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The Approximation of Laplace-Stieltjes Transforms in the Half Plane

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Abstract: In this paper, we study the growth of the analytic function represented by Laplace-Stieltjes transform of infinite order which is convergent in the right half plane. We also investigate the error in approximation defined on Laplace-Stieltjes transform of finite γ_U-order in the half plane, and some relations between the error and growth of Laplace-Stieltjes transform of finite γ_U-order.

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1. Introduction

Let Laplace-Stieltjes transform

$$F(s) = \int_0^{\infty} e^{-sx} d\alpha(x), \quad s = \sigma + it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y]$ $(0 < Y < +\infty)$, σ and t are real variables. If $\alpha(t)$ is a step function and satisfies,

$$\alpha(t) = \begin{cases} a_1 + a_2 + \cdots + a_n, & \lambda_n < x < \lambda_{n+1} \\ 0, & 0 \leq x < \lambda_1 \\ \frac{\alpha(x+)+\alpha(x-)}{2}, & x > 0 \end{cases}$$

where the sequence $\{\lambda_n\}_{n=0}^{\infty}$ satisfies

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots, \quad \lambda_n \longrightarrow \infty \quad as \quad n \longrightarrow \infty, \tag{2}$$

where $\alpha(x)$ is stated in (1) and $\{\lambda_n\}$ satisfy (2),

$$\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty, \quad \limsup_{n\to\infty} \frac{n}{\lambda_n} = E < +\infty. \tag{3}$$

Set

$$A_n^* = \sup_{\lambda_n \leq x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|,$$

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$$\limsup_{n\to\infty}\frac{\log A_n^*}{\lambda_n}=0. \tag{4}$$

Thus, $F(s)$ becomes Dirichlet series,

$$F(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_n s},\quad s=\sigma+it, \tag{5}$$

where σ,t are real variables and a_n are non-zero complex numbers.

The author studied the growth and value distribution of Laplace-Stieltjes transform (1) in 1963, J. R. Yu [9], and we get Valiron-Knopp-Bohr formula with associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transform and to investigate the singular direction-borel line of Laplace-Stieltjes transform. After his work, some mathematician investigated properties on the growth and the value distribution of Laplace-Stieltjes transforms in ([3, 5, 6, 14, 16, 17]) and J. R. Yu, L. N. Shang, Z. S. Gao, and H. Y. Xu investigated the value distribution of such functions ([7–9, 11]). Furthermore, for Dirichlet series (3), a special form of Laplace-Stieltjes transform, authors paid considerable attention to the growth and value distribution of analytic functions defined by Dirichlet series. They founded many interesting results in ([1, 2, 4, 10, 12, 13, 15, 18–20]).

In 1963, Yu [9] proved the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace-Stieltjes tranforms;

Theorem 1.1 ([9])**.** *Suppose that Laplace-Stieltjes transform (1) satisfy (3) and*

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}<+\infty,$$

then

$$\limsup_{n\to\infty}\frac{\log A_n^*}{\lambda_n}<\sigma_u^F\le\limsup_{n\to\infty}\frac{\log A_n^*}{\lambda_n}+\limsup_{n\to\infty}\frac{\log n}{\lambda_n},$$

where σ_u^F is called the abscissa of uniformly convergent.

It follows that from (3), (4) and Theorem 1.1 such that $\sigma_u^F=0$, i.e., $F(s)$ is analytic in the right half plane. Put

$$\begin{aligned}
M(\sigma,F)&=\sup_{-\infty<t<+\infty}|F(\sigma+it)|,\\
\mu(\sigma,F)&=\max_{n\in N}\{A_n^*e^{-\lambda_n\sigma}\},\\
M_u(\sigma,F)&=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{-(\sigma+it)y}d\alpha(y)\right|\quad(\sigma>0).
\end{aligned}$$

Definition 1.2 ([7])**.** *If the Laplace-Stieltjes transform (1) satisfy $\sigma_u^F=0$, then*

$$\limsup_{\sigma\to\infty}\frac{\log^+\log M_u(\sigma,F)}{-\log\sigma}=\rho,$$

we call $F(s)$ is of order ρ in the right half plane, where $\log^+ x=\max(\log x,0)$.

