n \\ f_k(x) & \text{if } x \in E_{n,k} \setminus \bigcup_{j=1}^{k-1} E_{n,j}, \ k = 2, 3, \ldots, m_n. \end{cases}$$

Then

$$\begin{aligned} b_n(x) &= \phi_2(2^{-n}|\theta(x)|\tilde{f}_n(x)) - 2^n \phi_1(\tilde{f}_n(x)) \\ &\geq 0. \end{aligned} \tag{1}$$

Therefore,

$$\begin{aligned} \int_X \phi_2(2^{-n}|\theta(x)|\tilde{f}_n(x))d\mu(x) &= 2^n \int_X \phi_1(\tilde{f}_n(x))d\mu(x) + \int_X b_n(x)d\mu(x) \\ &\geq \int_X b_n(x)d\mu(x) \\ &\geq 2^n. \end{aligned}$$

Thus by Lemma **??**, with $a_n(x) = \phi_2(2^{-n}|\theta(x)|\tilde{f}_n(x))$ and $\alpha_n = 1$, we obtain an increasing sequence $\{n_k\}$ and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that

$$\int_{A_k} \phi_2(2^{-n_k}|\theta(x)|\tilde{f}_{n_k}(x))d\mu(x) = 1, \quad k = 1, 2, \ \ldots \tag{2}$$

Put

$$f(x) = \begin{cases} \tilde{f}_{n_k}(x) & \text{if } x \in A_k \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Then for any $\lambda > 0$, using (2) and (3), we obtain

$$\begin{aligned} \int_X \phi_2\left(\lambda M_\theta f(x)\right) d\mu(x) &= \int_X \phi_2\left(\lambda|\theta(x)|f(x)\right) d\mu(x) \\ &= \sum_{k=1}^{\infty} \int_{A_k} \phi_2(\lambda|\theta(x)|\tilde{f}_{n_k}(x))d\mu(x) \\ &\geq \sum_{k=p}^{\infty} \int_{A_k} \phi_2(2^{-n_k}|\theta(x)|\tilde{f}_{n_k}(x))d\mu(x) \\ &= \infty, \end{aligned}$$

where p is so large that $2^{-n_p} \leq \lambda$. And using (1), (2) and (3), we have

$$\begin{aligned} \int_X \phi_1(f(x))d\mu(x) &= \sum_{k=1}^{\infty} \int_{A_k} \phi_1(\tilde{f}_{n_k}(x))d\mu(x) \\ &= \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} \phi_2(2^{-n_k}|\theta(x)|\tilde{f}_{n_k}(x))d\mu(x) - \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} b_{n_k}(x)d\mu(x) \\ &\leq \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} \phi_2(2^{-n_k}|\theta(x)|\tilde{f}_{n_k}(x))d\mu(x) \\ &= \sum_{k=1}^{\infty} 2^{-n_k} \end{aligned}$$

$$\leq 1.$$

Thus, $f \in L^{\phi_1}(x)(X)$ but $M_\theta(f) \notin L^{\phi_2}(X)$, which is a contradiction. Hence,

$$\int_X h_n(x)d\mu(x) < \infty \text{ for some } n$$

This completes the proof. □

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