An Extension of Hadamard-Rybezynski Solution

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Abstract: This paper is concerned with the motion of a liquid drop surrounded by a shell of immiscible liquid, having the same density but different viscosities, and both being embedded in a liquid medium differing from them in both density and viscosity. This is an extension of Hadamard and Rybezynski Levich [7].

Keywords: Slow viscous flow, drops and bubbles.

1. Introduction

The study of liquid drop having density \(\rho\) and viscosity \(\mu\) surrounded by a concentric liquid shell having the same density and viscosity \(\mu\), both being immersed in a uniform stream of viscosity \(U\) is considered here.

2. Formulation of the Problem

A viscous drop of radius \(b\) and a concentric liquid shell in a uniform stream of velocity \(U\) are considered. The Reynolds number of the flow in the three regions \([I, II, III]\), as shown in figure 1, is assumed to be small. Consequently the equation of motions in three regions are taken to be given by tokes flow for slow motion, namely

\[-\nabla \rho + \mu \nabla^2 v = 0\]  \hspace{1cm} (1)

And the equation of continuity

\[\nabla \cdot v = 0\]  \hspace{1cm} (2)

Spherical polar coordinates \((r, \theta, \phi)\) are used, and since the flow is axisymmetric

\[v = (v_r, v_\theta, 0)\]  \hspace{1cm} (3)

At large distance from origin the flow is uniform which implies

\[\begin{align*}
v_r &\approx U \cos \theta \\
v_\theta &\approx -U \sin \theta
\end{align*}\]  \hspace{1cm} (4)

The liquid drop, the liquid shell and the external uniform stream will be called \(III, II\) and \(I\) which implies respectively. The boundary conditions at the surfaces of the liquid drop and sell are:

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(i). Continuity of pressure.

(ii). The normal components both inner and outer velocities must be zero.

(iii). The component of the rate of strain tensor $e_{r\theta}$ must be continuous, where

$$e_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta}$$

(iv). The velocity must be finite in region (III).

(v). The total pressure in the outer liquid (region I) is $(p - \pi_1)$. The boundary conditions for the normal and tangential stress on $r = a$ are:

$$- (p - \pi_1) + 2\mu \left( \frac{\partial v_r}{\partial r} \right) = -p' + 2\mu' \left( \frac{\partial v_r'}{\partial r} \right) \quad \text{at } r = a
\mu \left( \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{v_r'}{r} \right) - v_\theta \frac{\partial v_r}{\partial \theta} = \mu' \left( \frac{1}{r} \frac{\partial v_\theta'}{\partial r} + \frac{v_r'}{r} \right) - v_\theta' \frac{\partial v_r'}{\partial \theta} \quad \text{at } r = a$$

where $\pi = (p - p') gx$ and at $r = b$ are:

$$- (p - \pi_1) + 2\mu \left( \frac{\partial v_r}{\partial r} \right) = -p'' + 2\mu'' \left( \frac{\partial v_r''}{\partial r} \right) \quad \text{at } r = a
\mu' \left( \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{v_r'}{r} \right) - v_\theta \frac{\partial v_r}{\partial \theta} = \mu'' \left( \frac{1}{r} \frac{\partial v_\theta''}{\partial r} + \frac{v_r'}{r} \right) - v_\theta' \frac{\partial v_r'}{\partial \theta} \quad \text{at } r = a$$

where $\pi_2 = (p' - p'') gx$. Notice that $p' = p''$ at $r = b$.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure1.png}
  \caption{Drop and sell immersed in fluid}
  \end{figure}

### 3. Method of Solution

The solution of equation (1) and (2) which satisfy the above boundary conditions can be found by standard techniques so that

In Region (I)

$$v_r = \left[ \frac{A}{r^3} + \frac{B}{r^2} - C + Dr^2 \right] \cos \theta$$

$$v_\theta = \left[ \frac{A}{r^3} + \frac{B}{r^2} - C - 2Dr^2 \right] \sin \theta$$

$$p = \frac{\mu B}{r^2} \cos \theta$$

\[ (8) \]
In Region (II)

\[
\begin{align*}
\nu' &= \left[ A' + B' - C + D' r^2 \right] \cos \theta \\
\nu'_{\theta} &= \left[ A' + B' - C - 2D' r^2 \right] \sin \theta \\
\nu &= \mu' \left( B' + 10D' r \right) \cos \theta
\end{align*}
\]

In Region (III)

\[
\begin{align*}
\nu'' &= \left[ A'' + B'' - C + D'r^2 \right] \cos \theta \\
\nu''_{\theta} &= \left[ A'' + B'' - C - 2D'r^2 \right] \sin \theta \\
\nu'' &= \mu'' \left( 10D'' r \right) \cos \theta
\end{align*}
\]

Now solving equations (6) - (10) for the arbitrary constants and with the appropriate substitutions we obtain, after lengthy calculations, the velocities and pressure components. Applying the boundary conditions (i)-(v), one finally obtains

\[
U = \frac{2 \left( p - p' \right) g a^2}{3\mu} \frac{\left( 3\mu'' + 2\mu' \right) \left( \mu + \mu' \right) a^5 - \left( \mu' - \mu'' \right) \left( 2 \mu' - 3 \mu \right) b^5}{\left( 3\mu'' + 2\mu' \right) \left( 3\mu'' + 2\mu' \right) a^5 - 6 \left( \mu' - \mu'' \right) \left( \mu' - \mu'' \right) b^5}
\]

Which is an extension to Hadamard-Rybczynski Solution.

4. Conclusion

On putting \( \mu = \mu' \) or \( b = 0 \), in equation (11), one recovers the solution

\[
U = \frac{2 \left( p - p' \right) g a^2 \left( \mu + \mu' \right)}{3\mu \left( 3\mu'' + 2\mu' \right)}
\]

This conjecture may have far reaching effects in multi-phase flows, or other related ones, such as in the velocity of rise or fall, and on the drag force on the drop.

References