For $\rho=\infty$, we get the definition of γ-order of Laplace-Stieltjes transform (1) as follows that.

Definition 1.3 ([8])**.** *If Laplace-Stieltjes transform (1) of γ-order satisfy,*

$$\limsup_{\sigma\to0^+}\frac{\gamma(\log^+ M_u(\sigma,F))}{-\log\sigma}=\rho_\gamma,$$

where $\gamma(x)\in\Im$, then ρ_γ is called the γ-order of $F(s)$, and $\Im$ is the class of all functions $\gamma(x)$ satisfies the following conditions:

(i). $\gamma(x)$ is positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to +\infty$and is defined on $[a, \infty), a > 0$,

(ii). $x\gamma^{'}(x) = o(1)$ as $x \to +\infty$.

Theorem 1.4 ([8]). *Let Laplace-Stieltjes transformation$F(s) \in \overline{L_\beta}$ of infinite order has finite γ-order ρ_γ $(0 < \rho_\gamma < +\infty)$ and the sequence (2) satisfies (3) and (4), then*

$$\limsup_{\sigma \to 0^+} \frac{\gamma(\log \mu(\sigma, F))}{-\log \sigma} = \rho_\gamma \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log M_u(\sigma, F))}{-\log \sigma} = \rho_\gamma,$$

Theorem 1.5 ([8]). *Let Laplace-Stieltjes transformation $F(s) \in \overline{L_\beta}$ of infinite order has finite γ-order ρ_γ $(0 < \rho_\gamma < +\infty)$ and the sequence (2) satisfies (3) and (4), then*

$$\limsup_{n \to \infty} \frac{\gamma(\lambda_n)}{\log \lambda_n - \log^+ \log^+ A_n^*} = \rho_\gamma \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log M_u(\sigma, F))}{-\log \sigma} = \rho_\gamma.$$

Theorem 1.6 ([4]). *Let Laplace-Stieltjes transform $F(s) \in \overline{L_\beta}$ is of infinite γ-order, then*

$$\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} = T \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ \mu(\sigma, F))}{\log U(\frac{1}{\sigma})} = T$$

where $0 < T < \infty$ and $U(x) = x^{\rho(x)}$ satisfies the following conditions,

(i). $\rho(x)$ is monotone and $\lim_{x \to \infty} \rho(x) = \infty$;

(ii). $\lim_{x \to \infty} \frac{\log U(x')}{\log U(x)} = 1$,where $x' = x(1 + \frac{1}{\log U(x)})$.

Definition 1.7 ([7]). *Let Laplace-Stieltjes transform $F(s)$ of infinite order has infinite γ-order and satisfies,*

$$\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} = T,$$

then T is called γ_U-order of Laplace-stieltjes transform $F(s)$.

We denote L to be the class of all the functions $F(s)$ of the form (1) which is analytic in the half plane $Re(s) > 0$ and the sequence $\{\lambda_n\}$ satisfies (2), (3) and (4), and denote $\overline{L_\beta}$ to the class of all the functions $F(s)$ of the form (1) which is analytic in the half plane $Re(s) \le \beta(-\infty < \beta < +\infty)$ and the sequence $\{\lambda_n\}$ satisfies (2) and (3).
Thus, if $-\infty < \beta < 0$ and $F(s) \in L$, then $F(s) \in \overline{L_\beta}$; if $0 < \beta < +\infty$ and $F(s) \in \overline{L_\beta}$ then $F(s) \in L$. If $A_n^* = 0$ for $n \ge k+1$, and $A_n^* \ne 0$, then $F(s)$ will be called an exponential polynomial of degree k usually denoted by P_k i.e., $P_k(s) = \int_0^{\lambda_k} \exp(-sy) d\alpha(y)$. Since, $F(s)$ is an analytic in the half plane, $H = \{s = \sigma + it, \sigma > 0, t \in \Re\}$. We denote $\prod_n$ to the class of all exponential polynomial of degree n i.e., $\prod_n = \{\sum_{i=1}^n b_i \exp(-s\lambda_i); (b_1, b_2, \dots, b_n) \in \mathbb{C}^n\}$.
For $F(s) \in \overline{L_\beta}, -\infty < \beta < +\infty$, we denote $E_n(F, \beta)$ be the error in approximating the function $F(s)$ by exponential polynomial of degree n in the uniform norms.

