On Generalization of $\delta$-Primary Elements in Multiplicative Lattices

Ashok V. Bingi$^{1,*}$

1 Department of Mathematics, St. Xavier’s College (Autonomous), Mumbai, Maharashtra, India.

Abstract: In this paper, we introduce $\varphi$-$\delta$-primary elements in a compactly generated multiplicative lattice $L$ and obtain its characterizations. We prove many of its properties and investigate the relations between these structures. By a counter example, it is shown that a $\varphi$-$\delta$-primary element of $L$ need not be $\delta$-primary and found conditions under which a $\varphi$-$\delta$-primary element of $L$ is $\delta$-primary.

MSC: 06B99.

Keywords: expansion function, $\delta$-primary element, $\varphi$-$\delta$-primary element, 2-potent $\delta$-primary element, $n$-potent $\delta$-primary element, global property.

© JS Publication.

1. Introduction

Prime ideals play a central role in commutative ring theory. In the literature, we find that there are several ways to generalize the notions of a prime ideal and a primary ideal of a commutative ring $R$ with unity. A prime ideal $P$ of $R$ is an ideal with the property that for all $a, b \in R$, $ab \in P$ implies either $a \in P$ or $b \in P$. We can either restrict or enlarge where $a$ and/or $b$ lie or restrict or enlarge where $ab$ lies. Same can be thought for primary ideals too. As a generalization of prime ideals of $R$, $\varphi$-prime ideals were introduced in [2] and [6] while as a generalization of primary ideals of $R$, $\varphi$-primary ideals were introduced in [4]. In an attempt to unify the prime and primary ideals of $R$ under one frame, $\delta$-primary ideals of $R$ were introduced in [12]. Further, the concept of $\delta$-primary ideals of $R$ was generalized by introducing the notion of $\varphi$-$\delta$-primary ideals of $R$ in [7].

As an extension of these concepts of a commutative ring $R$ to a multiplicative lattice $L$, C. S. Manjarekar and A. V. Bingi introduced $\delta$-primary elements of $L$ in [8] and introduced $\varphi$-prime, $\varphi$-primary elements of $L$ in [9]. In this paper, we introduce and study, $\varphi$-$\delta$-primary elements of $L$ as a generalization of $\delta$-primary elements of $L$ and unify $\varphi$-prime and $\varphi$-primary elements of $L$ under one frame.

A multiplicative lattice $L$ is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $e \in L$ is called meet principal if $a \wedge be = ((a : e) \wedge b)e$ for all $a, b \in L$. An element $e \in L$ is called join principal if $ae \vee b : e = (b : e) \vee a$ for all $a, b \in L$. An element $e \in L$ is called principal if $e$ is both meet principal and join principal. A multiplicative lattice $L$ is said to be principally generated(PG) if every element of $L$ is a join of principal elements of $L$. An element $a \in L$ is called compact if for $X \subseteq L$, $a \subseteq \vee X$ implies
the existence of a finite number of elements \(a_1, a_2, \cdots, a_n\) in \(X\) such that \(a \leq a_1 \lor a_2 \lor \cdots \lor a_n\). The set of compact elements of \(L\) will be denoted by \(L_c\). If each element of \(L\) is a join of compact elements of \(L\), then \(L\) is called a compactly generated lattice or simply a CG-lattice. An element \(a \in L\) is said to be proper if \(a < 1\). The radical of \(a \in L\) is denoted by \(\sqrt{a}\) and is defined as \(\lor\{x \in L_+ | x^n \leq a, \text{ for some } n \in \mathbb{Z}_+\}\). A proper element \(m \in L\) is said to be maximal if for every element \(x \in L\) such that \(m < x \leq 1\) implies \(x = 1\). A proper element \(p \in L\) is called a prime element if \(ab \leq p\) implies \(a \leq p\) or \(b \leq p\) where \(a, b \in L\) and is called a primary element if \(ab \leq p\) implies \(a \leq p\) or \(b \leq \sqrt{p}\) where \(a, b \in L\). For \(a, b \in L\), \((a : b) = \lor\{x \in L | xb \leq a\}\). A multiplicative lattice is called as a Noether lattice if it is modular, principally generated and satisfies ascending chain condition. An element \(a \in L\) is called a zero divisor if \(ab = 0\) for some \(0 \neq b \in L\) and is called idempotent if \(a = a^2\). A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice \(L\) with maximal element \(m\) is denoted by \((L, m)\). A Noether lattice \(L\) is local if it contains precisely one maximal prime. In a Noether lattice \(L\), an element \(a \in L\) is said to satisfy restricted cancellation law if for all \(b, c \in L\), \(ab = ac \neq 0\) implies \(b = c\) (see [11]). According to [8], an expansion function on \(L\) is a function \(\delta : L \rightarrow L\) which satisfies the following two conditions: (1) \(a \leq \delta(a)\) for all \(a \in L\), (2) \(a \leq b\) implies \(\delta(a) \leq \delta(b)\) for all \(a, b \in L\) and a proper element \(p \in L\) is called \(\delta\)-primary if for all \(a, b \in L\), \(ab \leq p\) implies either \(a \leq p\) or \(b \leq \delta(p)\). According to [9], a proper element \(p \in L\) is said to be \(\phi\)-prime if for all \(a, b \in L\), \(ab \leq p\) and \(ab \notin \phi(p)\) implies either \(a \leq p\) or \(b \leq p\) and a proper element \(p \in L\) is said to be \(\phi\)-primary if for all \(a, b \in L\), \(ab \leq p\) and \(ab \notin \phi(p)\) implies either \(a \leq p\) or \(b \leq \sqrt{p}\) where \(\phi : L \rightarrow L\) is a function on \(L\). The reader is referred to [1] and [5] for general background and terminology in multiplicative lattices.

