The Approximation of Laplace-Stieltjes Transforms in the Half Plane

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\textbf{Abstract:} In this paper, we study the growth of the analytic function represented by Laplace-Stieltjes transform of infinite order which is convergent in the right half plane. We also investigate the error in approximation defined on Laplace-Stieltjes transform of finite $\gamma_U$-order in the half plane, and some relations between the error and growth of Laplace-Stieltjes transform of finite $\gamma_U$-order.

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1. Introduction

Let Laplace-Stieltjes transform

\begin{equation}
F(s) = \int_0^\infty e^{-sx} d\alpha(x), \quad s = \sigma + it,
\end{equation}

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y]$ ($0 < Y < +\infty$), $\sigma$ and $t$ are real variables. If $\alpha(t)$ is a step function and satisfies,

\begin{equation}
\alpha(t) = \begin{cases} 
a_1 + a_2 + \cdots + a_n, & \lambda_n < x < \lambda_{n+1} \\
0, & 0 \leq x < \lambda_1 \\
\frac{a(x+1)+a(x-)}{2}, & x > 0
\end{cases}
\end{equation}

where the sequence $\{\lambda_n\}_{n=0}^\infty$ satisfies

\begin{equation}
0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to \infty \text{ as } n \to \infty,
\end{equation}

where $\alpha(x)$ is stated in (1) and $\{\lambda_n\}$ satisfy (2),

\begin{equation}
\limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h < +\infty, \quad \limsup_{n \to \infty} \frac{n}{\lambda_n} = E < +\infty.
\end{equation}

Set

\begin{equation}
A^*_n = \sup_{\lambda_n \leq x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{\lambda_{n+1}} e^{-it\alpha(y)} dy \right|,
\end{equation}

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Thus, \( F(s) \) becomes Dirichlet series,
\[
F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it,
\]
where \( \sigma, t \) are real variables and \( a_n \) are non-zero complex numbers.

The author studied the growth and value distribution of Laplace-Stieltjes transform (1) in 1963, J. R. Yu [9], and we get Valiron-Knopp-Bohr formula with associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transform and to investigate the singular direction-borel line of Laplace-Stieltjes transform. After his work, some mathematician investigated properties on the growth and the value distribution of Laplace-Stieltjes transforms in ([3, 5, 6, 14, 16, 17]) and J. R. Yu, L. N. Shang, Z. S. Gao, and H. Y. Xu investigated the value distribution of such functions ([7–9, 11]). Furthermore, for Dirichlet series (3), a special form of Laplace-Stieltjes transform, authors paid considerable attention to the growth and value distribution of analytic functions defined by Dirichlet series. They founded many interesting results in ([1, 2, 4, 10, 12, 13, 15, 18–20]).

In 1963, Yu [9] proved the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence; Theorem 1.1 ([9]).

\[
\limsup_{n \to \infty} \frac{\log A_n}{\lambda_n} = 0.
\]

Thus, \( F(s) \) becomes Dirichlet series,
\[
F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it,
\]
where \( \sigma, t \) are real variables and \( a_n \) are non-zero complex numbers.

### Definition 1.2 ([7])
If the Laplace-Stieltjes transform (1) satisfy \( \sigma_u^F = 0 \), then
\[
\limsup_{\sigma \to \infty} \frac{\log^+ \log M_u(\sigma, F)}{-\log \sigma} = \rho,
\]
we call \( F(s) \) is of order \( \rho \) in the right half plane, where \( \log^+ x = \max(\log x, 0) \).

For \( \rho = \infty \), we get the definition of \( \gamma \)-order of Laplace-Stieltjes transform (1) as follows that.

### Definition 1.3 ([8])
If Laplace-Stieltjes transform (1) of \( \gamma \)-order satisfy,
\[
\limsup_{\sigma \to \infty^+} \frac{\gamma(\log^+ M_u(\sigma, F))}{-\log \sigma} = \rho_\gamma,
\]
where \( \gamma(x) \in \mathbb{R} \), then \( \rho_\gamma \) is called the \( \gamma \)-order of \( F(s) \), and \( \mathbb{R} \) is the class of all functions \( \gamma(x) \) satisfies the following conditions:
(i). \( \gamma(x) \) is positive, strictly increasing, differentiable and tends to \(+\infty\) as \( x \to +\infty \) and is defined on \([a, \infty), a > 0\),

(ii). \( x^n \gamma(x) = o(1) \) as \( x \to +\infty \).