$$E_n(F, \beta) = \inf_{P \in \prod_n} \|F - P\|_\beta, n = 1, 2, \dots$$

where

$$\|F - P\|_\beta = \max_{-\infty < t < +\infty} |F(\beta + it) - P(\beta + it)|.$$

The authors ([2, 11]) investigated the approximation of analytic function defined by Laplace-Stieltjes transforms of finite order. In this paper, we study the approximation of analytic function defined by Laplace-Stieltjes transform and obtain relation between the error $E_n(F, \beta)$ and growth order of $F(s)$, when $F(s)$ is of infinite order.
To prove our results we use the following Lemma's;

Lemma 1.8 ([7]). *Let $\gamma(x) \in \Im$ and c be a constant, and $\psi(x)$ be the function such that*

$$\limsup_{x\to+\infty} \frac{log^+\psi(x)}{logx} = \rho, \quad (0 \le \rho < \infty)$$

and if the real function $M(x)$ satisfies

$$\limsup_{x\to+\infty} \frac{\gamma(logM(x))}{logx} = \nu(>0).$$

Then we have

$$\limsup_{x\to+\infty} \frac{\gamma(logM(x)+c)}{logx} = \nu, \qquad \limsup_{x\to+\infty} \frac{\gamma(\psi(x)logM(x))}{logx} = \nu$$

Lemma 1.9 ([7]). *If the abscissa $\sigma_u^F = 0$, of the uniform convergent Laplace-Stieltjes transformation and the sequence (2) satisfies (3), then for any given $\epsilon \in (0,1)$, and for $\sigma(>0)$ sufficiently reaching 0 we have*

$$\frac{1}{3}\mu(\sigma, F) \le M_u(\sigma, F) \le K(\epsilon)\mu((1-\epsilon)\sigma, F)\frac{1}{\sigma}.$$

Where $K(\epsilon)$ is a constant depending on ϵ.

2. Main Results

Theorem 2.1. *Let Laplace-Stieltjes transform $F(s) \in \overline{L_\beta}$ of infinite order has infinite γ-order, then*

$$\limsup_{\sigma\to0^+} \frac{\gamma(\log^+ M_u(\sigma,F))}{\log U(\frac{1}{\sigma})} = T \iff \limsup_{n\to\infty} \frac{\gamma(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = T,$$

Proof. We want to proof only sufficient part.

Suppose that

$$\limsup_{n\to\infty} \frac{\gamma(\log^+ A_n^*)}{\log U\left(\frac{\lambda_n}{log^+ A_n^*}\right)} = T. \tag{6}$$

Then, for any positive real number $\epsilon > 0$, for sufficiently large n, we have

$$\log^+ A_n^* < J\left((T+\epsilon)\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)\right),$$

where $J(x)$ is the inverse of $\gamma(x)$. Let $V(x)$ is the inverse function of $U(x)$, then

$$\frac{\gamma(\log^+ A_n^*)}{T+\epsilon} < \log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)$$
$$\log A_n^* < \frac{\lambda_n}{V\left(\exp\left(\frac{\gamma(\log^+ A_n^*)}{T+\epsilon}\right)\right)}$$
$$\log A_n^* < \lambda_n \left[V\left(\exp\left(\frac{\gamma(\log^+ A_n^*)}{T+\epsilon}\right)\right)\right]^{-1}.$$

Thus, we have

$$\log A_n^* e^{-\lambda_n\sigma} < \lambda_n \left[\left(V\left(\exp\left(\frac{\gamma(\log^+ A_n^*)}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right]. \tag{7}$$

For any fixed and sufficiently small $\sigma > 0$, set

$$I = J\left[(T+\epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)\right]$$

$$\gamma(I) = (T+\epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)$$
$$\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})} = V\left(\exp\left(\frac{\gamma(I)}{T+\epsilon}\right)\right). \tag{8}$$