This paper is motivated by [7]. In this paper, we define a \(\phi\)-\(\delta\)-primary element in \(L\) and obtain their characterizations. Various \(\phi_n\)-\(\delta\)-primary elements of \(L\) are introduced and relations among them are obtained. By counter examples, it is shown that a \(\phi\)-\(\delta\)-primary element of \(L\) need not be \(\phi\)-prime, a \(\phi\)-\(\delta\)-primary element of \(L\) need not be prime and a \(\phi\)-\(\delta\)-primary element of \(L\) need not be \(\delta\)-primary. In 7 different ways, we have proved that a \(\phi\)-\(\delta\)-primary element of \(L\) is \(\delta\)-primary under certain conditions. We define a 2-potent \(\delta\)-primary element of \(L\) and a n-potent \(\delta\)-primary element of \(L\).

We investigate some properties of \(\phi\)-\(\delta\)-primary elements of \(L\) with respect to lattice homomorphism and global property. Finally, we show that every idempotent element of \(L\) is \(\phi_2\)-\(\delta\)-primary but converse need not be true. Throughout this paper, (1) \(L\) denotes a compactly generated multiplicative lattice with greatest compact element \(1\) in which every finite product of compact elements is compact, (2) \(\delta\) denotes an expansion function on \(L\) and (3) \(\phi\) denotes a function defined on \(L\).

2. \(\phi\)-\(\delta\)-primary Elements of \(L\)

We begin with introducing the notion of \(\phi\)-\(\delta\)-primary elements of \(L\) which is the generalization of the concept of \(\delta\)-primary elements of \(L\).

**Definition 2.1.** Given an expansion function \(\delta : L \rightarrow L\) and a function \(\phi : L \rightarrow L\), a proper element \(p \in L\) is said to be \(\phi\)-\(\delta\)-primary if for all \(a, b \in L\), \(ab \leq p\) and \(ab \notin \phi(p)\) implies either \(a \leq p\) or \(b \leq \delta(p)\).

If \(\phi_n : L \rightarrow L\) is a function on \(L\), then \(\phi_n\)-\(\delta\)-primary elements of \(L\) are defined by following settings in the Definition 2.1 of a \(\phi\)-\(\delta\)-primary element of \(L\).

- \(\phi_0(p) = 0\). Then \(p \in L\) is called a weakly \(\delta\)-primary element.

- \(\phi_2(p) = p^2\). Then \(p \in L\) is called a 2-almost \(\delta\)-primary element or a \(\phi_2\)-\(\delta\)-primary element or simply an almost \(\delta\)-primary element.
• \( \phi_n(p) = p^n \) (\( n \geq 2 \)). Then \( p \in L \) is called an \( n \)-almost \( \delta \)-primary element or a \( \phi_n,\delta \)-primary element (\( n \geq 2 \)).

• \( \phi_\infty(p) = \prod_{n=1}^{\infty} p^n \). Then \( p \in L \) is called a \( \omega,\delta \)-primary element or \( \phi_\infty,\delta \)-primary element.

Since for an element \( a \in L \) with \( a \leq q \) but \( a \notin \phi(q) \) implies that \( a \notin q \land \phi(q) \), there is no loss generality in assuming that \( \phi(q) \leq q \). We henceforth make this assumption.

**Definition 2.2.** Given any two functions \( \gamma_1, \gamma_2 : L \rightarrow L \), we define \( \gamma_1 \leq \gamma_2 \) if \( \gamma_1(a) \leq \gamma_2(a) \) for each \( a \in L \).

Clearly, we have the following order:

\[ \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \]

Further as \( \phi(p) \leq p \) and \( p \leq \delta(p) \) for each \( p \in L \), the relation between the functions \( \delta \) and \( \phi \) is \( \phi \leq \delta \).

According to [8], \( \delta_0 \) is an expansion function on \( L \) defined as \( \delta_0(p) = p \) for each \( p \in L \) and \( \delta_1 \) is an expansion function on \( L \) defined as \( \delta_1(p) = \sqrt{p} \) for each \( p \in L \). Further, note that by Theorem 2.2 in [8], a proper element \( p \in L \) is \( \delta_1 \)-primary if and only if it is prime and by Theorem 2.3 in [8], a proper element \( p \in L \) is \( \delta_1 \)-primary if and only if it is primary.

The following 2 results relate \( \phi \)-prime and \( \phi \)-primary elements of \( L \) with some \( \phi,\delta \)-primary elements of \( L \).

**Theorem 2.3.** A proper element \( p \in L \) is \( \phi,\delta_0 \)-primary if and only if \( p \) is a prime.

**Proof.** The proof is obvious.

**Theorem 2.4.** A proper element \( p \in L \) is \( \phi,\delta_1 \)-primary if and only if \( p \) is a \( \phi \)-primary.

**Proof.** The proof is obvious.

**Theorem 2.5.** Let \( \delta, \gamma : L \rightarrow L \) be expansion functions on \( L \) such that \( \delta \leq \gamma \). Then every \( \phi,\delta \)-primary element of \( L \) is \( \phi,\gamma \)-primary. In particular, a \( \phi \)-prime element of \( L \) is \( \phi,\delta \)-primary for every expansion function \( \delta \) on \( L \).

**Proof.** Let a proper element \( p \in L \) be \( \phi,\delta \)-primary. Suppose \( ab \leq p \) and \( ab \notin \phi(p) \) for \( a, b \in L \). Then either \( a \leq p \) or \( b \leq \delta(p) \leq \gamma(p) \) and so \( p \) is \( \phi,\gamma \)-primary. Next, for any expansion function \( \delta \) on \( L \), we have \( \delta_0 \leq \delta \). So a \( \phi,\delta_0 \)-primary element of \( L \) is \( \phi,\delta \)-primary and we are done since a \( \phi \)-prime element of \( L \) is \( \phi,\delta_0 \)-primary.

**Corollary 2.6.** A prime element of \( L \) is \( \phi,\delta \)-primary for every expansion function \( \delta \) on \( L \).

