A Note on Multiplication Operators Between Orlicz Spaces

Heera Saini\(^1,∗\) and Isha Gupta\(^2\)

\(^1\) Department of Mathematics, G.C.E.T, Jammu, Jammu and Kashmir, India.
\(^2\) Department of Statistics, University of Jammu, Jammu, Jammu and Kashmir, India.

Abstract: In this paper, we study the boundedness of multiplication operators between any two Orlicz spaces.

MSC: 47B38, 46E30.

Keywords: Multiplication operators, Boundedness, Orlicz spaces, Musielak-Orlicz spaces.

\(∗\) E-mail: heerasainihs@gmail.com

1. Introduction

Let \(X = (X, \Sigma, \mu)\) be a \(\sigma\)-finite complete measure space. A nondecreasing continuous convex function \(\phi : [0, \infty) \rightarrow [0, \infty)\) for which \(\phi(0) = 0\) and \(\lim_{x \to \infty} \phi(x) = \infty\) is said to be an Orlicz function. For any \(f \in L^0(X)\), we define the modular

\[I_\phi(f) = \int_X \phi(|f(x)|)d\mu(x)\]

and the Orlicz space

\[L_\phi(\mu) = \{f \in L^0(X) \mid I_\phi(\lambda f) < \infty \text{ for some } \lambda = \lambda(f) > 0\}.
\]

This space is a Banach space with two norms: the Luxemburg – Nakano norm

\[\|f\|_\phi = \inf\{\lambda > 0 \mid I_\phi(f/\lambda) \leq 1\}\]

and the Orlicz norm

\[\|f\|_\phi^0 = \inf_{k > 0} (1 + I_\phi(kf))/k.\]

For the study of Orlicz spaces, one can refer to \([11, 13–15]\). Multiplication operators on Orlicz spaces have also been studied in \([5], [8]\) and references therein. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another (see, \[13, \text{ Page 45}\] for details).
2. Boundedness of Multiplication Operators

In this section, we study the boundedness of multiplication operators on weighted Orlicz spaces.

Lemma 2.1 ([13, Lemma 8.3]). Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite nonatomic measure space, \(\{\alpha_n\}\) a sequence of positive numbers and \(\{s_n\}\) a sequence of measurable, finite, non-negative functions on \(X\) such that for \(n = 1, 2, \ldots\)

\[
\int_X s_n(x) d\mu(x) \geq 2^n \alpha_n.
\]

Then there exist an increasing sequence \(\{n_k\}\) of integers and a sequence \(\{A_k\}\) of pairwise disjoint measurable sets such that for \(k = 1, 2, \ldots\)

\[
\int_{A_k} s_{n_k}(x) d\mu(x) = \alpha_{n_k}.
\]

Theorem 2.2. Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite nonatomic measure space and \(\theta : X \to \mathbb{C}\) be a measurable function. Then the multiplication operator \(M_\theta : L^{\theta^1}(X) \to L^{\theta^2}(X)\) is bounded if and only if there exist \(a, b > 0\) and \(0 \leq h \in L^1(X)\) such that \(\phi_2(a(\theta(x))u) \leq b\phi_1(u) + h(x)\) for almost all \(x \in X\) and for all \(u \geq 0\).

Proof. Suppose that the converse holds. Let \(0 \neq f \in L^{\theta^1}(X)\) and choose any \(M \geq 1\) such that \(M(b + \|h\|_1) \geq 1\). Then

\[
I_{\theta^2} \left( \frac{Mf}{M(b + \|h\|_1)} \frac{\phi_2}{\phi_1} \right) = \int_X \phi_2 \left( \frac{a(\theta(x))f(x)}{M(b + \|h\|_1)\|f\|_{\phi_1}} \right) d\mu(x)
\]

\[
\leq \frac{1}{M(b + \|h\|_1)} \int_X \phi_2 \left( \frac{a(\theta(x))f(x)}{\|f\|_{\phi_1}} \right) d\mu(x)
\]

\[
\leq \frac{1}{M(b + \|h\|_1)} \int_X \left( b \phi_1 \left( \frac{f(x)}{\|f\|_{\phi_1}} \right) + h(x) \right) d\mu(x)
\]

Thus \(\|M_\theta f\|_{\phi_2} \leq \frac{M}{b + \|h\|_1} \|f\|_{\phi_1}\) and hence \(M_\theta\) is bounded. Consider the function

\[
h_n(x) = \sup_{u \geq 0} \left( \phi_2(2^{-n}|\theta(x)|u) - 2^n \phi_1(u) \right).
\]

Write \(X = \bigcup_{i=1}^{\infty} X_i\), where \(\{X_i\}_{i=1}^{\infty}\) is a pairwise disjoint sequence of measurable subsets of \(X\) with \(\mu(X_i) < \infty\) for every \(i = 1, 2, \ldots\). For every \(q \in \mathbb{Q}^+\), we put \(f_q,i(x) = q\chi_{X_i}(x)\), where \(\chi_{X_i}\) is the characteristic function of \(X_i\). Then it can be shown that

\[
h_n(x) = \sup_{r \in \mathbb{Q}^+} (\phi_2(2^{-n}|\theta(x)|f_{r,i}(x)) - 2^n \phi_1(f_{r,i}(x))).
\]