If $\log A_n^* \leq I$, then for sufficiently large n, let $V\left(\exp\left(\frac{\gamma(I)}{T+\epsilon}\right)\right) \geq 1$, for $\sigma > 0$, from (7), (8) and definition of $U(x)$, we get

$$\log^+(A_n^* e^{-\lambda_n \sigma}) \leq I\left[\left(V\left(\exp\left(\frac{\gamma(\log^+ A_n^*)}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right]$$
$$\leq J\left((T+\epsilon)\log\left((1+o(1))U\left(\frac{1}{\sigma}\right)\right)\right) \tag{9}$$

If $log^+ A_n^* > I$ then from (7) and (8), we get

$$\log(A_n^* e^{-\lambda_n \sigma}) \leq \lambda_n\left(\left(V\left(\exp\left(\frac{\gamma(\log^+ A_n^*)}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right)$$
$$< 0 \tag{10}$$

For sufficiently large n from (9) and (10), we get

$$\log\ \mu(\sigma, F) \leq J\left((T+\epsilon)\log\left((1+o(1))U\left(\frac{1}{\sigma}\right)\right)\right)$$
$$\leq J\left((T+\epsilon)\log U\left(\frac{1}{\sigma}\right)\right).$$

Since ϵ is arbitrary by Theorem C and Lemma 2.2, we get

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} \leq \limsup_{n\to\infty} \frac{\gamma(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = T.$$

Suppose that

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} < T. \tag{11}$$

Then, there exist a real number $\epsilon(0 < \epsilon < \frac{T}{2})$. For any positive number n and sufficiently small $\sigma > 0$ from Lemma 1.2, we have

$$\log^+(A_n^* e^{-\lambda_n \sigma}) \leq \log M_u(\sigma, F) \leq J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma}\right)\right). \tag{12}$$

From (6), there exist a subsequence $\{n(p)\}$ for sufficiently large p, we have

$$\gamma(\log^+ A_{n(p)}^*) > (T-\epsilon)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*}\right). \tag{13}$$

Taking a sequence $\{\sigma_p\}$ satisfy

$$J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right) = \frac{\log^+(A_{n(p)}^*)}{1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*}\right)}. \tag{14}$$

From (12) and (13), we get

$$\log A_{n(p)}^* - \lambda_{n(p)}\sigma_p \ \leq\ J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right) = \frac{\log^+(A_{n(p)}^*)}{1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*}\right)},$$

that is,

$$\frac{1}{\sigma_p} \leq \frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}} \left(1 + \frac{1}{\log U\left(\frac{\lambda_{n(p)}}{\log A^*_{n(p)}}\right)}\right).$$

$$\begin{aligned} U\left(\frac{1}{\sigma_p}\right) &\leq U\left(\frac{\lambda_{n(p)}}{log^+ A^*_{n(p)}}\left(1+\frac{1}{logU\left(\frac{\lambda_{n(p)}}{logA^*_{n(p)}}\right)}\right)\right) \\ &\leq U^{(1+o(1))}\left(\frac{\lambda_{n(p)}}{log^+ A^*_{n(p)}}\right). \end{aligned} \tag{15}$$

From (14) and (15), we get

$$\begin{aligned} \log^+(A^*_{n(p)}) &= J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right) \\ &= J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right). \end{aligned}$$

Thus, from the Cauchy mean value theorem and there exist a real number ξ between x_1 and x_2, where

$$\begin{aligned} x_1 &= J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right) \quad and \\ x_2 &= J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right) \end{aligned}$$

such that

$$\begin{aligned} \gamma(log^+(A^*_{n(p)})) &= \gamma\left\{\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right\} \\ &= \gamma\left(J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right) + \log\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\xi\gamma'(\xi) \end{aligned}$$

Since,

$$\lim_{p\to\infty} \frac{\log\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)}{\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)} = 0.$$

Then for sufficiently large p, we have

$$\gamma(\log^+(A^*_{n(p)})) = (T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right) + K_1\xi\gamma'(\xi)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right), \tag{16}$$

where K_1 is a constant. From (13) and (16), we get a contradiction. Thus,

$$\limsup_{\sigma\to 0^+} \frac{\gamma(logM_u(\sigma,F))}{logU\left(\frac{1}{\sigma}\right)} = T.$$

Hence, the sufficient part is completed. The necessary part is similar. □

Now we establish some relation between $E_n(F,\beta)$ and growth of $F(s)$.