**Proof.** The proof follows by using Theorem 2.5 to the fact that every prime element of \( L \) is \( \phi \)-prime.

The following example shows that (by taking \( \phi \) as \( \phi_2 \) and \( \delta \) as \( \delta_1 \) for convenience)

1. A \( \phi,\delta \)-primary element of \( L \) need not be \( \phi \)-prime,
2. A \( \phi,\delta \)-primary element of \( L \) need not be prime.

**Example 2.7.** Consider the lattice \( L \) of ideals of the ring \( R = \langle Z_{24}, +, \cdot \rangle \). Then the only ideals of \( R \) are the principal ideals \((0),(2),(3),(4),(6),(8),(12),(1)\). Clearly, \( L = \{(0),(2),(3),(4),(6),(8),(12),(1)\} \) is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 1. It is easy to see that the element \((4) \in L \) is \( \phi_2,\delta_1 \)-primary while \((4) \) is not \( \phi_2 \)-prime because though \((2) \cdot (6) \subseteq (4), (2) \cdot (6) \subsetneq (4)^2 \) but \((2) \subsetneq (4) \) and \((6) \nsubseteq (4) \). Also, \((4) \) is not prime.
Now before obtaining the characterizations of a $\phi$-$\delta$-primary element of $L$, we state the following essential lemma which is outcome of Lemma 2.3.13 from [3].

**Lemma 2.8.** Let $a_1, a_2 \in L$. Suppose $b \in L$ satisfies the following property:

(\ast). If $h \in L_+$ with $h \leq b$, then either $h \leq a_1$ or $h \leq a_2$.

Then either $b \leq a_1$ or $b \leq a_2$.

**Theorem 2.9.** Let $q$ be a proper element of $L$. Then the following statements are equivalent:

1. $q$ is $\phi$-$\delta$-primary.
2. for every $a \in L$ such that $a \nleq \delta(q)$, either $(q : a) = q$ or $(q : a) = (\phi(q) : a)$.
3. for every $r, s \in L_+$, $rs \leq q$ and $rs \nleq \phi(q)$ implies either $s \leq q$ or $r \leq \delta(q)$.

**Proof.** 1$\Rightarrow$2. Suppose 1 holds. Let $h \in L_+$ be such that $h \leq (q : a)$ and $a \nleq \delta(q)$. Then $ah \leq q$. If $ah \leq \phi(q)$, then $h \leq (\phi(q) : a)$. If $ah \nleq \phi(q)$, then since $q$ is $\phi$-$\delta$-primary and $a \nleq \delta(q)$, it follows that $h \leq q$. Hence by Lemma 2.8, either $(q : a) \leq (\phi(q) : a)$ or $(q : a) \leq q$. Consequently, either $(q : a) = (\phi(q) : a)$ or $(q : a) = q$.

2$\Rightarrow$3. Suppose 2 holds. Let $rs \leq q$, $rs \nleq \phi(q)$ and $r \nleq \delta(q)$ for $r, s \in L_+$. Then by 2, either $(q : r) = (\phi(q) : r)$ or $(q : r) = q$. If $(q : r) = (\phi(q) : r)$, then as $s \leq (q : r)$, it follows that $s \leq (\phi(q) : r)$ which contradicts $rs \nleq \phi(q)$ and so we must have $(q : r) = q$. Therefore $s \leq (q : r)$ gives $s \leq q$.

3$\Rightarrow$1. Suppose 3 holds. Let $ab \leq q$, $ab \nleq \phi(q)$ and $a \nleq \delta(q)$ for $a, b \in L$. Then as $L$ is compactly generated, there exist $x, x', y' \in L_+$ such that $x \leq a$, $x' \leq a$, $y' \leq b$, $x \nleq \delta(q)$ and $x'y' \nleq \phi(q)$. Let $y \leq b$ be any compact element of $L$. Then $(x \vee x'), (y \vee y') \in L_+$ such that $(x \lor x')(y \lor y') \leq q$, $(x \lor x')(y \lor y') \leq (\phi(q) : q)$ and $(x \lor x') \nleq \delta(q)$. So by 3, it follows that $(y \lor y') \leq q$ which implies $b \leq q$ and therefore $q$ is $\phi$-$\delta$-primary.

**Theorem 2.10.** A proper element $q \in L$ is $\phi$-$\delta$-primary if and only if for every $a \in L$ such that $a \nleq q$ either $(q : a) \leq \delta(q)$ or $(q : a) = (\phi(q) : a)$.

**Proof.** Assume that a proper element $q \in L$ is $\phi$-$\delta$-primary. Let $h \in L_+$ be such that $h \leq (q : a)$ and $a \nleq q$. Then $ah \leq q$.

If $ah \leq \phi(q)$, then $h \leq (\phi(q) : a)$. If $ah \nleq \phi(q)$, then since $q$ is $\phi$-$\delta$-primary and $a \nleq q$, it follows that $h \leq \delta(q)$. Hence by Lemma 2.8, either $(q : a) \leq (\phi(q) : a)$ or $(q : a) \leq \delta(q)$. But as $(\phi(q) : a) \leq (q : a)$ we have either $(q : a) \leq \delta(q)$ or $(q : a) = (\phi(q) : a)$. Conversely, assume that for every $a \in L$ such that $a \nleq q$, either $(q : a) \leq \delta(q)$ or $(q : a) = (\phi(q) : a)$.

Let $rs \leq q$, $rs \nleq \phi(q)$ and $r \nleq q$ for $r, s \in L$. Then either $(q : r) = (\phi(q) : r)$ or $(q : r) \leq \delta(q)$. If $(q : r) = (\phi(q) : r)$, then as $s \leq (q : r)$, it follows that $s \leq (\phi(q) : r)$ which contradicts $rs \nleq \phi(q)$ and so we must have $(q : r) \leq \delta(q)$. Therefore $s \leq (q : r)$ gives $s \leq \delta(q)$. Hence $q$ is $\phi$-$\delta$-primary.
Theorem 2.11. Let \((L, m)\) be a quasi-local Noether lattice. If a proper element \(p \in L\) is such that \(p^2 = m^2 \leq p \leq m\), then \(p\) is \(\phi_2 \delta_1\)-primary.