**Theorem 1.4 ([8])**. Let Laplace-Stieltjes transformation \( F(s) \in \mathcal{L}_\beta \) of infinite order have finite \( \gamma \)-order \( \rho_\gamma \) \( (0 < \rho_\gamma < +\infty) \) and the sequence (2) satisfies (3) and (4), then

\[
\limsup_{\sigma \to 0^+} \frac{\gamma(\log \mu(\sigma, F))}{-\log \sigma} = \rho_\gamma \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log M_\mu(\sigma, F))}{-\log \sigma} = \rho_\gamma,
\]

**Theorem 1.5 ([8])**. Let Laplace-Stieltjes transformation \( F(s) \in \mathcal{L}_\beta \) of infinite order have finite \( \gamma \)-order \( \rho_\gamma \) \( (0 < \rho_\gamma < +\infty) \) and the sequence (2) satisfies (3) and (4), then

\[
\limsup_{n \to \infty} \frac{\gamma(\log M_\mu(n, F))}{\log \lambda_n - \log^+ \log^+ A_n^+} = \rho_\gamma \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log M_\mu(\sigma, F))}{-\log \sigma} = \rho_\gamma.
\]

**Theorem 1.6 ([4])**. Let Laplace-Stieltjes transform \( F(s) \in \mathcal{L}_\beta \) is of infinite \( \gamma \)-order, then

\[
\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M_\mu(\sigma, F))}{\log U(\frac{1}{n})} = T \iff \limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ \mu(\sigma, F))}{\log U(\frac{1}{n})} = T
\]

where \( 0 < T < \infty \) and \( U(x) = x^\rho(x) \) satisfies the following conditions,

(i). \( \rho(x) \) is monotone and \( \lim_{x \to \infty} \rho(x) = \infty \);

(ii). \( \lim_{x \to \infty} \frac{\log U(x')}{\log U(x)} = 1 \), where \( x' = x(1 + \frac{1}{\log U(x)}) \).

**Definition 1.7 ([7])**. Let Laplace-Stieltjes transform \( F(s) \) of infinite order has infinite \( \gamma \)-order and satisfies,

\[
\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M_\mu(\sigma, F))}{\log U(\frac{1}{n})} = T,
\]

then \( T \) is called \( \gamma \)-order of Laplace-Stieltjes transform \( F(s) \).

We denote \( L \) to be the class of all the functions \( F(s) \) of the form (1) which is analytic in the half plane \( \text{Re}(s) > 0 \) and the sequence \( \{\lambda_n\} \) satisfies (2), (3) and (4), and denote \( \mathcal{L}_\beta \) to the class of all the functions \( F(s) \) of the form (1) which is analytic in the half plane \( \text{Re}(s) \leq \beta (-\infty < \beta < +\infty) \) and the sequence \( \{\lambda_n\} \) satisfies (2) and (3).

Thus, if \( -\infty < \beta < 0 \) and \( F(s) \in L \), then \( F(s) \in \mathcal{L}_\beta \); if \( 0 < \beta < +\infty \) and \( F(s) \in \mathcal{L}_\beta \) then \( F(s) \in L \). If \( A_n^+ = 0 \) for \( n \geq k + 1 \), and \( A_n^* \neq 0 \), then \( F(s) \) will be called an exponential polynomial of degree \( k \) usually denoted by \( P_k \) i.e.,

\[
P_k(s) = \sum_{i=0}^{k} \exp(-sg_i)\alpha_i(g).
\]

Since, \( F(s) \) is an analytic in the half plane, \( H = \{s = \sigma + it, \sigma > 0, t \in \mathbb{R}\} \). We denote \( \Pi_n \) to the class of all exponential polynomial of degree \( n \) i.e.,

\[
\Pi_n = \{\sum_{i=1}^{n} b_i \exp(-s\lambda_i); (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\}.
\]

For \( F(s) \in L, -\infty < \beta < +\infty \), we denote \( E_n(F, \beta) \) be the error in approximating the function \( F(s) \) by exponential polynomial of degree \( n \) in the uniform norms.

\[
E_n(F, \beta) = \inf_{P \in \Pi_n} \|F - P\|_B, n = 1, 2, \ldots
\]

where

\[
\|F - P\|_B = \max_{-\infty < t < +\infty} |F(\beta + it) - P(\beta + it)|.
\]

The authors ([2, 11]) investigated the approximation of analytic function defined by Laplace-Stieltjes transforms of finite order. In this paper, we study the approximation of analytic function defined by Laplace-Stieltjes transform and obtain relation between the error \( E_n(F, \beta) \) and growth order of \( F(s) \), when \( F(s) \) is of infinite order.