Theorem 2.2. *Let Laplace-Stieljes transform $F(s) \in \overline{L_\beta}, (0 < \beta < +\infty)$ is of order ρ, then*

$$\rho = \limsup_{n\to\infty} \frac{log^+log^+(E_n(F,\beta)e^{\beta\lambda_{n+1}}}{log^+\lambda_{n+1} - log^+log^+(E_n(F,\beta)e^{\beta\lambda_{n+1}}}.$$

Theorem 2.3. *Let Laplace-Stieltjes transform $F(s) \in \overline{L_\beta}$ is of finite γ-order ρ_γ, for any real number $0 < \beta < +\infty$, then*

$$\limsup_{n\to\infty} \frac{\gamma(\lambda_n)}{log\lambda_n - log^+log^+(E_{n-1}(F,\beta)e^{\beta\lambda_n})} = \rho_\gamma.$$

Theorem 2.4. *Let $F(s) \in \overline{L_\beta}$ is of infinite γ-order, for any real number $0 < \beta < +\infty$, then*

$$\limsup_{\sigma\to 0^+} \frac{\gamma(log^+M(\sigma,F))}{logU(\frac{1}{\sigma})} = T \iff \limsup_{n\to+\infty} \phi_n(F,\beta,\lambda_n) = T;$$

where

$$\phi_n(F,\beta,\lambda_n) = \frac{\gamma(log^+(E_{n-1}(F,\beta)e^{\beta\lambda_n}))}{logU\left(\frac{\lambda_n}{log^+(E_{n-1}(F,\beta)e^{\beta\lambda_n})}\right)}$$

Proof. We want to proof only sufficient part of the theorem.

Suppose that

$$\lim_{n\to\infty} \phi_n(F,\beta,\lambda_n) = \limsup_{n\to\infty} \frac{\gamma(\log^+(E_{(n-1)}e^{\beta\lambda_n}))}{\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{\beta\lambda_n})}\right)} = T. \tag{17}$$

For sufficiently large positive integer n and any positive real number $\epsilon > 0$, we have

$$\log^+(E_{n-1}e^{\beta\lambda_n}) < J\left((T+\epsilon)\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{\beta\lambda_n})}\right)\right).$$

By using the similar argument of Theorem , we have

$$\log^+(E_{n-1}e^{-(\sigma-\beta)\lambda_n}) \le \lambda_n\left(\left(V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{-\beta\lambda_n}))}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right). \tag{18}$$

For any fixed and sufficiently small $\sigma > 0$, Set

$$I = J\left((T+\epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right),$$

i.e

$$\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})} = V\left(\exp\left(\frac{\gamma(I)}{T+\epsilon}\right)\right). \tag{19}$$

If $\log^+(E_{n-1}e^{\beta\lambda_n}) \le I$, for sufficiently large positive integer n, let $V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{\beta\lambda_n}))}{T+\epsilon}\right)\right) \ge 1$. Since $\sigma > 0$, from (17) and (18), and definition of $U(x)$, we have

$$\begin{aligned}\log^+(E_{n-1}e^{-(\sigma-\beta)\lambda_n}) &\le \lambda_n\left(\left(V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{\beta\lambda_n}))}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right)\\ &\le I = J\left((T+\epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right)\\ &\le J\left((T+\epsilon)\log\left((1+o(1))U(\frac{1}{\sigma})\right)\right).\end{aligned} \tag{20}$$

If $\log^+(E_{n-1e}e^{\beta\lambda_n}) > I$, it follows that from (17) and (18),

$$\begin{aligned}\log^+(E_{n-1}e^{-(\sigma-\beta)\lambda_n}) &\le \lambda_n\left(\left(V\left(\exp\left(\frac{\gamma(I)}{T+\epsilon}\right)\right)\right)^{-1} - \sigma\right)\\ &\le \lambda_n\left(\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)^{-1} - \sigma\right)\\ &< 0.\end{aligned} \tag{21}$$