Proof. Let \(xy \leq p\) and \(xy \notin \phi_2(p)\) for \(x, y \in L\). If \(x \notin m\), then \(x = 1\). So \(xy \leq p\) gives \(y \leq p\). Similarly, \(y \notin m\) gives \(x \leq p\). Now if \(x \leq m\), then \(x^2 \leq m^2 = p^2 \leq p\) and hence \(x \leq \delta_1(p)\). Similarly, \(y \leq m\) gives \(y \leq \delta_1(p)\). Hence in any case, \(p\) is \(\phi_2 \delta_1\) primary.

To obtain the relation among \(\phi_n \delta_1\)-primary elements of \(L\), we prove the following lemma.

Lemma 2.12. Let \(\gamma_1, \gamma_2 : L \rightarrow L\) be functions such that \(\gamma_1 \leq \gamma_2\) and \(\delta\) be an expansion function on \(L\). Then every proper \(\gamma_1 \delta\)-primary element of \(L\) is \(\gamma_2 \delta\)-primary.

Proof. Let a proper element \(p \in L\) be \(\gamma_1 \delta\)-primary. Suppose \(ab \leq p\) and \(ab \notin \gamma_2(p)\) for \(a, b \in L\). Then as \(\gamma_1 \leq \gamma_2\), we have \(ab \leq p\) and \(ab \notin \gamma_1(p)\). Since \(p\) is \(\gamma_1 \delta\)-primary, it follows that either \(a \leq p\) or \(b \leq \delta(p)\) and hence \(p\) is \(\gamma_2 \delta\)-primary.

Theorem 2.13. For a proper element \(p\) of \(L\), consider the following statements:

(a). \(p\) is a \(\delta\)-primary element of \(L\).

(b). \(p\) is a \(\phi_0 \delta\)-primary element of \(L\).

(c). \(p\) is a \(\phi_n \delta\)-primary element of \(L\).

(d). \(p\) is a \(\phi(n+1) \delta\)-primary element of \(L\).

(e). \(p\) is a \(\phi_n \delta\)-primary element of \(L\) where \(n \geq 2\).

(f). \(p\) is a \(\phi_2 \delta\)-primary element of \(L\).

Then \((a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (f)\).

Proof. Obviously, every \(\delta\)-primary element of \(L\) is weakly \(\delta\)-primary and hence \((a) \implies (b)\). The remaining implications follow by using Lemma 2.12 to the fact that \(\phi_0 \leq \phi_1 \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2\).

Corollary 2.14. Let \(p \in L\) be a proper element. Then \(p\) is \(\phi_n \delta\)-primary if and only if \(p\) is \(\phi_n \delta\)-primary for every \(n \geq 2\).

Proof. Assume that \(p \in L\) is \(\phi_n \delta\)-primary for every \(n \geq 2\). Let \(ab \leq p\) and \(ab \notin \bigwedge_{n=1}^{\infty} p^n\) for \(a, b \in L\). Then \(ab \leq p\) and \(ab \notin p^n\) for some \(n \geq 2\). Since \(p\) is \(\phi_n \delta\)-primary, we have either \(a \leq p\) or \(b \leq \delta(p)\) and hence \(p\) is \(\phi_n \delta\)-primary. The converse follows from Theorem 2.13.

Now we show that under a certain condition, a \(\phi_n \delta\)-primary element of \(L\) \((n \geq 2)\) is \(\delta\)-primary.

Theorem 2.15. Let \(L\) be a local Noetherian domain. A proper element \(p \in L\) is \(\phi_n \delta\)-primary for every \(n \geq 2\) if and only if \(p\) is \(\delta\)-primary.

Proof. Assume that a proper element \(p \in L\) is \(\phi_n \delta\)-primary for every \(n \geq 2\). Let \(ab \leq p\) for \(a, b \in L\). If \(ab \notin \phi_n(p)\) for \(n \geq 2\), then as \(p \in L\) is \(\phi_n \delta\)-primary, we have \(a \leq p\) or \(b \leq \delta(q)\). If \(ab \leq \phi_n(p) = p^n\) for all \(n \geq 1\), then as \(L\) is local Noetherian, by Corollary 3.3 of [5], it follows that \(ab \leq \bigwedge_{n=1}^{\infty} p^n = 0\) and so \(ab = 0\). Since \(L\) is domain, we have either \(a = 0\) or \(b = 0\) which implies either \(a \leq p\) or \(b \leq \delta(q)\) and hence \(p\) is \(\delta\)-primary. Converse follows from Theorem 2.13.

Corollary 2.16. Let \(L\) be a local Noetherian domain. A proper element \(p \in L\) is \(\phi_2 \delta\)-primary if and only if \(p\) is \(\delta\)-primary.
Proof. The proof follows from Theorem 2.15 and Corollary 2.14.

Clearly, every $\delta$-primary element of $L$ is $\phi$-$\delta$-primary. The following example shows that its converse need not be true (by taking $\phi$ as $\phi_2$ and $\delta$ as $\delta_1$ for convenience).

Example 2.17. Consider the lattice $L$ of ideals of the ring $R = \langle \mathbb{Z}_{30} , + , \cdot \rangle$. Then the only ideals of $R$ are the principal ideals $(0),(2),(3),(5),(6),(10),(15),(1)$. Clearly $L = \{(0),(2),(3),(5),(6),(10),(15),(1)\}$ is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 2. It is easy to see that the element $(6) \in L$ is $\phi_2$-$\delta_1$-primary but not $\delta_1$-primary.