To prove our results we use the following Lemma's;
Lemma 1.8 ([7]). Let $\gamma(x) \in \Im$ and $c$ be a constant, and $\psi(x)$ be the function such that

$$\limsup_{x \to +\infty} \frac{\log^+ \psi(x)}{\log x} = \rho, \quad (0 \leq \rho < \infty)$$

and if the real function $M(x)$ satisfies

$$\limsup_{x \to +\infty} \frac{\gamma(\log M(x))}{\log x} = \nu > 0.$$

Then we have

$$\limsup_{x \to +\infty} \frac{\gamma(\log M(x) + c)}{\log x} = \nu, \quad \limsup_{x \to +\infty} \frac{\gamma(\psi \log M(x))}{\log x} = \nu.$$

Lemma 1.9 ([7]). If the abscissa $\sigma_{F}^0 = 0$, of the uniform convergent Laplace-Stieltjes transformation and the sequence (2) satisfies (3), then for any given $\epsilon \in (0, 1)$, and for $\sigma(>0)$ sufficiently reaching 0 we have

$$\frac{1}{3} \mu(\sigma, F) \leq M_{\epsilon}(\sigma, F) \leq K(\epsilon) \mu((1-\epsilon)\sigma, F) \frac{1}{\sigma},$$

where $K(\epsilon)$ is a constant depending on $\epsilon$.

2. Main Results

Theorem 2.1. Let Laplace-Stieltjes transform $F(s) \in L_{\beta}$ of infinite order has infinite $\gamma$-order, then

$$\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M_{\epsilon}(\sigma, F))}{\log U(\frac{1}{\sigma})} = T \iff \limsup_{n \to \infty} \frac{\gamma(\log^+ A_{\sigma}^*)}{\log U\left(\frac{\lambda_{n}}{\log^+ A_{\sigma}^*}\right)} = T.$$

Proof. We want to proof only sufficient part.

Suppose that

$$\limsup_{n \to \infty} \frac{\gamma(\log^+ A_{\sigma}^*)}{\log U\left(\frac{\lambda_{n}}{\log^+ A_{\sigma}^*}\right)} = T. \quad (6)$$

Then, for any positive real number $\epsilon > 0$, for sufficiently large $n$, we have

$$\log^+ A_{\sigma}^* < J\left((T + \epsilon) \log U\left(\frac{\lambda_{n}}{\log^+ A_{\sigma}^*}\right)\right),$$

where $J(x)$ is the inverse of $\gamma(x)$. Let $V(x)$ is the inverse function of $U(x)$, then

$$\frac{\gamma(\log^+ A_{\sigma}^*)}{T + \epsilon} < \log U\left(\frac{\lambda_{n}}{\log^+ A_{\sigma}^*}\right)$$

$$\log A_{\sigma}^* < \frac{\lambda_{n}}{V\left(\exp\left(\frac{\gamma(\log^+ A_{\sigma}^*)}{T + \epsilon}\right)\right)}$$

$$\log A_{\sigma}^* < \lambda_{n} \left[\exp\left(\frac{\gamma(\log^+ A_{\sigma}^*)}{T + \epsilon}\right)\right]^{-1}.$$  

Thus, we have

$$\log A_{\sigma}^* e^{-\lambda_{n} \sigma} < \lambda_{n} \left[\exp\left(\frac{\gamma(\log^+ A_{\sigma}^*)}{T + \epsilon}\right)\right]^{-1} - \sigma. \quad (7)$$

For any fixed and sufficiently small $\sigma > 0$, set

$$I = J\left((T + \epsilon) \log U\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)\right)$$
\[ \gamma(I) = (T + \epsilon) \log U \left( \frac{1}{\sigma} + \frac{1}{\sigma \log U \left( \frac{1}{T} \right)} \right) \]
\[ \frac{1}{\sigma} + \frac{1}{\sigma \log U \left( \frac{1}{T} \right)} = V \left( \exp \left( \frac{\gamma(I)}{T + \epsilon} \right) \right). \] (8)