Hence from (19) and (20) for sufficiently large positive integer n, we get

$$\log^+(E_{n-1}e^{-(\sigma-\beta)\lambda_n}) \le J\left((T+\epsilon)\log\left((1+o(1))U\left(\frac{1}{\sigma}\right)\right)\right). \tag{22}$$

For any $\beta > 0$, then from the definition of $E_k(\beta, F)$, then exist $P_1 \in \prod_{n-1}$ satisfying

$$\|F - P_1\| \le K_2 E_{n-1}. \tag{23}$$

Since,

$$\begin{aligned}
A_n^* \exp(-\beta\lambda_n) &= \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty<t<+\infty} \left|\int_{\lambda_n}^{x} \exp\{-ity\}d\alpha(y)\right| \exp(-\beta\lambda_n) \\
&\le \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty<t<+\infty} \left|\int_{\lambda_n}^{x} \exp\{-\beta - ity\}d\alpha(y)\right| \\
&\le \sup_{-\infty<t<+\infty} \left|\int_{\lambda_n}^{\infty} \exp\{-\beta - ity\}d\alpha(y)\right|,
\end{aligned}$$

then for any $P \in \prod_{n-1}$, we have

$$\begin{aligned}
A_n^* \exp(-\beta\lambda_n) &\le |F(\beta + it) - P(\beta + it)| \\
&\le \|F - P\|_\beta.
\end{aligned} \tag{24}$$

Hence, for any $\beta > 0$, and $F(s) \in \overline{L_\beta}$, it follows that from (22) and (23)

$$\begin{aligned}
A_n^* \exp(-\beta\lambda_n) &\le K_2 E_{n-1}, \\
i.e., \quad A_n^* &\le K_2 E_{n-1} \exp(\beta\lambda_n) \\
A_n^* e^{-\sigma\lambda_n} &\le K_2 E_{n-1} e^{-(\sigma-\beta)\lambda_n}.
\end{aligned} \tag{25}$$

Thus from (22), (24), by Lemma 1.8 and Theorem C as $\epsilon \to 0$, we have

$$\limsup_{\sigma\to 0} \frac{\gamma(\log^+ M_u(\sigma, F))}{\log U\left(\frac{1}{\sigma}\right)} \le T.$$

Suppose that

$$\limsup_{\sigma\to 0} \frac{\gamma(\log^+ M_u(\sigma, F))}{\log U\left(\frac{1}{\sigma}\right)} < T.$$

Then there exist any real number ϵ $(0 < \epsilon < \frac{T}{2})$, and for any sufficiently small $\sigma > 0$, we get

$$\log^+(M_u(\sigma, F)) \le J\left((T - 2\epsilon)\log U\left(\frac{1}{\sigma}\right)\right). \tag{26}$$

Since,

$$\begin{aligned}
E_{n-1}(\beta, F) &\le \|F - P_{n-1}\|_\beta \\
&\le |F(\beta + it) - P_{n-1}(\beta + it)| \\
&\le \left|\int_{\lambda_n}^{+\infty} \exp\{-(\beta + it)y\}d\alpha(y)\right|,
\end{aligned} \tag{27}$$

for $0 < \beta < \sigma$, and

$$\left|\int_{\lambda_k}^{+\infty} \exp\{-(\beta+it)y\}d\alpha(y)\right| = \lim_{b\to+\infty}\left|\int_{\lambda_k}^{b} \exp\{-(\beta+it)y\}d\alpha(y)\right|.$$

Set,

$$I_{j+k}(b\,;it) = \int_{\lambda_{j+k}}^{b} \exp(-ity)d\alpha(y), (\lambda_{j+k} \le b \le \lambda_{j+k+1}),$$

then we have $|I_{j+k}(b\,;it)| \le A^*_{j+k}$. Thus, it follows

$$\begin{aligned}\left|\int_{\lambda_k}^{b} \exp\{-(\beta+it)y\}d\alpha(y)\right| &= \left|\sum_{j=k}^{n+k-1}\int_{\lambda_j}^{\lambda_{j+1}} \exp(-\beta y)dyI_j(y\,;it) + \int_{\lambda_{n+k}}^{b} \exp(-\beta y)dyI_{n+k}(y\,;it)\right|\\ &\le 2\sum_{j=k}^{n+k} A^*_n e^{-\beta\lambda_{n+1}}.\end{aligned}$$