![Figure 2.](image)
$q \in L$ is $\phi_n$-δ-primary ($n$-almost δ-primary) and for all $n \geq 2$. Let $xy \leq q$ for $x, y \in L$. If $xy \not\leq \phi_n(q)$ for some $n \geq 2$, then as $q$ is $\phi_n$-δ-primary, we have either $x \leq q$ or $y \leq \delta(q)$ and we are done. So let $xy \leq \phi_n(q) = q^n$ for all $n \geq 2$. Consider $(x \lor q)y = xy \lor qy \leq q$. If $(x \lor q)y \not\leq \phi_n(q)$, then as $q$ is $\phi_n$-δ-primary, we have either $x \leq (x \lor q) \leq q$ or $y \leq \delta(q)$. So assume that $(x \lor q)y \leq \phi_n(q)$. Then $qy \leq q^n \leq q^2 \neq 0$ as $n \geq 2$. This implies $y \leq q \leq \delta(q)$ by Lemma 1.11 of [11]. Hence $q$ is δ-primary.

**Corollary 2.21.** Every non-zero and non-nilpotent $\phi_n$-δ-primary element ($\forall n \geq 2$) of a Noether lattice $L$ satisfying the restricted cancellation law is δ-primary.

**Proof.** The proof follows from proof of the Theorem 2.20.

**Definition 2.22.** A proper element $p \in L$ is said to be 2-potent δ-primary if for all $a, b \in L$, $ab \leq p^2$ implies either $a \leq p$ or $b \leq \delta(p)$.

Obviously, every 2-potent δ₀-primary element of $L$ is 2-potent prime and vice versa. Also, every 2-potent δ₀-primary element of $L$ is 2-potent δ-primary.

**Theorem 2.23.** Let a proper element $q \in L$ be 2-potent δ-primary. Then $q$ is $\phi$-δ-primary for some $\phi \leq \phi_2$ if and only if $q$ is δ-primary.

**Proof.** Assume that $q \in L$ is a δ-primary element. Then obviously, $q$ is $\phi$-δ-primary for every $\phi$ and hence for some $\phi \leq \phi_2$. Conversely, let $q \in L$ be $\phi$-δ-primary for some $\phi \leq \phi_2$. Then by Lemma 2.12, $q \in L$ is $\phi_2$-δ-primary (almost δ-primary). Let $xy \leq q$ for $x, y \in L$. If $xy \not\leq \phi_2(q)$, then as $q$ is $\phi_2$-δ-primary, we have either $x \leq q$ or $y \leq \delta(q)$. If $xy \leq \phi_2(q) = q^2$, then as $q$ is 2-potent δ-primary, we have either $x \leq q$ or $y \leq \delta(q)$. Hence $q$ is δ-primary.

**Corollary 2.24.** Every $\phi_2$-δ-primary element of $L$ which is 2-potent δ-primary is δ-primary.

**Proof.** The proof follows from proof of the Theorem 2.23.

**Theorem 2.25.** Let a proper element $q \in L$ be 2-potent δ₀-primary. Then $q$ is $\phi$-δ-primary for some $\phi \leq \phi_2$ if and only if $q$ is δ-primary.

**Proof.** The proof follows by using Theorem 2.23 to the fact that every 2-potent δ₀-primary element of $L$ is 2-potent δ-primary.

**Corollary 2.26.** Every $\phi_2$-δ-primary element of $L$ which is 2-potent δ₀-primary is δ-primary.

**Definition 2.27.** Let $n \geq 2$. A proper element $p \in L$ is said to be $n$-potent δ-primary if for all $a, b \in L$, $ab \leq p^n$ implies either $a \leq p$ or $b \leq \delta(p)$.

Obviously, every $n$-potent δ₀-primary element of $L$ is $n$-potent δ-primary.

The following result is general form of Theorem 2.23.

**Theorem 2.28.** A proper element $q \in L$ is $\phi$-δ-primary for some $\phi \leq \phi_n$, where $n \geq 2$ if and only if $q$ is δ-primary, provided $q$ is $k$-potent δ-primary for some $k \leq n$.

**Proof.** Assume that $q \in L$ is a δ-primary element. Then obviously, $q$ is $\phi$-δ-primary for every $\phi$ and hence for some $\phi \leq \phi_n$ where $n \geq 2$. Conversely, let $q \in L$ be $\phi$-δ-primary for some $\phi \leq \phi_n$ where $n \geq 2$. Then by Lemma 2.12, $q \in L$ is $\phi_n$-δ-primary ($n$-almost δ-primary). Let $xy \leq q$ for $x, y \in L$. If $xy \not\leq \phi_n(q) = q^k$, then $xy \not\leq \phi_n(q) = q^n$ as $k \leq n$. Since $q$ is $\phi_n$-δ-primary, we have either $x \leq q$ or $y \leq \delta(q)$. If $xy \leq \phi_n(q) = q^k$, then as $q$ is $k$-potent δ-primary, we have either $x \leq q$ or $y \leq \delta(q)$. Hence $q$ is δ-primary.
Corollary 2.29. Every $\phi_n$-$\delta$-primary element of $L$ which is $k$-potent $\delta$-primary is $\delta$-primary where $k \leq n$.

Theorem 2.30. Let a proper element $q \in L$ be $\phi$-$\delta$-primary. If $q^2 \not\subseteq \phi(q)$, then $q$ is $\delta$-primary.

\textbf{Proof.} Let $ab \leq q$ for $a, b \in L$. If $ab \not\subseteq \phi(q)$, then as $q$ is $\phi$-$\delta$-primary, we have either $a \leq q$ or $b \leq \delta(q)$. So assume that $ab \leq \phi(q)$. First suppose $aq \not\subseteq \phi(q)$. Then $ad \not\subseteq \phi(q)$ for some $d \leq q$ in $L$. Also $a(b \vee d) = ab \vee ad \leq q$ and $a(b \vee d) \not\subseteq \phi(q)$.