Taking \((f_k)\) to be a rearrangement of \((f_{q,i})\) with \(f_1 = f_{0,i}\), the above equation can be rewritten as

\[
h_n(x) = \sup_{k \in \mathbb{N}} (\phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x))).
\]

It is clear that \(h_n\) are measurable and \(h_n(x) \geq 0\) for each \(x \in X\). To complete the proof, we need only to show that \(\int_X h_n(x) d\mu(x) < \infty\) for some \(n\). Suppose this is not true. Denote

\[
b_{m,n}(x) = \max_{1 \leq k \leq m} \left( \phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x)) \right).
\]

Then \(b_{m,n}\) are measurable, \(b_{m,n}(x) \geq 0\) and \(b_{m,n}(x)\) is a non-decreasing sequence tending to \(h_n(x)\) as \(m \to \infty\) for every \(x \in X\). Thus for any \(n\), there exists \(m_n\) such that \(\int_X b_{m,n}(x) d\mu(x) \geq 2^n\). Writing \(b_n = b_{m,n}\), we have \(\int_X b_n(x) d\mu(x) \geq 2^n\) for \(n = 1, 2, \ldots\). Let

\[
E_{n,k} = \{ x \in X \mid \phi_2(2^{-n}|\theta(x)|f_k(x)) - 2^n \phi_1(f_k(x)) = b_n(x) \}
\]
and

\[ E_n = X \setminus (E_{n,1} \cup E_{n,2} \cup \cdots \cup E_{n,m_n}). \]

Then \( \mu(E_n) = 0 \). Let

\[
\tilde{f}_n(x) = \begin{cases} 
0 & \text{if } x \in E_{n,1} \cup B_n \\
f_k(x) & \text{if } x \in E_{n,k} \setminus \bigcup_{j=1}^{k-1} E_{n,j}, \ k = 2, 3, \ldots, m_n.
\end{cases}
\]

Then

\[
b_n(x) = \phi_2(2^{-k} |\theta(x)| \tilde{f}_n(x)) - 2^n \phi_1(\tilde{f}_n(x)) \geq 0. \tag{1}
\]

Therefore,

\[
\int_X \phi_2(2^{-n} |\theta(x)| \tilde{f}_n(x)) d\mu(x) = 2^n \int_X \phi_1(\tilde{f}_n(x)) d\mu(x) + \int_X b_n(x) d\mu(x) \geq \int_X b_n(x) d\mu(x) \geq 2^n.
\]

Thus by Lemma ??, with \( \alpha_n(x) = \phi_2(2^{-n} |\theta(x)| \tilde{f}_n(x)) \) and \( \alpha_n = 1 \), we obtain an increasing sequence \( \{n_k\} \) and a sequence \( \{A_k\} \) of pairwise disjoint measurable sets such that

\[
\int_{A_k} \phi_2(2^{-nk} |\theta(x)| \tilde{f}_{nk}(x)) d\mu(x) = 1, \quad k = 1, 2, \ldots \tag{2}
\]

Put

\[
f(x) = \begin{cases} 
\tilde{f}_{nk}(x) & \text{if } x \in A_k \\
0 & \text{otherwise.}
\end{cases}
\]

Then for any \( \lambda > 0 \), using (2) and (3), we obtain

\[
\int_X \phi_2(\lambda M_n f(x)) d\mu(x) = \int_X \phi_2(\lambda |\theta(x)| f(x)) d\mu(x) = \sum_{k=1}^{\infty} \int_{A_k} \phi_2(\lambda |\theta(x)| \tilde{f}_{nk}(x)) d\mu(x) \geq \sum_{k=p}^{\infty} \int_{A_k} \phi_2(2^{-nk} |\theta(x)| \tilde{f}_{nk}(x)) d\mu(x) = \infty,
\]

where \( p \) is so large that \( 2^{-np} \leq \lambda \). And using (1), (2) and (3), we have

\[
\int_X \phi_1(f(x)) d\mu(x) = \sum_{k=1}^{\infty} \int_{A_k} \phi_1(\tilde{f}_{nk}(x)) d\mu(x) = \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} \phi_2(2^{-nk} |\theta(x)| \tilde{f}_{nk}(x)) d\mu(x) - \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} b_{nk}(x) d\mu(x) \leq \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} \phi_2(2^{-nk} |\theta(x)| \tilde{f}_{nk}(x)) d\mu(x) = \sum_{k=1}^{\infty} 2^{-nk} \]


Thus, \( f \in L^{\phi_1}(X) \) but \( M_{\phi}(f) \notin L^{\phi_2}(X) \), which is a contradiction. Hence,

\[
\int_X h_n(x)d\mu(x) < \infty \text{ for some } n
\]

This completes the proof. □

References