If \( A_n^* \leq I \), then for sufficiently large \( n \), let \( V \left( \exp \left( \frac{\gamma(I)}{T + \epsilon} \right) \right) \geq 1 \), for \( \sigma > 0 \), from (7), (8) and definition of \( U(x) \), we get
\[ \log^+ (A_n^* e^{-\lambda_n \sigma}) \leq I \left[ \left( V \left( \exp \left( \frac{\gamma(l^+ A_n^*)}{T + \epsilon} \right) \right) \right)^{-1} - \sigma \right] \]
\[ \leq J \left( (T + \epsilon) \log \left( (1 + o(1))U \left( \frac{1}{\sigma} \right) \right) \right) \] (9)

If \( l^+ A_n^* > I \) then from (7) and (8), we get
\[ \log(A_n^* e^{-\lambda_n \sigma}) \leq \lambda_n \left( \left( V \left( \exp \left( \frac{\gamma(l^+ A_n^*)}{T + \epsilon} \right) \right) \right)^{-1} - \sigma \right) \]
\[ < 0 \] (10)

For sufficiently large \( n \) from (9) and (10), we get
\[ \log \mu(\sigma, F) \leq J \left( (T + \epsilon) \log \left( (1 + o(1))U \left( \frac{1}{\sigma} \right) \right) \right) \]
\[ \leq J \left( (T + \epsilon) \log U \left( \frac{1}{\sigma} \right) \right). \]

Since \( \epsilon \) is arbitrary by Theorem C and Lemma 2.2, we get
\[ \lim \sup_{\sigma \to 0^+} \frac{\gamma(\log M_n(\sigma, F))}{\log U \left( \frac{1}{\sigma} \right)} \leq \lim \sup_{n \to \infty} \frac{\gamma(l^+ A_n^*)}{\log U \left( \frac{1}{\log l^+ A_n^*} \right)} = T. \]

Suppose that
\[ \lim \sup_{\sigma \to 0^+} \frac{\gamma(\log M_n(\sigma, F))}{\log U \left( \frac{1}{\sigma} \right)} < T. \] (11)

Then, there exist a real number \( \epsilon(0 < \epsilon < \frac{T}{2}) \). For any positive number \( n \) and sufficiently small \( \sigma > 0 \) from Lemma 1.2, we have
\[ \log^+ (A_n^* e^{-\lambda_n \sigma}) \leq \log M_n(\sigma, F) \leq J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma} \right) \right). \] (12)

From (6), there exist a subsequence \( \{n(p)\} \) for sufficiently large \( p \), we have
\[ \gamma(\log l^+ A_{n(p)}^*) > (T - \epsilon) \log U \left( \frac{\lambda_{n(p)}}{\log l^+ A_{n(p)}^*} \right). \] (13)

Taking a sequence \( \{\sigma_p\} \) satisfy
\[ J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma_p} \right) \right) = \frac{\log^+ (A_{n(p)}^*)}{1 + \log U \left( \frac{\lambda_{n(p)}}{\log l^+ A_{n(p)}^*} \right)} \] (14)

From (12) and (13), we get
\[ \log A_{n(p)}^* - \lambda_{n(p)} \sigma_p \leq J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma_p} \right) \right) = \frac{\log^+ (A_{n(p)}^*)}{1 + \log U \left( \frac{\lambda_{n(p)}}{\log l^+ A_{n(p)}^*} \right)}. \]
Then for sufficiently large $p$, we have

\[
\frac{1}{\sigma_p} \leq \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \left( 1 + \frac{1}{\log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right)} \right).
\]

and

\[
U \left( \frac{1}{\sigma_p} \right) \leq U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \left( 1 + \frac{1}{\log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right)} \right) \right) \leq U^{(1 + o(1))} \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right). \tag{15}
\]

From (14) and (15), we get

\[
\log^+ (A_{n(p)}^*) = J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma_p} \right) \right) \left( 1 + \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right).
\]

Thus, from the Cauchy mean value theorem and there exist a real number $\xi$ between $x_1$ and $x_2$, where

\[
x_1 = J \left( (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right) \text{ and }
\]

\[
x_2 = J \left( (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \left( 1 + \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right) \right)
\]

such that

\[
\gamma(\log^+ (A_{n(p)}^*)) = \gamma \left\{ 1 + \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right\} J \left( (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right) \}
\]

\[
= \gamma \left( J \left( (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right) \right) + \log \left( 1 + \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right) \xi^\gamma(\xi)
\]