As $q$ is $\phi$-$\delta$-primary, either $a \leq q$ or $(b \vee d) \leq \delta(q)$. Hence either $a \leq q$ or $b \leq \delta(q)$. Similarly, if $bq \not\subseteq \phi(q)$, we can show that either $a \leq q$ or $b \leq \delta(q)$. So we can assume that $aq \subseteq \phi(q)$ and $bq \subseteq \phi(q)$. Since $q^2 \not\subseteq \phi(q)$, there exist $r, s \leq q$ in $L$ such that $rs \not\subseteq \phi(q)$. Then $(a \vee r)(b \vee s) \leq q$ but $(a \vee r)(b \vee s) \not\subseteq \phi(q)$. As $q$ is $\phi$-$\delta$-primary, we have either $(a \vee r) \leq q$ or $(b \vee s) \leq \delta(q)$. Therefore either $a \leq q$ or $b \leq \delta(q)$ and hence $q$ is $\delta$-primary.

From the Theorem 2.30, it follows that,

- if a proper element $q \in L$ is $\phi$-$\delta$-primary but not $\delta$-primary, then $q^2 \subseteq \phi(q)$,
- a $\phi$-$\delta$-primary element $q < 1$ of $L$ with $q^2 \not\subseteq \phi(q)$ is $\delta$-primary.

Clearly, given an expansion function $\delta$ on $L$, $\delta(p) \leq \delta(\delta(p))$ for each $p \in L$. Moreover, for each $p \in L$, $\delta(\delta(1)) = \delta(1)$, by property ($p3$) of radicals in [10]. Also, obviously $\delta_0(\delta_0(p)) = \delta_0(p)$ for each $p \in L$.

Now we present the consequences of the Theorem 2.30 in the form of following corollaries.

Corollary 2.31. If a proper element $q \in L$ is $\phi$-$\delta$-primary but not $\delta$-primary, then $\delta_1(q) = \delta_1(\phi(q))$.

\textbf{Proof.} By Theorem 2.30, we have $q^2 \subseteq \phi(q)$. So $q \leq \delta_1(\phi(q))$ which gives $\delta_1(q) \leq \delta_1(\delta_1(\phi(q))) = \delta_1(\phi(q))$. Since $\phi(q) \leq q$, we have $\delta_1(\phi(q)) \leq \delta_1(q)$. Hence $\delta_1(q) = \delta_1(\phi(q))$.

Corollary 2.32. If a proper element $q \in L$ is $\phi$-$\delta$-primary where $\phi \leq \phi_3$, then $q$ is $\phi_n$-$\delta$-primary for every $n \geq 2$.

\textbf{Proof.} If $q$ is $\delta$-primary, then by Theorem 2.13, $q$ is $\phi_3$-$\delta$-primary. So assume that $q$ is not $\delta$-primary. Then by Theorem 2.30 and by hypothesis, we get $q^2 \subseteq \phi(q) \leq q^3$. Hence $\phi(q) = q^n$ for every $n \geq 2$. Consequently, $q$ is $\phi_n$-$\delta$-primary for every $n \geq 2$.

Corollary 2.33. If a proper element $q \in L$ is $\phi$-$\delta$-primary where $\phi \leq \phi_3$, then $q$ is $\phi_n$-$\delta$-primary.

\textbf{Proof.} The proof follows from Corollary 2.32 and Corollary 2.14.

Corollary 2.34. If a proper element $q \in L$ is $\phi_0$-$\delta$-primary but not $\delta$-primary, then $q^2 = 0$.

\textbf{Proof.} The proof is obvious.

Theorem 2.35. Let $q$ be a $\phi$-$\delta$-primary element of $L$. If $\phi(q)$ is a $\delta$-primary element of $L$, then $q$ is $\delta$-primary.

\textbf{Proof.} Let $ab \leq q$ for $a, b \in L$. If $ab \not\subseteq \phi(q)$, then as $q$ is $\phi$-$\delta$-primary, we have either $a \leq q$ or $b \leq \delta(q)$ and we are done. Now if $ab \leq \phi(q)$, then as $\phi(q)$ is $\delta$-primary, we have either $a \leq \phi(q)$ or $b \leq \delta(\phi(q))$. This implies that either $a \leq q$ or $b \leq \delta(q)$ because $\phi(q) \leq q$ and $\delta(\phi(q)) \leq \delta(q)$.

The next result shows that the join of a family of ascending chain of $\phi$-$\delta$-primary elements of $L$ is again $\phi$-$\delta$-primary.

Theorem 2.36. Let $\{p_i : i \in \Delta\}$ be a chain of $\phi$-$\delta$-primary elements of $L$ and let the function $\phi$ be such that $x \leq y$ imply $\phi(x) \leq \phi(y)$ for all $x, y \in L$. Then the element $p = \bigvee_{i \in \Delta} p_i$ is also $\phi$-$\delta$-primary.
Proof. Since $1 \in L$ is compact, $\forall p_i = p \neq 1$. Let $ab \leq p$, $ab \not\in \phi(p)$ and $a \not\in p$ for $a, b \in L$. Then as $\{p_i \mid i \in \Delta\}$ is a chain, we have $ab \leq p_i$ for some $i \in \Delta$ but $a \not\in p_i$ and $ab \not\in \phi(p_i)$ because for each $k \in \Delta$, we have $p_k \leq p$ and this implies $\phi(p_k) \leq \phi(p)$. As each $p_i$ is $\phi$-primary, it follows that $b \leq \delta(p_i)$. Since $p_i \leq p$, we have $\delta(p_i) \leq \delta(p)$ and so $b \leq \delta(p)$. Hence $p$ is $\phi$-primary.

The following theorem shows that a under certain condition, $(p : q) \in L$ is $\phi$-primary if $p \in L$ is $\phi$-primary element where $q \in L$.

**Theorem 2.37.** Let a proper element $p \in L$ be $\phi$-primary. Then $(p : q)$ is $\phi$-primary for all $q \in L$ if $(\phi(p) : q) \leq \phi(p : q)$.