Because $b \to +\infty$ as $n \to +\infty$, thus it follows that

$$\left|\int_{\lambda_k}^{+\infty} \exp\{-(\beta+it)y\}d\alpha(y)\right| \le 2\sum_{n=k}^{+\infty} A^*_n e^{-\beta\lambda_{n+1}}. \tag{28}$$

By Lemma 1.2, from (26) and (27), we have

$$E_{n-1}(\beta, F) \le 6M_u(\sigma, F)\sum_{k=n}^{\infty} \exp\{(-\beta+\sigma)\lambda_k\}. \tag{29}$$

From (3), we can take $h'(0 < h' < h)$ such that $(\lambda_{(n+1)} - \lambda_n) \ge h'$ for $n \ge 0$, then from (29) for $\sigma \le \frac{\beta}{2}$, we get

$$\begin{aligned}E_{n-1}(\beta, F) &\le 6M_u(\sigma, F)\exp\{(-\beta+\sigma)\lambda_n\}\sum_{n=k}^{+\infty}\exp\{(\lambda_k-\lambda_n)(-\beta+\sigma)\}\\ &\le 6M_u(\sigma, F)\exp\{(-\beta+\sigma)\lambda_n\}\left(1-\exp\left(-\frac{3}{2}\beta kh'\right)\right)^{-1},\\ E_{n-1}(\beta, F) &\le K_3M_u(\sigma, F)\exp\{(-\beta+\sigma)\lambda_n\},\end{aligned} \tag{30}$$

where K_3 is constant. Then for sufficiently small $\sigma > 0$ and $0 < \beta < \sigma < +\infty$, we have

$$M_u(\sigma, F) \ge K_3E_{n-1}(\beta, F)e^{\lambda_n(\beta-\sigma)} = K_3E_{n-1}(\beta, F)\exp(\beta\lambda_n)e^{(-\lambda_n\sigma)}, \tag{31}$$

where $K_4 = 1 - \exp\left(-\frac{3}{2}\beta h'\right)$. Hence, it follows that from (26) and (31)

$$\log^+\left[K_3E_{n-1}(\beta, F)\exp(\beta\lambda_n)e^{(-\lambda_n\sigma)}\right] \le \log M_u(\sigma, F) \le J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma}\right)\right). \tag{32}$$

From the assumption, there exist a subsequence $\{\lambda_{n(p)}\}$ such that for sufficiently large p,

$$\gamma\left(\log^+(E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\})\right) > (T-\epsilon)\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right). \tag{33}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right) = \frac{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}{1+\log U\left(\frac{\lambda_{n(p)}}{log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)}. \tag{34}$$

From (31) and (34), by using the similar argument of Theorem 2.1, we get

$$\log^+(E_{n(p-1)}e^{\beta\lambda_{n(p)}}) = J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)\right). \tag{35}$$

Then by Cauchy mean value theorem, then there exist a real number $\xi \in (x_1, x_2)$ where

$$x_1 = J\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_p}\right)\right)$$
$$x_2 = x_1\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)\right),$$

such that

$$\gamma\left(\log^+(E_{n(p-1)}e^{\beta\lambda_{n(p)}})\right) = \gamma\left(J\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)\right)\right)$$
$$+\log\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)\right)\xi\gamma'(\xi),$$

since

$$\lim_{p\to\infty}\frac{\log\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)\right)}{\log U\left(\frac{\lambda_{n(p)}}{\log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right)} = 0,$$

then for $p \to +\infty$ and let $\sigma \to 0^+$, it follows that

$$\gamma\left(log^+(E_{n(p-1)}e^{\beta\lambda_{n(p)}})\right) = (T-2\epsilon)(1+o(1))logU\left(\frac{\lambda_{n(p)}}{log^+ E_{n(p-1)}\exp\{\beta\lambda_{n(p)}\}}\right) + o(1). \tag{36}$$

From (26) and (36) by Lemma 1.8, we obtain a contradiction with the assumption $0 < \epsilon < \frac{T}{2}$. Thus,

$$\limsup_{\sigma\to 0^+}\frac{\gamma(\log^+ M_u(\sigma, F))}{\log U\left(\frac{1}{\sigma}\right)} = T.$$

Hence, the sufficient part is proved. The necessary part is similar. □

References

[1] A. Nautiyal, *On the coefficients of analytic Dirichlet series of fast growth*, Indian Journal of Pure and Appl. Math., 155(10), 1984, 1102-1114.