Since,

\[
\lim_{p \to \infty} \frac{\log \left( 1 + \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) \right)}{\log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right)} = 0.
\]

Then for sufficiently large $p$, we have

\[
\gamma(\log^+ (A_{n(p)}^*)) = (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right) + K_1 \xi^\gamma(\xi) \log U \left( \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right), \tag{16}
\]

where $K_1$ is a constant. From (13) and (16), we get a contradiction. Thus,

\[
\limsup_{\sigma \to 0^+} \frac{\gamma(\log M_n(\sigma, F))}{\log U \left( \frac{1}{T} \right)} = T.
\]

Hence, the sufficient part is completed. The necessary part is similar. \qed

Now we establish some relation between $E_n(F, \beta)$ and growth of $F(s)$.

**Theorem 2.2.** Let Laplace-Stieljes transform $F(s) \in \mathcal{L}_{\beta}$, $0 < \beta < +\infty$ is of order $\rho$, then

\[
\rho = \limsup_{n \to \infty} \frac{\log^+ \log^+ (E_n(F, \beta)e^{\beta \lambda_n})}{\log^+ \lambda_n + \log^+ \log^+ (E_n(F, \beta)e^{\beta \lambda_n})}.
\]

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Theorem 2.3. Let Laplace-Stieltjes transform $F(s) \in \mathbb{L}_\beta$ is of finite $\gamma$-order $\rho_1$, for any real number $0 < \beta < +\infty$, then
\[
\limsup_{n \to \infty} \frac{\log \lambda_n}{\log U(\frac{1}{\beta})} = \rho_1.
\]

Theorem 2.4. Let $F(s) \in \mathbb{L}_\beta$ is of infinite $\gamma$-order, for any real number $0 < \beta < +\infty$, then
\[
\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M(\sigma, F))}{\log U(\frac{1}{\beta})} = T \iff \limsup_{n \to \infty} \phi_n(F, \beta, \lambda_n) = T;
\]
where
\[
\phi_n(F, \beta, \lambda_n) = \frac{\gamma(\log^+(E_{n-1}e^{\beta \lambda_n}))}{\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{\beta \lambda_n})}\right)}.
\]

Proof. We want to prove only sufficient part of the theorem.

Suppose that
\[
\lim_{n \to \infty} \phi_n(F, \beta, \lambda_n) = \limsup_{n \to \infty} \frac{\gamma(\log^+(E_{n-1}e^{\beta \lambda_n}))}{\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{\beta \lambda_n})}\right)} = T.
\]

For sufficiently large positive integer $n$ and any positive real number $\epsilon > 0$, we have
\[
\log^+(E_{n-1}e^{\beta \lambda_n}) < J\left((T + \epsilon)\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{\beta \lambda_n})}\right)\right).
\]

By using the similar argument of Theorem 2.3, we have
\[
\log^+(E_{n-1}e^{-(\sigma - \beta)\lambda_n}) \leq \lambda_n \left(V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{-\beta \lambda_n}))}{T + \epsilon}\right)\right)\right)^{-1} - \sigma.
\]

For any fixed and sufficiently small $\sigma > 0$, Set
\[
I = J\left((T + \epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)\right),
\]
\[
i.e
\]
\[
\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})} = V\left(\exp\left(\frac{\gamma(I)}{T + \epsilon}\right)\right).
\]

If $\log^+(E_{n-1}e^{\beta \lambda_n}) \leq I$, for sufficiently large positive integer $n$, let $V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{\beta \lambda_n}))}{T + \epsilon}\right)\right) \geq 1$. Since $\sigma > 0$, from (17) and (18), and definition of $U(x)$, we have
\[
\log^+(E_{n-1}e^{-(\sigma - \beta)\lambda_n}) \leq \lambda_n \left(V\left(\exp\left(\frac{\gamma(\log^+(E_{n-1}e^{\beta \lambda_n}))}{T + \epsilon}\right)\right)\right)^{-1} - \sigma
\]
\[
\leq I = J\left((T + \epsilon)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)\right)
\]
\[
\leq J\left((T + \epsilon)\log \left(1 + o(1)\right)U\left(\frac{1}{\sigma}\right)\right).
\]