**Proof.** Clearly, $pq \leq p$ implies $p \leq (p : q)$ and so $\delta(p) \leq (p : q)$. Now let $ab \leq (p : q)$, $ab \not\in \phi(p : q)$ and $a \not\in (p : q)$ for $a, b \in L$. Then $abq \leq p$, $abq \not\in \phi(p)$ and $aq \not\in p$ since $ab \not\in (\phi(p) : q)$. Now as $p$ is $\phi$-primary, we have $b \leq \delta(p) \leq \delta(p : q)$ and hence $(p : q)$ is $\phi$-primary.

In the next result, we show that under a certain condition $\delta_1(p) \leq \delta(p)$, for every $\phi$-primary element $p \in L$.

**Theorem 2.38.** If a proper element $p \in L$ is $\phi$-primary element such that $\delta_1(\phi(p)) \leq \delta(p)$, then $\delta_1(p) \leq \delta(p)$.

**Proof.** Assume that a proper element $p \in L$ is $\phi$-primary. For $a \leq L$, let $a \leq \delta_1(p) = \sqrt{\beta}$. Then there exists a least positive integer $k$ such that $a^k \leq p$. If $k = 1$, then $a \leq p \leq \delta(p)$. Now let $k > 1$. If $a^k \not\in \phi(p)$, then $a \leq \delta_1(\phi(p)) \leq \delta(p)$. So let $a^k \not\in \phi(p)$. Clearly, $a^{k-1}a \leq p$ where $a^{k-1} \notin p$. As $p \in L$ is $\phi$-primary, it follows that $a \leq \delta(p)$. Thus in any case, we have $\delta_1(p) \leq \delta(p)$.

Note that, if $p \in L$ is $\delta$-primary, then by consequence of Theorem 2.5 of [8], we have $\phi(p) \leq p$ implies $\delta_1(\phi(p)) \leq \delta_1(p) \leq \delta(p)$ and hence $\delta_1(\phi(p)) \leq \delta(p)$.

**Corollary 2.39.** If a proper element $p \in L$ is $\phi$-primary element such that $\delta_1(\phi(p)) \leq \delta(p)$ with $\delta(p) \leq \delta_1(p)$, then $\delta_1(p) = \delta(p)$.

**Proof.** The proof follows from Theorem 2.38.

According to [8], an expansion function $\delta$ on $L_1$ and on $L_2$ is said to have global property if for any lattice isomorphism $f : L_1 \rightarrow L_2$, $\delta(f^{-1}(a)) = f^{-1}(\delta(a))$ for all $a \in L_2$ where $L_1$ and $L_2$ are multiplicative lattices. Similarly, now we define global property of a function $\phi$ on multiplicative lattices.

**Definition 2.40.** Let $L_1$ and $L_2$ be multiplicative lattices. A function $\phi$ on $L_1$ and on $L_2$ is said to have **global property** if for any lattice isomorphism $f : L_1 \rightarrow L_2$, $\phi(f^{-1}(a)) = f^{-1}(\phi(a))$ for all $a \in L_2$.

**Lemma 2.41.** Let the function $\beta$ on $L_1$ and on $L_2$ have the global property where $L_1$ and $L_2$ are multiplicative lattices. If the function $g : L_1 \rightarrow L_2$ is a lattice isomorphism, then $g(\beta(q)) = \beta(g(q))$ for all $q \in L_1$.

**Proof.** For $q \in L_1$, the global property of $\beta$ gives $\beta(q) = \beta(g^{-1}(g(q))) = g^{-1}(\beta(g(q)))$. Then since $g$ is onto, we have $g(\beta(q)) = \beta(g(q))$.

The next result shows that if $q \in L$ is $\phi$-primary with some conditions on $\delta$ and $\phi$, then $\delta(q) \in L$ is $\phi$-prime.

**Theorem 2.42.** Let the expansion function $\delta$ on $L$ be a lattice isomorphism. Let the function $\phi$ on $L$ have the global property. If a proper element $q \in L$ is $\phi$-primary and satisfies $\delta(\delta(q)) \leq \delta(q)$, then $\delta(q)$ is a $\phi$-prime element of $L$. 
Proof. By Lemma 2.41, we have $\delta(\phi(q)) = \phi(\delta(q))$. Let $xy \leq \delta(q)$, $xy \notin \phi(\delta(q)) = \delta(\phi(q))$ and $x \notin \delta(q)$ for $x, y \in L$. So $\delta^{-1}(x) \cdot \delta^{-1}(y) = \delta^{-1}(xy) \leq q$, $\delta^{-1}(x) \cdot \delta^{-1}(y) = \delta^{-1}(xy) \notin \phi(q)$ and $\delta^{-1}(x) \leq q$. As $q$ is $\phi$-$\delta$-primary, we have $\delta^{-1}(y) \leq \delta(q)$ which implies $y \leq \delta(\delta(q)) \leq \delta(q)$ and hence $\delta(q)$ is a $\phi$-prime element of $L$.

Note that, in the Theorem 2.42, the idea behind taking the expansion function $\delta$ on $L$ as a lattice isomorphism and the function $\phi$ on $L$ with the global property is to get $\delta(\phi(q)) = \phi(\delta(q))$. The following theorem is a similar version of Theorem 2.42.

**Theorem 2.43.** If a proper element $q \in L$ is $\phi$-$\delta_1$-primary such that $\delta_1(\phi(q)) = \phi(\delta_1(q))$, then $\delta_1(q)$ is a $\phi$-prime element of $L$.