[2] A. Nautiyal and D. P. Shukla, *On the approximation of an analytic function by exponential polynomials*, Indian J. of Pure and App. Math., 14(6)(1983),722-727.

[3] C. J. K. Batty, *Tauberian theorem for the Laplace-Stieltjes transform*, Trans. Amer. Math.Soc., 322(2)(1990), 783-804.

[4] D. C. Sun, *On the distribution of values of random Dirichlet Series II*, Chin. Ann. Math. Ser. B, 11(1)(1990), 33-44.

[5] H. Y. Xu, *The logarithmic order and logarithmic type of Laplace-Stieltjes transform*, J. Jiangxi Norm. Univ. Nat. Sci., 41(2017), 180-183.

[6] H. Y. Xu, C. F. Yi and T. B. Cao, *On proximate order and type function of Laplace - Stieltjes transformation convergent in the right half plane*, Math.Commun., 17(2012), 355-369.

[7] H. Y. Xu and Z. X. Xuan, *The growth and value distribution of Laplace -Stieltjes transformation with infinite order in the right half plane*, Journal of Inequalities and Applications 2013(2013), Art. 273.

[8] H. Y. Xu and Z. X. Xuan, *The singular points of analytic functions with finite X-order defined by Laplace-Stieltjes transformation*, Journal of Functional Spaces, 2015(2015), Art.ID 865069, 9 pages.

[9] J. R. Yu, *Borel's line of entire functions represented by Laplace-Stieltjes transformation (in chinese)*, Acta Math. Sinica, 13(1963), 471-484.

[10] J. R. Yu, X. Q. Ding and F. J. Tian, *On the distribution of values of Dirichlet series and random Dirichlet Series*, Wuhan: Press in Wuhan University, (2004).

[11] L. N. Shang and Z. S. Gao, *The growth of entire functions of infinite order represented by Laplace-Stieltjes transformation*, Acta Math. Sci., 27A(6)(2007), 1035-1043.

[12] M. S. Liu, *The regular growth of Dirichlet series of finite order in the half plane*, J. Syst. Sci. Math. Sci., 22(2)(2002), 229-238.

[13] J. R. Yu, *Dirichlet Series and the Random Dirichlet Series*, Science Press, Beijing, (1997).

[14] W. C. Lu, *On the λ^*-logarithmic type of analytic functions represented by Laplace-Stieltjes transformation*, J. Jiangxi Norm. Univ. Nat. Sci., 40(2016), 591-594.

[15] W. J. Tang, Y. Q. Cui, H. Q. Xu, *On some q-order and q-type of Taylor-Hadamard Product function*, J. Jiangxi Norm. Univ. Nat. Sci., 40(2016), 276-279.

[16] Y. Y. Kong and Y. Yang, *On the growth properties of the Laplace-Stieltjes transform*, Complex Variables and Elliptic Equation, 59(2014), 553-563.

[17] Y. Y. Kong and Y. Y. Huo, *On general order and type of Laplace-Stieltjes transforms analytic in the right half plane*, Acta Math. Sinica, 59A(2016), 91-98.

[18] X. Luo, X. Z. Liu and Y. Y. Kong, *The regular growth of Laplace-Stieltjes transforms*, J. of Math. (PRC) 34(2014), 1181-1186.

[19] X. Luo and Y. Y. Kong, *On the order and type Laplace-Stieltjes transforms of slow growth*, Acta Math. Sci., 32A(2012), 601-607.

[20] Z. S. Gao, *The growth of entire functions represented by Dirichlet series*, Acta Mathematica Sinica, 42A(1999), 741-748.