If $\log^+(E_{n-1}e^{\beta \lambda_n}) > I$, it follows that from (17) and (18),
\[
\log^+(E_{n-1}e^{-(\sigma - \beta)\lambda_n}) \leq \lambda_n \left(V\left(\exp\left(\frac{\gamma(I)}{T + \epsilon}\right)\right)\right)^{-1} - \sigma
\]
\[
\leq \lambda_n \left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})}\right)^{-1} - \sigma
\]
\[
< 0.
\]
Hence from (19) and (20) for sufficiently large positive integer \( n \), we get

\[
\log^+ (E_{n-1} e^{-(\sigma - \beta)\lambda_n}) \leq J \left( (T + \epsilon) \log \left( (1 + o(1))U \left( \frac{1}{\sigma} \right) \right) \right).
\]

(22)

For any \( \beta > 0 \), then from the definition of \( E_k(\beta, F) \), then exist \( P_1 \in \prod_{n-1} \) satisfying

\[
\|F - P_1\| \leq K_2 E_{n-1}.
\]

(23)

Since,

\[
A_n^* \exp (-\beta \lambda_n) = \sup_{\lambda_n < x \leq \lambda_{n+1}} \left| \int_x^{\infty} \exp(-\beta \lambda_n y) d\alpha(y) \right| \leq K_2 \exp(-\sigma \lambda_n)
\]

then for any \( P \in \prod_{n-1} \), we have

\[
A_n^* \exp (-\beta \lambda_n) \leq |F(\beta + it) - P(\beta + it)|
\]

\[
\leq \|F - P\| \beta.
\]

(24)

Hence, for any \( \beta > 0 \), and \( F(s) \in \overline{T} \), it follows that from (22) and (23)

\[
A_n^* \exp(-\beta \lambda_n) \leq K_2 E_{n-1},
\]

\[
i.e., \quad A_n^* \leq K_2 E_{n-1} \exp(\beta \lambda_n)
\]

\[
A_n^* e^{-\sigma \lambda_n} \leq K_2 E_{n-1} e^{-(\sigma - \beta)\lambda_n}.
\]

(25)

Thus from (22), (24), by Lemma 1.8 and Theorem C as \( \epsilon \to 0 \), we have

\[
\limsup_{\sigma \to 0} \frac{\gamma(\log^+ M_\sigma(\sigma, F))}{\log U \left( \frac{1}{\sigma} \right)} \leq T.
\]

Suppose that

\[
\limsup_{\sigma \to 0} \frac{\gamma(\log^+ M_\sigma(\sigma, F))}{\log U \left( \frac{1}{\sigma} \right)} < T.
\]

Then there exist any real number \( \epsilon \) \((0 < \epsilon < \frac{T}{2})\), and for any sufficiently small \( \sigma > 0 \), we get

\[
\log^+ (M_\sigma(\sigma, F)) \leq J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma} \right) \right).
\]

(26)

Since,

\[
E_{n-1}(\beta, F) \leq \|F - P_{n-1}\| \beta
\]

\[
\leq |F(\beta + it) - P_{n-1}(\beta + it)|
\]

\[
\leq \left| \int_{\lambda_n}^{+\infty} \exp(-(\beta + it)y) d\alpha(y) \right|.
\]