**Proof.** Assume that $ab \leq \delta_1(q)$, $ab \notin \phi(\delta_1(q))$ and $a \notin \delta_1(q)$ for $a, b \in L$. Then there exists $n \in \mathbb{Z}_+^+$ such that $a^n \cdot b^n = (ab)^n \leq q$. If $(ab)^n \leq \phi(q)$, then by hypothesis $ab \leq \delta_1(\phi(q)) = \phi(\delta_1(q))$, a contradiction. So we must have $a^n \cdot b^n = (ab)^n \notin \phi(q)$. Since $q$ is $\phi$-$\delta_1$-primary and $a^n \notin q$ for all $n \in \mathbb{Z}_+$, we have $b^n \leq \delta_1(q)$ and hence $b \leq \delta_1(\delta_1(q)) = \delta_1(q)$. This shows that $\delta_1(q)$ is a $\phi$-prime element of $L$.

**Lemma 2.44.** Let the expansion function $\delta$ on $L_1$ and on $L_2$ have the global property where $L_1$ and $L_2$ are multiplicative lattices. Let the function $\phi$ on $L_1$ and on $L_2$ have the global property. If $f : L_1 \rightarrow L_2$ is a lattice isomorphism, then for any $\phi$-$\delta$-primary element $p \in L_2$, $f^{-1}(p) \in L_1$ is $\phi$-$\delta$-primary.

**Proof.** Assume that a proper element $p \in L_2$ is $\phi$-$\delta$-primary. Let $ab \leq f^{-1}(p), ab \notin \phi(f^{-1}(p)) = f^{-1}(\phi(p))$ and $a \notin f^{-1}(p)$ for $a, b \in L_1$. Then $f(ab) = f(a) \cdot f(b) \leq p, f(ab) = f(a) \cdot f(b) \notin \phi(p)$ and $f(a) \notin p$. As $p$ is $\phi$-$\delta$-primary, we have $f(b) \leq \delta(p)$.

Now the global property of $\delta$ gives $b \leq f^{-1}(\delta(p)) = \delta(f^{-1}(p))$ showing that $f^{-1}(p) \in L_1$ is $\phi$-$\delta$-primary.

The following result gives another characterization of $\phi$-$\delta$-primary elements of $L$.

**Theorem 2.45.** Let the expansion function $\delta$ on $L_1$ and on $L_2$ have the global property where $L_1$ and $L_2$ are multiplicative lattices. Let the function $\phi$ on $L_1$ and on $L_2$ have the global property. Let $f : L_1 \rightarrow L_2$ be a lattice isomorphism. Then a proper element $a \in L_1$ is $\phi$-$\delta$-primary if and only if $f(a) \in L_2$ is $\phi$-$\delta$-primary.

**Proof.** Assume that a proper element $a \in L_1$ is $\phi$-$\delta$-primary. Clearly, by Lemma 2.41, the global property of $\delta$ gives $f(\delta(a)) = \delta(f(a))$. Also, by Lemma 2.41, the global property of $\phi$ gives $f(\phi(a)) = \phi(f(a))$. Now, let $xy \leq f(a), xy \notin \phi(f(a))$ and $x \notin f(a)$ for $x, y \in L_2$. Then there exists $b, c \in L_1$ such that $f(b) = x, f(c) = y$. So $f(bc) = f(b) \cdot f(c) = xy \leq f(a), f(bc) = f(b) \cdot f(c) = xy \notin \phi(f(a)) = f(\phi(a))$ and $f(b) = x \notin f(a)$. As $a$ is $\phi$-$\delta$-primary in $L_1, bc \leq a, bc \notin \phi(a)$ and $b \notin a$, we have $c \leq \delta(a)$. So $y = f(c) \leq f(\delta(a))$ and hence $y \leq \delta(f(a))$ showing that $f(a) \in L_2$ is $\phi$-$\delta$-primary. The converse follows from Lemma 2.44.

Now we relate idempotent element of $L$ with $\phi_n$-$\delta$-primary element ($n \geq 2$) of $L$.

**Theorem 2.46.** Every idempotent element of $L$ is $\phi_n$-$\delta$-primary and hence $\phi_n$-$\delta$-primary ($n \geq 2$).

**Proof.** Let $p$ be an idempotent element of $L$. Then $p = p^n$ for all $n \in \mathbb{Z}_+$. So $\phi_n(p) = p$. Therefore $p$ is a $\phi_n$-$\delta$-primary of $L$. Hence $p$ is a $\phi_n$-$\delta$-primary element ($n \geq 2$) of $L$ by Theorem 2.13.

As a consequence of Theorem 2.46, we have following result whose proof is obvious.

**Corollary 2.47.** Every idempotent element of $L$ is $\phi_2$-$\delta$-primary.
However, a $\phi_2\delta_1$-primary element of $L$ need not be idempotent as shown in the following example (by taking $\delta$ as $\delta_1$ for convenience).

**Example 2.48.** Consider the lattice $L$ of ideals of the ring $R = \langle \mathbb{Z}_8, +, \cdot \rangle$. Then the only ideals of $R$ are the principal ideals $(0),(2),(4),(1)$. Clearly, $L = \{(0),(2),(4),(1)\}$ is a compactly generated multiplicative lattice. Its lattice structure and multiplication table is as shown in Figure 3. It is easy to see that the element $(4) \in L$ is $\phi_2\delta_1$-primary but not idempotent.

![Figure 3](image)

We conclude this paper with the following examples, from which it is clear that,

1. A $\phi_2\delta_1$-primary element of $L$ need not be 2-potent $\delta_0$-primary,
2. A 2-potent $\delta_0$-primary element of $L$ which is $\phi_2\delta_1$-primary need not be prime.

**Example 2.49.** Consider $L$ as in Example 2.17. Here the element $(6) \in L$ is $\phi_2\delta_1$-primary but not 2-potent $\delta_0$-primary.

**Example 2.50.** Consider $L$ as in Example 2.48. Here the element $(4) \in L$ is 2-potent $\delta_0$-primary, $\phi_2\delta_1$-primary but not prime.

**Acknowledgement**

The author would like to thank the referee for careful reading of the manuscript. Also the author wish to thank the referee for his assistance in making this paper accessible to a broader audience.

**References**