(27)
for $0 < \beta < \sigma$, and
\[
\left| \int_{\lambda_k}^{+\infty} \exp\{- (\beta + it)y\} \, da(y) \right| = \lim_{b \to +\infty} \left| \int_{\lambda_k}^{b} \exp\{- (\beta + it)y\} \, da(y) \right|.
\]
Set,
\[
I_{j+k}(b : it) = \int_{\lambda_{j+k}}^{b} \exp(-ity) \, da(y), \quad (\lambda_{j+k} \leq b \leq \lambda_{j+k+1}),
\]
then we have $|I_{j+k}(b : it)| \leq A_{j+k}^*$. Thus, it follows
\[
\left| \int_{\lambda_k}^{b} \exp\{- (\beta + it)y\} \, da(y) \right| = \left| \sum_{j=k}^{n+k-1} \int_{\lambda_j}^{\lambda_{j+1}} \exp(-by) \, dy + \int_{\lambda_{n+k}}^{b} \exp(-by) \, dy \right|
\leq 2 \sum_{j=k}^{n+k} A_j^* e^{-\beta \lambda_{n+1}}.
\]
Because $b \to +\infty$ as $n \to +\infty$, thus it follows that
\[
\left| \int_{\lambda_k}^{+\infty} \exp\{- (\beta + it)y\} \, da(y) \right| \leq 2 \sum_{n=k}^{+\infty} A_n^* e^{-\beta \lambda_{n+1}}.
\]  
By Lemma 1.2, from (26) and (27), we have
\[
E_{n-1}(\beta, F) \leq 6M_u(\sigma, F) \sum_{k=n}^{+\infty} \exp\{- (\beta + \sigma) \lambda_k\}.
\]  
From (3), we can take $h'(0 < h' < h)$ such that $(\lambda_{n+1} - \lambda_n) \geq h'$ for $n \geq 0$, then from (29) for $\sigma \leq \frac{3}{2}$, we get
\[
E_{n-1}(\beta, F) \leq 6M_u(\sigma, F) \exp\{- (\beta + \sigma) \lambda_n\} \sum_{n=k}^{+\infty} \exp\{- (\lambda_k - \lambda_n) (\beta + \sigma)\}
\leq 6M_u(\sigma, F) \exp\{- (\beta + \sigma) \lambda_n\} \left( 1 - \exp\left( -\frac{3}{2} \beta h' \right) \right)^{-1},
\]
\[
E_{n-1}(\beta, F) \leq K_3 M_u(\sigma, F) \exp\{- (\beta + \sigma) \lambda_n\},
\]
where $K_3$ is constant. Then for sufficiently small $\sigma > 0$ and $0 < \beta < \sigma < +\infty$, we have
\[
M_u(\sigma, F) \geq K_4 E_{n-1}(\beta, F) e^{\lambda_n(\beta - \sigma)} = K_4 E_{n-1}(\beta, F) \exp(\beta \lambda_n) e^{-\lambda_n \sigma},
\]
where $K_4 = 1 - \exp\left(-\frac{3}{2} \beta h'\right)$. Hence, it follows that from (26) and (31)
\[
\log^+ \left[ K_4 E_{n-1}(\beta, F) \exp(\beta \lambda_n) e^{-\lambda_n \sigma} \right] \leq \log M_u(\sigma, F) \leq J \left( (T - 2c) \log U \left( \frac{1}{\sigma} \right) \right).
\]
From the assumption, there exist a subsequence $\{\lambda_n(p)\}$ such that for sufficiently large $p$,
\[
\gamma (\log^+ (E_{n(p-1)} \exp(\beta \lambda_n(p)))) > (T - \epsilon) \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right).
\]
Take a sequence $\{\sigma_p\}$ satisfying
\[
J \left( (T - 2c) \log U \left( \frac{1}{\sigma_p} \right) \right) = \frac{\log^+ E_{n(p-1)} \exp \{\beta \lambda_n(p)\}}{1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right)}.
\]
From (31) and (34), by using the similar argument of Theorem 2.1, we get
\[
\log^+ (E_{n(p-1)}e^{\beta \lambda_n(p)}) = J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma_p} \right) \right) \left( 1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right). \tag{35}
\]
Then by Cauchy mean value theorem, then there exist a real number \( \xi \in (x_1, x_2) \) where
\[
x_1 = J \left( (T - 2\epsilon) \log U \left( \frac{1}{\sigma_p} \right) \right) \left( 1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right),
\]
\[
x_2 = x_1 \left( 1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right),
\]
such that
\[
\gamma \left( \log^+ (E_{n(p-1)}e^{\beta \lambda_n(p)}) \right) = \gamma \left( J \left( (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right) \right) + \log \left( 1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right) \xi \gamma' (\xi),
\]
since
\[
\lim_{p \to \infty} \log \left( 1 + \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) \right) = 0,
\]
then for \( p \to +\infty \) and let \( \sigma \to 0^+ \), it follows that
\[
\gamma \left( \log^+ (E_{n(p-1)}e^{\beta \lambda_n(p)}) \right) = (T - 2\epsilon)(1 + o(1)) \log U \left( \frac{\lambda_n(p)}{\log^+ E_{n(p-1)} \exp(\beta \lambda_n(p))} \right) + o(1). \tag{36}
\]
From (26) and (36) by Lemma 1.8, we obtain a contradiction with the assumption \( 0 < \epsilon < \frac{T}{2} \). Thus,
\[
\limsup_{\sigma \to 0^+} \frac{\gamma(log^+ M_\sigma(\sigma, F))}{\log U \left( \frac{\sigma}{2} \right)} = T.
\]
Hence, the sufficient part is proved. The necessary part is similar. \( \square \)

References


